Naïve blowups and canonical birationally commutative factors

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Outline

Groundrules

- $\mathbb{C}$ = algebraically closed uncountable field of characteristic 0.
- A is always a commutative noetherian $\mathbb{C}$-algebra, and a skew polynomial extension $A[t;\tau]$ always has $\tau \in \mathrm{Aut}_{\mathbb{C}}(A)$.

Definition

A graded algebra R is birationally commutative *if $R \subseteq A[t;\tau]$.*

Key definition:

Definition

A $\mathbb{C}$*-algebra R is* strongly noetherian *if* $R \otimes_{\mathbb{C}} A$ *is noetherian for* any *commutative noetherian* $\mathbb{C}$*-algebra A.*

We are interested in noetherian algebras that might not be strongly noetherian.

$$\{ \text{ strongly noetherian } \} \subsetneq \{ \text{noetherian} \}$$

Goal: Extend results that hold for strongly noetherian algebras to the noetherian case.

Theorem (Rogalski-Zhang)

If $R = \mathbb{C} \oplus R_1 \oplus \cdots$ is a strongly noetherian connected graded algebra generated in degree 1 then there is a birationally commutative factor of R that is universal for maps from R to birationally commutative algebras.

- *That is, we have $\theta : R \to A[t;\tau]$.*
- *Any map $R \to A'[t';\tau']$ factors through $\theta(R)$ (up to finite dimension).*
- *$\theta(R)$ is, in large degree, a* twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$.

- We'll recall the definition of $B(X, \mathcal{L}, \sigma)$ in a moment. For now: a well-understood, well-behaved, birationally commutative algebra defined by geometric data, including a projective scheme X with $A = \mathbb{C}(X)$.
- The universal property above is slightly different from the universal property given by Rogalski-Zhang.

Example (Artin-Tate-Van den Bergh)

Let

$$S := \mathbb{C}\langle x, y, z\rangle/(axy-byx-cz^2, ayz-bzy-cx^2, azx-bxz-cy^2),$$

the 3-dimensional Sklyanin algebra. This is strongly noetherian, so the theorem applies. Further:

- X is (generically) an elliptic curve,
- $\theta : S \to B(X, \mathcal{L}, \sigma)$ is surjective,
- $\ker\theta = gS$, $g \in S_3$ is normal and regular.

That is, for the Skylanin algebra θ gives an embedding of an elliptic curve in a noncommutative $\mathbb{P}^2$.

But:

$$\{ \text{ strongly noetherian } \} \subsetneq \{ \text{noetherian} \}$$

Question

What if R is noetherian but not strongly noetherian?

- *Is there a map $\theta : R \to A[t; \tau]$?*
- *What can we say about the image $\theta(R)$?*
- *Does θ satisfy a universal property?*

Twisted homogeneous coordinate rings

- X a projective scheme
- $\sigma \in \mathrm{Aut}_{\mathbb{C}}(X)$
- $\mathcal{L}$ a σ-ample (appropriately positive) invertible sheaf

$B(X, \mathcal{L}, \sigma)$ is defined by

$$B_n := H^0(X, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^* \mathcal{L}) = H^0(X, \mathcal{L}_n).$$

Example

- $X = \mathbb{P}^2, \mathcal{L} = \mathcal{O}(1)$
- $B(\mathbb{P}^2, \mathcal{O}(1), \sigma) \cong \mathbb{C}[x_0, x_1, x_2]^\sigma =$

$$\mathbb{C}\langle x_0, x_1, x_2 \rangle / \Big(x_i \sigma^{-1}(x_j) = x_j \sigma^{-1}(x_i) \Big)_{i,j}$$

- That is, $B_n = \{$ n-forms in 3-variables $\}$, with multiplication twisted by σ.

Twisted homogeneous coordinate rings are strongly noetherian. They are also birationally commutative:

$$B(X, \mathcal{L}, \sigma) \subseteq \mathbb{C}(X)[t; \sigma].$$

Theorem (Rogalski-Zhang)

Let $R = \mathbb{C} \oplus R_1 \oplus \cdots$ be a strongly noetherian connected graded algebra generated in degree 1. Then there is a canonical map, surjective in large degree,

$$\theta : R \to B(X, \mathcal{L}, \sigma)$$

Here:

- *X is a projective scheme*
- *$\sigma \in \mathrm{Aut}_{\mathbb{C}}(X)$, and*
- *$\mathcal{L}$ is a σ-ample invertible sheaf on X.*

Any $R \to A[t; \tau]$ factors through θ (up to finite dimension).

Naïve blowup algebras

$$\{ \text{ strongly noetherian } \} \subsetneq \{ \text{ noetherian } \}$$

- X a projective variety of dimension ≥ 2
- $\mathcal{L}$, σ as before
- $p \in X$ of infinite order under σ

The *naïve blowup algebra*

$$R(X, \mathcal{L}, \sigma, p) \subseteq B(X, \mathcal{L}, \sigma)$$

is defined by:

$$R(X, \mathcal{L}, \sigma, p)_n = H^0(X, \mathcal{I}_{p, \ldots, \sigma^{-(n-1)}(p)} \mathcal{L}_n) \subseteq H^0(X, \mathcal{L}_n)$$

That is,

$$R_n = \{ f \in H^0(X, \mathcal{L}_n) | \ f \text{ vanishes at } p, \sigma^{-1}(p), \ldots, \sigma^{-(n-1)}(p) \}$$

Example

If $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}(1)$, and $p = [1 : 1 : 1]$ has a dense orbit then

$$R = \mathbb{C}\langle x_0 - x_1, x_1 - x_2 \rangle \subset B(\mathbb{P}^2, \mathcal{O}(1), \sigma) = \mathbb{C}[x_0, x_1, x_2]^\sigma.$$

Theorem (Rogalski)

If p has a (critically) dense orbit then R is noetherian, not strongly noetherian.

- Later generalised by Keeler-Rogalski-Stafford to apply to any $R(X, \mathcal{L}, \sigma, p)$ where p has a dense orbit and $X, \mathcal{L}, \sigma$ as before.
- Technical note: in our context dense = critically dense (Bell). Characteristic-free results have "critically dense" wherever "dense" appears.

Let $X, \mathcal{L}, \sigma, p$ as above and let

$$R := R(X, \mathcal{L}, \sigma, p).$$

Recall our questions:

- Does R have a birationally commutative factor?
  Of course: $R \subseteq \mathbb{C}(X)[t; \sigma]$ is birationally commutative.
- What can we say about the factor?
  Certainly not $B(X, \mathcal{L}, \sigma)$.
- Is there a universal property? Yes.

Point spaces, following Artin-Tate-Van den Bergh

Let $R = \mathbb{C} \oplus R_1 \oplus \cdots$ be connected graded, finitely generated in degree 1.

Definition

A point module *is a cyclic graded (right) R-module with Hilbert series* $1 + s + s^2 + \cdots$*.*

- We will construct the "point space" that parameterizes point modules.
- To do this, look at truncations: cyclic modules with Hilbert series $1 + s + \cdots + s^n$.
- These truncated point modules are parameterised by a projective scheme, which we always call Y_n.

Definition

The point space *of R is the object*

$$Y_\infty := \varprojlim Y_n.$$

- Can be thought of as a proscheme.
- Parameterizes (i.e., *represents*) point modules over R.
- More carefully, represents *embedded* point modules: point modules M together with a surjective map $R \to M$.

Point space of a twisted homogenous coordinate ring

Example

$R := B(\mathbb{P}^2, \mathcal{O}(1), \sigma) \cong \mathbb{C}[x_0, x_1, x_2]^\sigma$.

- Hilbert series 1: only one module, and $Y_0 = \{\mathrm{pt}\}$.
- Hilbert series $1 + s$:
  - here $M = R/(I_1 + R_{\geq 2})$
  - $I_1 \subseteq R_1$, codimension 1, and any such I_1 gives a truncated point module
  - so $Y_1 = \mathbb{P}^2 = \mathbb{P}(R_1^*)$
- Given any such I_1, then $I_1 R_1 \subseteq R_2$ has codimension 1, so $Y_2 \cong \mathbb{P}^2$
- And in fact all $Y_n \cong \mathbb{P}^2$
- So here $Y_\infty \cong \mathbb{P}^2$.

Theorem (Artin-Zhang)

If R is strongly noetherian and generated in degree 1, then the point schemes Y_n stabilize for $n \gg 0$. Thus $Y_\infty = Y_{n \gg 0}$ is an honest scheme.

- This Y_∞ is the scheme X that occurs in Rogalski-Zhang's result.

Point space of a naïve blowup algebra

Example

$S := R(\mathbb{P}^2, \mathcal{O}(1), \sigma, p) = \mathbb{C}\langle x_0 - x_1, x_1 - x_2 \rangle \subseteq B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$.

- $Y_0 = \{\mathrm{pt}\}$
- $Y_1 \cong \mathbb{P}^1 \cong \mathbb{P}(S_1^*)$
- $Y_2 \cong \mathrm{Bl}_{p, \sigma^{-1}(p)}(\mathbb{P}^2)$
- $Y_n \cong \mathrm{Bl}_{p, \dots \sigma^{-(n-1)}(p)}(\mathbb{P}^2)$
- So $Y_\infty = \mathrm{Bl}_{\{\text{ (one-sided) orbit of } p\ \}}(\mathbb{P}^2)$
- Not a projective scheme, but not too bad; in this case it's an (fpqc-algebraic) stack.
- Further, Y_∞ is noetherian iff the σ-orbit of p is dense.

Moduli of points

For $R = R(\mathbb{P}^2, \mathcal{O}(1), \sigma, p)$ with the orbit of p dense, we have

$$\begin{array}{ccc} \mathrm{Bl}_{\{\text{ orbit of } p\ \}}(\mathbb{P}^2) & = & Y_\infty \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & = & X. \end{array}$$

Here Y_∞ represents points (= point modules).

Theorem (Nevins-S., 2010)

In this example, and for a naïve blowup algebra $R(X, \mathcal{L}, \sigma, p)$ more generally, X also solves a moduli problem: X corepresents *equivalence classes $\{$ points $\}/\sim$, where $M \sim N$ if $M_{\geq n} \cong N_{\geq n}$.*

That is, X is a *coarse moduli scheme* for isomorphism classes of tails of points.

We have:

$$\begin{array}{ccc} \{\text{ points }\} & \stackrel{\cong}{\longleftrightarrow} & Y_\infty \\ \downarrow & & \downarrow \\ \{\text{ tails of points }\} & \longrightarrow & X. \end{array}$$

The bottom map is bijective on closed points and makes X a coarse moduli scheme for the functor of tails of points.

Theorem (Nevins-S.)

Let R be a connected graded noetherian algebra generated in degree 1. Suppose we have

$$\begin{array}{ccc} \{\text{ points }\} & \overset{\cong}{\longleftrightarrow} & Y_\infty \\ \downarrow & & \downarrow p \\ \{\text{ tails of points }\} & \longrightarrow & X \end{array}$$

where

- X *corepresents* $\{$ *tails of points* $\}$.
- X *is a* locally factorial *projective variety of dimension* ≥ 2.
- $p^{-1} : X \dashrightarrow Y_\infty$ *is defined and is a local isomorphism except at countably many points of* X.

Theorem (continued)

Then:

- *There is a map, surjective in large degree, from R to a naïve blowup algebra* $R(X, \mathcal{L}, \sigma, P)$.
- *Here:*
  - $\sigma \in \mathrm{Aut}(X)$,
  - $\mathcal{L}$ *is a* σ*-ample invertible sheaf on X, and*
  - *P is a 0-dimensional subscheme of X supported on dense orbits.*
- *The same universal property holds:* $\theta(R)$ *is universal for maps from R to birationally commutative algebras.*
- *Furthermore, the indeterminacy locus of* p^{-1} *is dense, and*
- *The point space* Y_∞ *is noetherian.*

Canonical birationally commutative factors

Let R be any connnected graded algebra finitely generated in degree 1.

- There are two maps $\phi, \psi : Y_{n+1} \to Y_n$.

$$\begin{aligned} \phi : Y_{n+1} &\to Y_n \\ M &\mapsto M/M_{n+1}. \end{aligned}$$

- $Y_\infty := \varprojlim_\phi Y_n$.

$$\begin{aligned} \psi : Y_{n+1} &\to Y_n \\ M &\mapsto M[1]_{\geq 0} \end{aligned}$$

- There is an induced automorphism $\Psi := \varprojlim \psi : Y_\infty \to Y_\infty$, where $\Psi(M) = M[1]_{\geq 0}$.

Let $\mathbb{P} := \mathbb{P}(R_1^*)$.

- Each $Y_n \hookrightarrow \mathbb{P}^{\times n}$.
- So we have an invertible sheaf $\mathcal{M}_n := \mathcal{O}(1, \ldots, 1)|_{Y_n}$.
- And in fact $B := \bigoplus H^0(Y_n, \mathcal{M}_n)$ is a ring (uses ϕ, ψ).
  - Called the *point parameter ring*
- Notice that we have:

$$\begin{array}{ccc} T(R_1) & & \bigoplus H^0(\mathbb{P}^{\times n}, \mathcal{O}(1, \ldots 1)) \\ \downarrow & & \downarrow \\ R & & B = \bigoplus H^0(Y_n, \mathcal{M}_n). \end{array}$$

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$$\begin{array}{ccc} T(R_1) & \xleftarrow{\cong}\!\!\rightarrow & \bigoplus H^0(\mathbb{P}^{\times n}, \mathcal{O}(1, \ldots 1)) \\ \downarrow & \searrow & \downarrow \\ R & & B = \bigoplus H^0(Y_n, \mathcal{M}_n). \end{array}$$

Theorem (Rogalski-Zhang)

The map $T(R_1) \to B$ defined above factors through R.

$$
\begin{array}{ccc}
T(R_1) & \overset{\cong}{\longleftrightarrow} & \bigoplus H^0(\mathbb{P}^{\times n}, \mathcal{O}(1, \dots 1)) \\
\downarrow & \searrow & \downarrow \\
R & \underset{\theta}{\dashrightarrow} & B.
\end{array}
$$

Theorem (Nevins-S.)

The homomorphism θ constructed above is universal for maps to birationally commutative algebras: any $R \to A[t; \tau]$ factors through $\theta(R)$ (up to finite dimension).

We call $\theta(R)$ the *canonical birationally commutative factor* of R (by slight abuse of notation).

Now suppose that the point space of R has the geometry of a naïve blowup algebra:

$$\begin{array}{ccc} \{\text{ points }\} & \overset{\cong}{\longleftrightarrow} & Y_\infty \\ \downarrow & & \downarrow p \\ \{\text{ tails of points }\} & \longrightarrow & X \end{array}$$

where X corepresents $\{$ tails of points $\}$.

- In this situation, there is an automorphism σ of X with

$$\begin{array}{ccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty \\ p\downarrow & & \downarrow p \\ X & \xrightarrow[\sigma]{} & X. \end{array}$$

- We have

$$\begin{array}{ccc} R & \xrightarrow{\theta} & B \\ & \searrow^{\theta'} & \downarrow \\ & & \mathbb{C}(X)[t;\sigma] \end{array}$$

- Given the geometry in the theorem, $\ker\theta$ and $\ker\theta'$ are equal in large degree, and $\theta'(R)$ is a naïve blowup algebra on X.
- Can deduce density of orbits (= indeterminacy points of p^{-1}) and noetherianness of Y_∞ from the fact that the naïve blowup algebra $\theta'(R)$ is noetherian.

Recall our questions one more time.

Question

Let R be a connected graded noetherian algebra generated in degree 1, and assume that the geometry of point modules looks like that of a naïve blowup algebra.

- *Is there a map $R \to A[t;\tau]$?* *Yes, constructed canonically*
- *What can we say about the image?*
  It is a naïve blowup algebra, up to finite dimension
- *Is there a universal property?*
  Same as for strongly noetherian algebras: universal for maps from R to birationally commutative algebras

Future directions

- We would like to weaken restrictions on the coarse moduli scheme X.
- Some noetherian rings have no scheme parameterizing tails of points in any way, so the geometry will be very different.
- The easiest example is a domain with 4 generators and 6 quadratic relations.

Question

What is the class of canonical birationally commutative factors of noetherian algebras?

- For connected graded noetherian R generated in degree 1 we always have $\theta : R \to B$ as constructed above.
- Describe $\{\theta(R)\}$.
- Class includes:
  - homogeneous coordinate rings,
  - naïve blowup algebras,
  - 4-generator algebra from the previous slide,
  - ???

Question

Can noetherianness be detected geometrically, by looking at noncommutative Hilbert schemes as in Artin-Zhang?

Question

Is there a universal object for maps to iterated skew extensions?

Happy Birthday!