Geometric idealizers and critical transversality

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Outline

The underlying geometric data

We start with the following data:

- Let X be a projective variety defined over an algebraically closed field k.
- Let ϕ be an automorphism of X.
- Let $\mathcal{L}$ be an invertible sheaf on X; as usual, we denote the product $(\phi^{n-1})^*\mathcal{L} \otimes \cdots \otimes \phi^*\mathcal{L} \otimes \mathcal{L}$ by $\mathcal{L}_n$.
- We require that $\mathcal{L}$ is ϕ-ample: this is a technical condition that means that $\{\mathcal{L}_n\}$ has the same good properties as the tensor powers of an ample invertible sheaf.
- Let $Z \subset X$ be an irreducible subvariety.
- *Standing assumption:* No power of ϕ fixes Z.

Constructing a ring

We define a ring $R = R(X, \mathcal{L}, \phi, Z)$ as follows:

- Form the twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \phi)$ defined by $B_n = H^0(X, \mathcal{L}_n)$.
- Recall that B is a ring via the maps

$$H^0(X, \mathcal{L}_n) \otimes H^0(X, \mathcal{L}_m) \xrightarrow{1 \otimes \phi^n}$$

$$H^0(X, \mathcal{L}_n) \otimes H^0(X, (\phi^n)^* \mathcal{L}_m) \longrightarrow H^0(X, \mathcal{L}_{m+n}).$$

- Let I be the right ideal of B of sections vanishing on Z.
- Let $R = \{g \in B \mid gI \subseteq I\}$. R is the *idealizer* in B of I; we write $R = \mathbb{I}_B(I)$.
- Notice that $R = k + I$, since Z is of infinite order under ϕ.

Technical properties

What are the properties of $R = R(X, \mathcal{L}, \phi, Z)$? How do they depend on the underlying geometry?

Definition

R is *strongly right Noetherian* if for any commutative Noetherian ring C, the ring $R \otimes_k C$ is right Noetherian.

Definition

We say that R satisfies *(right)* χ_r if (roughly speaking) $\underline{\mathrm{Ext}}^r_R(k, M)$ is finite dimensional for all finitely generated right R-modules M.

Rogalski's result

Rogalski studied the case where $X = \mathbb{P}^d$, $\mathcal{L} = \mathcal{O}(1)$, and $Z = \{c\}$, a point. He found that R has unusual properties.

Definition

The set $\{\phi^n(c)\}$ is *critically dense* if any infinite subset of $\{\phi^n(c)\}$ is Zariski dense in $\mathbb{P}^d$.

Theorem (Rogalski)

$R = R(\mathbb{P}^d, \mathcal{O}(1), \phi, c)$ is strongly right Noetherian. If $\{\phi^n(c)\}$ is critically dense then R is left Noetherian but not strongly left Noetherian, and satisfies right χ_{d-1} but not right χ_d. R always fails left χ_1.

This generalizes an example of Stafford and Zhang of an idealizer given by a point in $\mathbb{P}^1$.

Critical transversality

We seek to understand R for more general X, $\mathcal{L}$, ϕ, and Z. In particular, is there an appropriate analogue of critical density for subschemes that are not just points?

Definition

We say the set $\{\phi^n Z\}$ is *critically transverse* if for any Y, for $|n| \gg 0$, we have that $\mathcal{T}or_j^X(\mathcal{O}_{\phi^n Z}, \mathcal{O}_Y) = 0$ for $j \geq 1$

Algebraic motivation: a result of Rogalski that says R is left Noetherian if and only if $\operatorname{Tor}_1^B(B/I, B/J)$ is finite dimensional for all left ideals J in ${}_B B$. (Recall that I is the right ideal of B corresponding to Z.)

- $\operatorname{Tor}_1^B(B/I, B/J) = (I \cap J)/IJ$. This governs extensions and contractions of left ideals between R and B.

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Geometric motivation

Definition

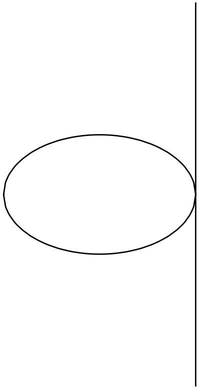
Two closed subschemes Y and Z of X are *homologically transverse* if for all $j \geq 1$, we have $\mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y) = 0$.

Homological transversality says that intersection formulae are simple. Recall Serre's definition of the intersection multiplicity of Y and Z along a component P of their intersection:

$$i(Y, Z; P) = \sum (-1)^i \operatorname{len} \mathcal{T}or_i^X(\mathcal{O}_Y, \mathcal{O}_Z)_P$$

where the length is taken over the local ring at P.
Thus if Y and Z are homologically transverse, then $i(Y, Z; P) = \operatorname{len}(\mathcal{O}_Y \otimes_X \mathcal{O}_Z)_P$; that is, the intersection multiplicity is given by the length of the scheme-theoretic intersection of Y and Z.

Understanding homological transversality



- Two distinct irreducible curves on a smooth surface are always homologically transverse. Thus "transverse" is really not the right word, but it gives a flavour of what we're after.

More examples

- A point P is homologically transverse to a subscheme Z exactly when $P \notin Z$.
- More generally: Y and Z are not homologically transverse if their intersection has unexpectedly high codimension or if Z meets the non-Cohen-Macaulay locus of Y badly.
- The standard example involves the intersection of three 2-planes meeting at a point in $\mathbb{P}^4$, which is hard to draw.

First main theorem

Theorem

Let Z be an irreducible proper subvariety of X of codimension $d > 1$. Let $R = R(X, \mathcal{L}, \phi, Z)$.

- *If Z intersects all orbits only finitely many times, then R is strongly right Noetherian.*
- *R always fails left χ_1.*

If $\{\phi^n Z\}$ is critically transverse in X, then:

- *R is left Noetherian but not strongly left Noetherian.*
- *If X and Z are smooth, then R satisfies right χ_{d-1} and fails right χ_d.*

In particular, if $X = \mathbb{P}^d$, $\mathcal{L} = \mathcal{O}(1)$, and $Z = \{c\}$, we obtain Rogalski's result.

Definition

- Recall the definition of the category

$$\text{qgr-}R = \{ \text{ graded right } R\text{-modules } \}/\text{torsion}.$$

- There is a canonical functor $\pi : \text{gr-}R \to \text{qgr-}R$.
- Consider the functor $\operatorname{Hom}_{\text{qgr-}R}(\pi R, \_\_)$.
- The *right cohomological dimension* of R is the cohomological dimension of the functor $\operatorname{Hom}_{\text{qgr-}R}(\pi R, \_\_)$.

Example: Let T be commutative, $Y = \operatorname{Proj} T$.

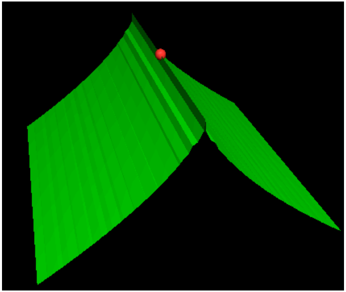
- Then $\text{qgr-}T \simeq \mathcal{O}_Y\text{-mod}$, and $\pi T \cong \mathcal{O}_Y$.
- Thus $\operatorname{Hom}_{\text{qgr-}T}(\pi T, \_\_) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \_\_) = H^0(Y, \_\_)$, the global section functor.
- So the cohomological dimension of T is the cohomological dimension of $Y = \operatorname{Proj} T$.

Finite cohomological dimension

Stafford and van den Bergh ask: Do all Noetherian connected graded rings have finite cohomological dimension?

- No known counterexample.
- A (finitely generated) commutative graded ring has finite cohomological dimension.
- Proof:
  1. We are really working with local cohomology, which is the same as Čech cohomology. So $H^n = 0$ for $n > \dim Y$.
  2. Induct directly on $\dim Y$.
- These proofs both depend on the geometry of the underlying space of Y. Thus they fail for noncommutative rings, where there is no "space" to work with.

A right Noetherian counterexample



- Let C be the cuspidal cubic in $\mathbb{P}^2$ and let $X = C \times \mathbb{P}^1$.
- Let P be the singular point of C and let $Z = P \times [0 : 1]$.
- Let $\phi : X \to X$ be the automorphism defined by $\phi(x, [s : t]) = (x, [s + t : t])$.

Then if $\mathcal{L}$ is any ϕ-ample invertible sheaf on X, the ring $R = R(X, \mathcal{L}, \phi, Z)$ is right (but not left) Noetherian, and the right cohomological dimension of R is infinite.

Understanding the counterexample

- Z meets orbits at most once, so R is right Noetherian.
- Since all locally free resolutions of $\mathcal{O}_Z$ are infinite, the right cohomological dimension of R is infinite. (Here qgr-$R \simeq \mathcal{O}_X$-mod, with πR corresponding to $\mathcal{I}$.)
- All $\phi^n(Z)$ are in $P \times \mathbb{P}^1$, so critical transversality fails. Thus R is not left Noetherian.

Theorem

For a general $X, \mathcal{L}, \phi, Z$, if $R = R(X, \mathcal{L}, \phi, Z)$ is left Noetherian (technically, if the corresponding "sheafified" object $\mathcal{R}$ is left Noetherian), then R has finite right cohomological dimension.

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Understanding the theorem

Theorem

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- Recall that R being left Noetherian is controlled by the critical transversality of $\{\phi^n Z\}$.
- A counterexample would require Z to have infinite homological dimension and to satisfy critical transversality.
- It turns out (Hochster) that none such exist.

What does critical transversality mean, geometrically?

Theorem

Assume characteristic 0 and that $\mathrm{Aut}(X)$ *is an algebraic group. Then* $\{\phi^n Z\}$ *is critically transverse if and only if* Z *is homologically transverse to all* ϕ*-fixed subschemes of* X*.*

- A generalization of the result of Keeler, Rogalski, and Stafford that in this situation $\{\phi^n(c)\}$ is critically dense exactly when $\{\phi^n(c)\}$ is (Zariski) dense.

Example: $X = \mathbb{P}^d$ and ϕ a diagonal automorphism with algebraically independent eigenvalues. If Z is homologically transverse to the coordinate subspaces, then $\{\phi^n(Z)\}$ is critically transverse.

Second main theorem

A corollary of the following purely geometric result:

Theorem

Let G be an algebraic group acting on a variety X. Let Z be a closed subscheme that is homologically transverse to the orbits of G.

- *Then for any closed subscheme Y, the generic translate of Z is homologically transverse to Y.*
- *That is, there is a dense open subset $U \subseteq G$ such that, if $g \in U$, then gZ is homologically transverse to Y.*

This generalizes a result of Miller and Speyer that says that homological transversality is generic for transitive group actions, and ultimately goes back to the Kleiman-Bertini theorem.