A dichotomy in GK-dimension 5

Susan J. Sierra

University of Edinburgh

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Outline

This is a report on work in progress. Conventions:

- $\mathbb{C}$ = ground field
- A connected graded ring R is:
  - $\mathbb{N}$-graded: $R = R_0 \oplus R_1 \oplus \cdots$ with $R_i R_j \subseteq R_{i+j}$
  - connected: $R_0 = \mathbb{C}$.
  - In addition, we assume $\dim_{\mathbb{C}} R_n < \infty$ for all n.

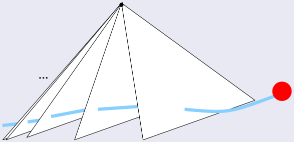
Let R be a $\mathbb{C}$-algebra. We study

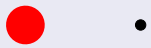
$$\operatorname{Spec}(R) = \{ \text{ prime ideals of } R \ \}$$

under the Zariski topology.

Theorem

If R is a <u>*GK-dimension 5 birationally commutative projective surface*</u>*, then* Spec *R is either*





a very strange space *or* *just two points.*

What is a GK-5 birationally commutative projective surface?

Definition

Let $A \subseteq B$ be an inclusion of graded Ore domains. Then A is a big subalgebra of B if $Q_{\text{gr}}(A) = Q_{\text{gr}}(B)$.

Theorem (Rogalski)

Let K be a finitely generated field of transcendence degree 2 and let $\sigma \in \text{Aut}(K)$. Then every finitely generated big subalgebra $R \subseteq K[t; \sigma]$ has the same GK-dimension,

$$d = d(\sigma) \in \{3, 4, 5, \infty\}.$$

Suppose $d < \infty$. Then $d \in \{3, 5\} \iff$ there is a projective surface X with $\mathbb{C}(X) = K$ and $\sigma \in \text{Aut}(X)$. (We say σ is geometric.)

Definition

A GK-5 birationally commutative projective surface is:

- *GK-5*
- *connected graded*
- *noetherian*
- *subalgebra of some* $K[t; \sigma]$,

where K *is a finitely generated field of transcendence degree 2.*

We will call this a GK-5 surface for short.

Twisted homogeneous coordinate rings

Recall:

- X a projective scheme
- $\sigma \in \mathrm{Aut}(X)$
- $\mathcal{L}$ an invertible sheaf on X

Then $B(X, \mathcal{L}, \sigma)$ is defined by

$$B_n := H^0(X, \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}) = H^0(X, \mathcal{L}_n).$$

Example

- $X = \mathbb{P}^2, \mathcal{L} = \mathcal{O}(1)$, σ arbitrary
- $B(\mathbb{P}^2, \mathcal{O}(1), \sigma) \cong \mathbb{C}[x_0, x_1, x_2]^\sigma =$

$$\mathbb{C}\langle x_0, x_1, x_2\rangle / \Big(x_i\sigma^{-1}(x_j) = x_j\sigma^{-1}(x_i) \Big)_{i,j}$$

- That is, $B_n = \{$ n-forms in 3-variables $\}$, multiplication twisted by σ.

- This example has GK-dimension 3.
- To get GK-5 need a $d = 5$ automorphism

Example

Let E be an elliptic curve. Define

$$\begin{aligned} \tau : E \times E &\to E \times E \\ (x, y) &\mapsto (x, x + y) \end{aligned}$$

Then $d(\tau) = 5$.

Theorem (Artin-Van den Bergh)

- *Let X be a projective surface and $\sigma \in \mathrm{Aut}(X)$. One can read off $d(\sigma)$ from the action of σ on $\mathrm{Pic}(X)$. Both 3 and 5 occur.*
- *In particular, $d(\tau) = 5$.*
- *If $d < \infty$, and $\mathcal{L}$ is any ample invertible sheaf on X, then $B(X, \mathcal{L}, \sigma)$ is noetherian (with GK-dim d).*

In fact, if we take $\mathcal{M} := \mathcal{O}_{E \times E}(\Delta)$, the theorem still applies. We will do that, to be more explicit.

Example

Let $S := B(E \times E, \mathcal{M}, \tau)$. This is GK-5. We compute Spec S.

Spec of a twisted homogeneous coordinate ring

Let $B := B(X, \mathcal{L}, \sigma)$.

- (Non-irrelevant) homogeneous ideals of B correspond to σ-invariant subschemes of X
- So non-irrelevant homogeneous primes $\longleftrightarrow$ reduced and "irreducible" σ-invariant subvarieties.
- (Bell-Rogalski-S.) σ has infinite order $\Rightarrow$ the height 1 primes of B are homogeneous.
- (Bell-Rogalski-S.) If X is a curve or a surface, then a prime ideal P of B is primitive $\iff$ P is locally closed in Spec B.

Consequences for Spec S

Recall:

$$\tau : (x, y) \mapsto (x, x + y)$$

- Homogeneous primes of S correspond to τ-invariant irreducible subvarieties of $E \times E$.
- In particular, for any $x \in E$ there is a homogeneous prime

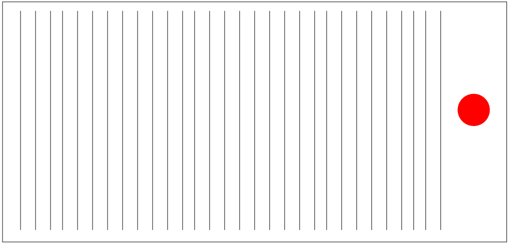
$$P_x \longleftrightarrow x \times E.$$

- Since

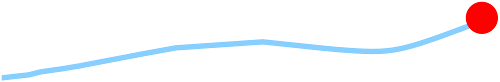
$$E \times E \supsetneq x \times E$$

  there are no homogeneous primes between 0 and P_x.
- Since height 1 primes are homogeneous, P_x is height 1.

That is, Spec S is at least $\{P_x\}$ (together with 0):



That is, Spec $S \supseteq$



What is Spec S/P_x?

$S/P_x \cong B(x \times E, \mathcal{M}_x, \tau_x)$, where $\tau_x = \tau|_{x \times E}$

- $\tau(x, y) = (x, x + y)$, so τ_x is translation by x.

Suppose x has infinite order.

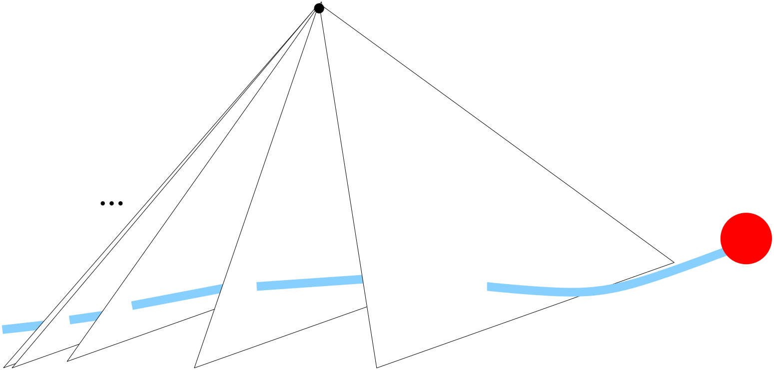
- $x \times E$ has no invariant subschemes
- This means Spec(S/P_x) is very small:

$$\text{Spec}(S/P_x) = \{P_x, S_+\} =$$

- P_x is locally closed and thus primitive.

Suppose x has finite order.

- Then $S/P_x \cong B(E, \mathcal{M}_x, \tau_x)$ is PI, of PI-degree $n =$ order of x.
- S/P_x is GK-2 and PI $\Rightarrow$ there is a 2-dimensional space of maximal ideals containing P_x.
- This is one of the “sheets” in the picture.
- Countably many points of finite order $\Rightarrow$ countably many co-PI height 1 primes.



Why does S have GK 5?

Riemann-Roch:

$$\dim S_n \sim \dim H^0(\mathcal{M}_n) \sim c(\Delta + \tau(\Delta) + \cdots + \tau^{n-1}(\Delta))^2.$$

- This is $O(n^2) \cdot (\Delta \cap \tau^{n-1}(\Delta))$.
- And $\tau^{n-1}(\Delta) = \{(x, [n]x)\}$
- So
$$\Delta \cap \tau^{n-1}(\Delta) = \#\{x \mid [n]x = x\} = (n-1)^2.$$
- That is, $\dim S_n$ is $O(n^2) \cdot O(n^2) = O(n^4) \Rightarrow$ GK-dim S = 5.

Theorem (Diller-Favre, Gizatullin)

Let X be a projective surface, $\sigma \in \mathrm{Aut}(X)$ with $d(\sigma) = 5$. Then X is an elliptic surface:

$$\begin{array}{c} X \\ \downarrow \\ C \end{array} \qquad \textit{fibers are (generically) elliptic curves}$$

Further, σ preserves the fibration:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ C & \xrightarrow[\sigma_C]{} & C. \end{array}$$

Proposition

Let $\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ C & \xrightarrow[\sigma_C]{} & C \end{array}$ *be a* $d = 5$ *automorphism of an elliptic surface* X.

- *If* σ_C *has infinite order then up to a finite cover we have* $X \cong E \times E$, *and*

$$\sigma(x, y) = (x + p, x + y + q)$$

  where $p \in E$ *has infinite order. Thus* σ *has no invariant subschemes at all!*

- *If* σ_C *has finite order then* X *has a 1-dimensional family of invariant curves, and points of arbitrarily large finite order.*

Let $R \subseteq K[t;\sigma]$ be a GK-5 birationally commutative projective surface.

- Since $d(\sigma) = 5$, there is a surface X with $\sigma \in \operatorname{Aut}(X)$.
- We have $$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ C & \xrightarrow[\sigma_C]{} & C \end{array}$$

Suppose σ_C has finite order.

~~Suppose σ_C has finite order~~. Suppose $\sigma_C = \mathrm{Id}_C$.

- The geometry of σ on X is very like the geometry of τ on $E \times E$.
- We again have primes P_x corresponding to fibers
- Points of arbitrarily big order $\Rightarrow R/P_x$ can have arbitrarily big PI-degree.
- Can show that R is actually a twisted homogeneous coordinate ring $R = B(X', \mathcal{L}', \sigma')$, where X' is very close to X.

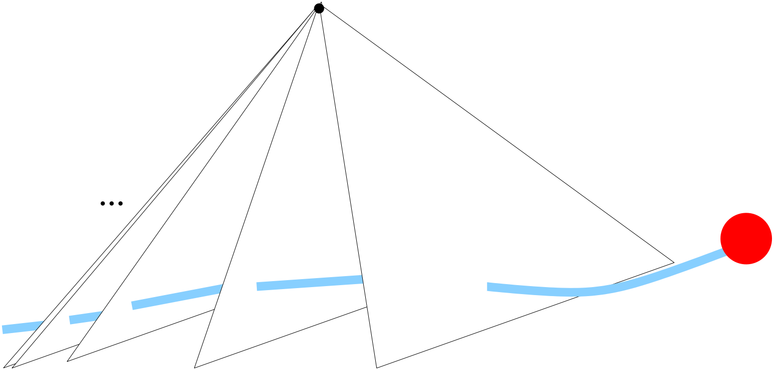
Summary of the finite order case

Theorem

Let R be a GK-5 birationally commutative projective surface, and suppose that σ_C has finite order.

- *R is (up to finite dimension) a twisted homogeneous coordinate ring.*
- *R has a 1-dimensional space of height 1 primes, all but countably many of which are primitive.*
- *The rest are co-PI, and the factors are GK-2 of arbitrarily large PI-degree.*

Spec $R =$



Summary of the infinite order case

Theorem

Let R be a GK-5 birationally commutative projective surface and suppose that σ_C has infinite order.

- *R is a subring of some $B := B(E \times E, \mathcal{L}, \tau')$, where*

$$\tau'(x, y) = (x + p, x + y + q)$$

  (for $p \in E$ of infinite order) and $Q_{\mathrm{gr}}(R) \subseteq Q_{\mathrm{gr}}(B)$ is a finite extension.
- Spec $R = \{(0), R_+\}$ *is two points. (R is projectively simple)*

- R need not be a twisted homogeneous coordinate ring
- The possibilities for R can be enumerated (idealisers, naïve blowups, etc.)

$\operatorname{Spec} R =$

The classification of noncommutative surfaces

Definition

A birationally commutative projective surface is a connected graded noetherian domain R so that $Q_{gr}(R) = K[t^{\pm}; \sigma]$ *for K a field of transcendence degree 2.*

These have been (almost) classified, at least in the case of a geometric automorphism.

- If σ geometric and R generated in degree 1, classified (Rogalski-Stafford, 2006)
- If $d(\sigma) = 3$, classified (S., 2008)
- The classification of GK-5 surfaces not generated in degree 1 follows from the work in this talk.
- The only remaining noetherian case is GK-4, when the automorphism is not geometric (eg. recent work of Rogalski-S.)