What are the noncommutative projective surfaces?

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25 January 2013

Outline

Conventions for the talk

- $k = \overline{k}$ = algebraically closed ground field
- all rings are k-algebras, etc.
- A connected graded (cg) ring R is:
  - $\mathbb{N}$-graded: $R = R_0 \oplus R_1 \oplus \cdots$ with $R_i R_j \subseteq R_{i+j}$
  - connected: $R_0 = k$.
  - In addition, we assume $\dim_k R_n < \infty$ for all n.

Commutative projective geometry

X=commutative projective variety of dimension d.

We have:

- the **geometric object** X
- the **category** $\mathcal{O}_X$-mod of sheaves on X (roughly speaking, what makes algebraic geometry algebraic)
- a **graded ring**: the homogeneous coordinate ring of X

All three carry (roughly) equivalent information.

Homogeneous coordinate rings

X is projective, so there is an embedding $i : X \hookrightarrow \mathbb{P}^n$.

Definition

The homogeneous coordinate ring of X is $k[x_0, \ldots, x_n]/I(X)$.

This depends on X and i. To keep track of i:

- $\mathbb{P}^n$ carries a special invertible sheaf (or line bundle), $\mathcal{O}(1)$.
- Let $\mathcal{L} := \mathcal{O}(1)|_X$. This is a (very) ample invertible sheaf on X.
- Can recover i from $\mathcal{L}$.

Then write:

$$B(X, \mathcal{L}) = k[x_0, \ldots, x_n]/I(X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n}).$$

Dimension data

We can recover X and information about X from $B(X, \mathcal{L})$.

To recover X:

$$X = \operatorname{Proj} B(X, \mathcal{L}).$$

To recover $\dim X$:

$$\dim X = d \qquad \Longleftrightarrow \qquad \dim_k B_n \sim n^d.$$

Definition

If $\dim_k B_n \sim n^d$, then the Gelfand-Kirillov dimension or GK-dimension of B is $d + 1$.

The correct definition was in James Zhang's talk.

Serre's theorem

Theorem (Serre)

Let R be a commutative cg ring generated in degree 1. Let $X = \operatorname{Proj} R$. Then

$$\mathcal{O}_X\text{-mod} \simeq \{\textit{graded } R\textit{-modules}\}/\{\textit{finite dimensional modules}\}.$$

That is: there is a functor

$$\text{gr-}R \to \mathcal{O}_X\text{-mod} \qquad M \mapsto \mathcal{M}$$

We have $\mathcal{M} = \mathcal{M}' \iff M_{\geq n} = M'_{\geq n}$ for $n \gg 0$.

Definition

For any cg ring R, the category

$$\{\textit{graded } R\textit{-modules}\}/\{\textit{finite dimensional modules}\}$$

makes sense and is called qgr-R *(or R-*qgr*).*

What is a NC projective d-fold?

It is hard to see how to make X noncommutative. However, the ring and the category can be made noncommutative:

Definition

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Definition

2. *A (smooth) NC projective d-fold is a category that behaves like $\mathcal{O}_X$-mod for a (smooth) projective d-fold X:*

- *a Grothendieck category*
- *locally noetherian*
- *homological dimension d*
- *....*

Twisted homogeneous coordinate rings

Definition

Let X be a projective d-fold, $\sigma \in \mathrm{Aut}(X)$, $\mathcal{L}$ an ample invertible sheaf on X. The twisted homogeneous coordinate ring of X is

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^* \mathcal{L}).$$

Theorem (Artin-Van den Bergh, Keeler)

If $\sigma \in \mathrm{Aut}^o(X)$, then $B(X, \mathcal{L}, \sigma)$ is a NC projective d-fold, where $d = \dim X$. Further, Serre's Theorem holds:

$$\text{qgr-}B(X, \mathcal{L}, \sigma) \simeq \mathcal{O}_X\text{-mod}\,.$$

In terms of definition 2, these rings do not give new categories. In that sense, they are not very noncommutative, although their ring theory can be quite noncommutative!

A (very noncommutative) NC projective surface

Example

Let $a, b, c \in k$. Define:

$$\begin{aligned} S_{abc} = k\langle x, y, z\rangle/(&axy + byx + cz^2, \\ &ayz + bzy + cx^2, \\ &azx + bxz + cy^2). \end{aligned}$$

Theorem (Artin-Tate-Van den Bergh)

S_{abc} is a noetherian domain of GK-dim 3, i.e. a NC projective surface.

S_{abc} is the famous Sklyanin algebra.

- It is thought of as (the coordinate ring of) a "NC $\mathbb{P}^2$" because it shares many properties with $k[x, y, z]$.
- qgr-S is not commutative.

What are the NC projective curves?

Theorem (Artin-Stafford)

Let R be a NC projective curve, i.e. a cg noetherian domain of GK-dim 2. For simplicity assume R is generated in degree 1. Then there are a projective curve C, an ample invertible sheaf $\mathcal{L}$ on C, and $\sigma \in \mathrm{Aut}(C)$, so that (up to a finite-dimensional vector space)

$$R = B(C, \mathcal{L}, \sigma).$$

Corollary

If R is a NC projective curve, then $\text{qgr-}R \simeq \mathcal{O}_C\text{-mod}$ *for a projective curve C.*

In terms of definition 2, our NC projective curves are commutative!

Function fields of NC d-folds

The classification of NC surfaces is much harder!

To begin, it's natural to work "birationally" (as in commutative geometry).

Definition

If R is a graded ring (of finite GK-dimension) then we may form the *graded quotient ring*

$$Q_{\mathrm{gr}}(R) := R\langle h^{-1} \mid 0 \neq h \in R \text{ is homogeneous} \rangle.$$

This is a *graded division ring*, *i.e.*

$$Q_{\mathrm{gr}}(R) = D[t, t^{-1}; \tau]$$

for some division ring D and $\tau \in \mathrm{Aut}(D)$. We write $D = D(R)$ and say it is the *function (skew) field of R*.

Why “function field"?

Example

If $B = B(X, \mathcal{L}, \sigma)$ then $D(B) = k(X)$, the field of rational functions on X.

Definition

If $D(R)$ is a field, we say that R is *<u>birationally commutative</u>.*

So all NC projective curves are birationally commutative.

- In fact, Artin and Stafford proved this first and then proved their classification theorem.

Artin's conjecture

The birational classification of NC projective surfaces is unknown.

Conjecture (Artin, 1996)

Let R be a NC projective surface. Then $D(R)$ is either:

1. *A finite module over its centre K (which must be a field of transcendence degree 2).*
2. *A skew extension $K(t; \sigma, \delta)$ where $\operatorname{trdeg} K = 1$.*
3. *$D(S_{abc})$, the Sklyanin function field.*

The story so far:

- A great deal of progress on understanding rings within various birational classes;
- Much less progress on proving (or disproving) the conjecture.

Birationally commutative surfaces: case (1a)

Birationally commutative NC projective surfaces are classified:

Theorem (Rogalski-Stafford, S.)

Let R be a NC projective surface with $D(R) = K$, a field of trdeg 2. Then R determines and is determined by geometric data:

1. *a projective surface X*
2. $\sigma \in \mathrm{Aut}(X)$
3. *an (appropriately ample) invertible sheaf $\mathcal{L}$ on X*
4. *some other data*

In particular, $R \subseteq B(X, \mathcal{L}, \sigma)$ and is "close to" B.

Point modules

Definition

Let R be a cg ring. A point module over R is a cyclic graded module M so that

$$\dim_k M_n = \begin{cases} 1 & \textit{if } n \geq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

That is M has the Hilbert series $1/(1-s) = 1 + s + s^2 + \cdots$ of a point in $\mathbb{P}^n$.

(If we replace 1 by m in the definition above we have an m-point.)

One way to find X from a birationally commutative projective surface is that X is the (coarse) moduli space of point modules.Roughly speaking, R has a "surface of points."

FACT: a NC projective surface R has a surface of points $\iff$ R is birationally commutative (Rogalski-Zhang, Nevins-S.).

Birationally PI surfaces: case (1b)

If R is in case (1) but not case (1a) then $D(R)$ is a finite module over its centre or, equivalently, satisfies a polynomial identity. We say R is birationally PI.

Example

(D. Chan) Let X be a surface and $\mathcal{A}$ be an order on X (a sheaf of NC algebras finite over X). Let $\mathcal{L}$ be an invertible $\mathcal{A}$-bimodule. The twisted ring

$$B = B(\mathcal{A}, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is a NC projective surface and is birationally PI. We have

$$D(B) = \mathcal{A} \otimes k(X),$$

a division ring finite over $k(X)$.

NC surfaces with a surface of m-points

Theorem (D. Chan)

If R is a NC surface whose m-points are parameterized by a surface (and some technical conditions) then there are $\mathcal{A}, \mathcal{L}$ as above so that $R \subseteq B(\mathcal{A}, \mathcal{L})$, with the same graded quotient ring.

Question

Can algebras $R \subseteq B(\mathcal{A}, \mathcal{L})$ be classified, similarly to the classification of birationally commutative surfaces?

Question

If R is birationally PI, does it have a surface of m-points?

q-ruled surfaces: case (2)

Here $D(R) = K(t; \sigma, \delta)$, where trdeg $K = 1$.

- A surface X is (birationally) ruled, i.e., birational to $C \times \mathbb{P}^1$, iff $k(X) = k(C)(t)$.
- Thus case (2) is called q-ruled by Artin.

Proposition (Bell-Rogalski)

If $D = K(t; \sigma, \delta)$ with trdeg $K = 1$ *then we either have $D \cong K(t; \sigma')$ or $D \cong K(t; \delta')$. That is, either:*

1. *D is the full quotient ring of the THCR of a curve C with $k(C) = K$; or*
2. *D is the full quotient ring of the ring $\mathcal{D}(C)$ of differential operators on a curve C.*

Question

Can algebras in either subcase be classified?

An example

Let $B = B(C, \mathcal{L}, \sigma)$ for some curve C. Let

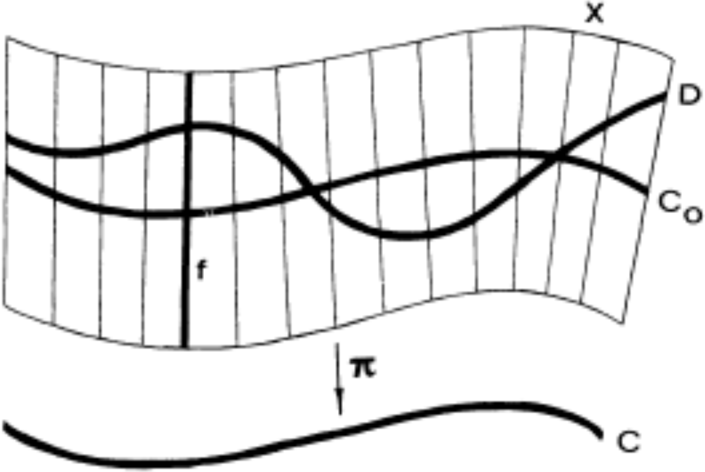
$$R = B[t; \tau],$$

graded so that $R_1 = B_1 + k \cdot t$.

Fact: Right ideals of B whose factor is a point module correspond to points on C.

- Let I be such a right ideal.
- B/I has Hilbert series $1/(1-s)$
- R/IR has Hilbert series $1/(1-s)^2$ and is a "line module."
- There is a component of the “line scheme” of R that is parameterized by C:

Here's a picture



which is of course the classical picture of a ruled surface.

A curve of points

How many points does R have?

- Recall that point modules over B are parameterized by C.
- Since B is a factor of R, all B-modules are R-modules.
- So R has at least a curve worth of points.

That is, there is a section, C_0, of the line scheme, as in the previous picture.

Question

If R is q-ruled, does it have a curve of points?

Some functors

Theorem (D. Chan)

If R is a cg domain of GK-dim 3 so that qgr-R *is "nice" and so that R has a "well-behaved" family of "rational curve modules" parameterised by a curve C, then there are well-behaved adjoint functors:*

$$\pi^* : \mathcal{O}_C\text{-mod} \leftrightarrows \text{qgr-}R : \pi_*$$

and R is a "NC ruled surface."

This:

- is the right way to define a "noncommutative morphism"
- happens in the previous example.

BUT:

- It's very hard to say anything about the ring theory of R from these functors.
- In fact, R might not even be q-ruled!

NC quadric surfaces

Recall that the 3-dimensional Sklyanin algebra S_{abc} is a NC $\mathbb{P}^2$.

Theorem (Smith-Van den Bergh)

There is a NC projective surface R which has the following properties

- *R is a "quadric surface in a NC $\mathbb{P}^3$"*
- *R is birational to S_{abc}.*
- *R is a NC $\mathbb{P}^1 \times \mathbb{P}^1$.*
- *R has a $\mathbb{P}^1$-worth of line modules and the previous theorem applies.*
- *That is, R is a NC ruled surface that is not q-ruled.*

Question

Is there a module-theoretic criterion for q-ruledness?

q-rational surfaces: case (3)

There are many interesting examples:

- S_{abc} itself
- The quadric surfaces above
- Subalgebras of $S = S_{abc}$ that are "noncommutative blowups" of S

Theorem (Rogalski-S.-Stafford)

If $R \subseteq$ (a Veronese of) S is a maximal order with the same graded quotient ring, then R is a blowup of S at ≤ 8 points.

Question

What about algebras that are birational to S but not contained in (some Veronese of) S. Can these be classified?

Artin's conjecture is hard

No counterexample is known, so this suggests the conjecture is true!

The same argument has been made to "prove" the existence of God.

(A bit) more is known about division rings of trdeg 2 than in 1996.

Transcendence degree and dimension

A number of different ways to measure "transcendence degree" of a division algebra:

- GK-transcendence degree
- Lower transcendence degree (Zhang)
- Homological transcendence degree (Yekutieli-Zhang)
- ....

Relations among these have been studied.

Theorem (Smoktunowicz)

There are no graded domains with GK-dimension strictly between 2 and 3.

Question

Is this still true if we remove the word "graded"?

Free subalgebras of division rings

PI algebras do not contain free subalgebras.

Theorem (Makar-Limanov)

Let $D = Q(A_1)$ be the quotient division ring of the Weyl algebra. Then D contains a free subalgebra on two generators: $k\langle x, y\rangle$, where x and y have no relations.

Theorem (Bell-Rogalski)

Let A be any non-PI domain of GK-dimension 2. Then $Q(A)$ contains a free subalgebra on two generators.

Question

This strongly suggests that any function field of a NC projective surface must be either PI or contain a free subalgebra. Is this true?

Finite degree subskewfields of skew extensions

Fact: All division rings on Artin's list are contained in $K(t;\sigma)$ or $K(t;\delta)$ where $\operatorname{trdeg} K = 1$.

- $D(S_{abc}) \subset k(E)(t;\sigma)$ where E is an elliptic curve; the extension is finite.

Question

If $D \subset^{finite} K(t;\sigma)$ or $K(t;\delta)$ where $\operatorname{trdeg} K = 1$ must D be one of the division rings on Artin's list?

Valuations

Artin has a programme to attack the conjecture based on valuations.

Question

Do all function fields of NC surfaces have a valuation?

All the division rings on Artin's list have valuations.

Presumably finite degree subskewfields of $K(t;\sigma)$, $K(t;\delta)$ have valuations.

Question

If D has valuations, is it on Artin's list?

Characterizing $\mathcal{D}(C)$

Theorem (Bell-Smoktunowicz)

If D contains a GK-dimension 2 subalgebra R with $D = Q(R)$ and so that R has a locally nilpotent derivation, then $D = \mathcal{D}(C)$.

Such a D has a negative valuation.

Question

If D has a negative valuation must $D = \mathcal{D}(C)$?

Point modules

Except (possibly) for some q-ruled surfaces, all the algebras discussed so far have at least a curve of points or m-points. (Point modules on S_{abc} are parameterised by an elliptic curve.)

Question

If R is a NC projective surface, must R have a positive-dimensional space of points (m-points)?

This fails for NC 3-folds. Also, there is a non-noetherian domain of GK-dim 3 that is birational to the Sklyanin algebra and has exactly one point module.

Question

If R has a positive-dimensional space of points, is its function field on Artin's list?

Counterexamples must be strange!

A counterexample is likely to be sporadic (not in a family), and not to be a deformation of a commutative surface.

- A family of NC projective surfaces is likely to be a deformation of a commutative surface.
- Artin argues that if R is a domain of GK 3 that is a deformation of the (commutative) homogeneous coordinate ring of a surface X, then the function field of R is on the list.

If the previous questions have positive answers, then a counterexample has:

- Fewer points than any known NC surface.
- Its function field has no valuations.
- Does not deform to any commutative surface.

It would be unlike any known NC projective surface.

An exotic counterexample

Thank you!