The classification of birationally commutative projective surfaces

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The problem

Question

What are the graded noetherian domains of GK-dimension 3, i.e. "noncommutative projective surfaces"?

- Answer: Nobody knows!
- We give a partial answer: we classify those rings that are birational to a commutative surface.

Definitions

- We work over an algebraically closed field k of characteristic 0. (Makes some theorem statements simpler, but our results also work in positive characteristic.)
- A graded ring R is *connected graded* if $R_0 = k$.

Definition

Let R be an $\mathbb{N}$-graded noetherian domain. The graded quotient ring of R is of the form

$$Q_{\mathrm{gr}}(R) = D[z, z^{-1}; \sigma]$$

for some division ring D and automorphism σ of D.

- *D is the* function field *of R.*
- *R is* birationally commutative *if its function field is commutative.*

Main result

Definition

A birationally commutative projective surface *is a k-algebra that is:*

- *a connected $\mathbb{N}$-graded domain;*
- *noetherian;*
- *birationally commutative;*
- *GK-dimension 3.*

Theorem (S.)

R is a birationally commutative projective surface if and only if R falls into one of four infinite families. In particular, any such R is defined by geometric data and is canonically associated to a (unique) projective surface X.

Generalizes work of Rogalski and Stafford for rings generated in degree 1 (2 families).

Review: curves

Definition

A (noncommutative) projective curve *is a k-algebra that is:*

- *a connected $\mathbb{N}$-graded domain;*
- *noetherian;*
- *GK-dimension 2.*

Theorem (Artin-Stafford, 1995)

R is a projective curve if and only if R falls into one of two infinite families. In particular, any such R is birationally commutative and canonically associated to a (unique) projective curve.

Family (1): Twisted homogeneous coordinate rings

- Let X be a projective variety.
- Let σ be an automorphism of X.
- Let $\mathcal{L}$ be an invertible sheaf on X. As usual, let $\mathcal{L}^\sigma = \sigma^*\mathcal{L}$ and let $\mathcal{L}_n$ denote the product $\mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$.
- The *twisted homogeneous coordinate ring* $B = B(X, \mathcal{L}, \sigma)$ is defined by

$$B = B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n)$$

- Multiplication on B is induced by σ.

Ampleness

- We say that $\mathcal{L}$ is σ*-ample* if $\{\mathcal{L}_n\}$ has the same good properties as the tensor powers of an ample invertible sheaf.

Theorem (Artin-Van den Bergh 90)

If $\mathcal{L}$ is σ-ample, then $B(X, \mathcal{L}, \sigma)$ is noetherian.

- Example: Let

$$\sigma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{P}GL_2.$$

Then

$$B(\mathbb{P}^1, \mathcal{O}(1), \sigma) \cong k\langle x, y\rangle/(xy - yx - x^2).$$

Family (1a): Geometric idealizers

- Let $X, \mathcal{L}, \sigma$ as before. Assume $\mathcal{L}$ is σ-ample. Let $B = B(X, \mathcal{L}, \sigma)$.
- Let Z be a closed subscheme of X, of infinite order under σ.
- Let $I \subseteq B$ be the right ideal generated by sections vanishing on Z.
- Let
$$R(X, \mathcal{L}, \sigma, Z) = k + I \subset B.$$
- In nice situations R is the *idealizer* of I in B: maximal subring in which I is a 2-sided ideal.

Idealizing on $\mathbb{P}^1$

- Example: Assume char $k = 0$. Let

$$B = B(\mathbb{P}^1, \mathcal{O}(1), \sigma) \cong k\langle x, y\rangle/(xy - yx - x^2)$$

and

$$R = R(\mathbb{P}^1, \mathcal{O}(1), \sigma, [1:0]) \cong k + yB$$

- (Stafford-Zhang 94)
  - R is noetherian.
  - In general $k + yB$ is noetherian if and only if char $k = 0$.
- Need $[1:0]$ to have infinite order under σ.

The classification of noncommutative curves

Theorem (Artin-Stafford 95)

R is a projective curve if and only if (up to replacing R by a Veronese subring), R is either

(1) *A twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ where X is a (commutative) projective curve and $\mathcal{L}$ is σ-ample; or*

(1a) *An idealizer $R(X, \mathcal{L}, \sigma, Z)$ on a projective curve X at points of infinite order.*

- All noncommutative curves are (birationally) commutative: all division rings of transcendence degree 1 are fields.

Artin's conjecture

Not many division rings of transcendence degree 2 are known.

Conjecture (Artin 95)

Let R be a (noncommutative) projective surface. Then the function field of R is either:

- *a field of transcendence degree 2;*
- *a division ring finite-dimensional over a central field of transcendence degree 2;*
- *the full quotient division ring of a skew polynomial extension of a field of transcendence degree 1 (a "quantum ruled surface"); or*
- $S(E, \sigma)$*, the function field of the* Sklyanin algebra $A(E, \sigma)$ *for some elliptic curve E and automorphism* σ *of E (a "quantum rational surface").*

Some birationally commutative projective surfaces

- Twisted homogeneous coordinate rings of surfaces;
  - GK 3 puts restrictions on the automorphism
- Geometric idealizers on surfaces

Theorem (S.)

A geometric idealizer $R(X, \mathcal{L}, \sigma, Z)$ of GK-dimension 3 is noetherian if and only if Z is "transverse" to all σ-invariant subschemes.

- Very weak definition of transverse: no component of Z can contain or be contained in any nontrivial invariant subscheme.
- In positive characteristic need "critically transverse."
- (Also understand higher-dimensional idealizers.)

Naïve blowups

- Let X be a projective surface (variety of dimension ≥ 2).
- Let $\sigma \in \mathrm{Aut}(X)$ and let $\mathcal{L}$ be a σ-ample invertible sheaf.
- Let $P \in X$ of infinite order; let $\mathfrak{m}$ be the ideal sheaf defining P.
- The *naïve blowup* of X at P is the ring

$$S(X, \mathcal{L}, \sigma, P) = \bigoplus_{n \geq 0} H^0(X, \mathfrak{m}\mathfrak{m}^{\sigma} \cdots \mathfrak{m}^{\sigma^{n-1}} \cdot \mathcal{L}_n).$$

- A noncommutative Rees ring; first studied by Keeler, Rogalski, and Stafford.
- Can also form $S(X, \mathcal{L}, \sigma, Z)$ for any 0-dimensional subscheme $Z \subset X$.

Theorem (Rogalski-Stafford 06; S.; Bell 08)

$S(X, \mathcal{L}, \sigma, Z)$ is noetherian if and only if Z is supported at points with (critically) dense orbits.

ADC rings

- A more general naïve blowup.
- Example: Let $X, \mathcal{L}, \sigma, P$ as above.
- Let $\mathfrak{a}$, $\mathfrak{d}$, $\mathfrak{c}$ be ideal sheaves cosupported at P satisfying $\mathfrak{a}\mathfrak{c} \subseteq \mathfrak{d}$. Let

$$S = \bigoplus_{n \geq 0} H^0(X, \mathfrak{a}\mathfrak{d}^{\sigma} \cdots \mathfrak{d}^{\sigma^{n-1}} \mathfrak{c}^{\sigma^n} \cdot \mathcal{L}_n)$$

- Recall the naïve blowup is

$$\bigoplus_{n \geq 0} H^0(X, \mathfrak{m}\mathfrak{m}^{\sigma} \cdots \mathfrak{m}^{\sigma^{n-1}} \cdot \mathcal{L}_n).$$

- If $\mathfrak{a}\mathfrak{c} = \mathfrak{d}$ the ADC ring is the naïve blowup at the scheme defined by $\mathfrak{a}\mathfrak{c}^{\sigma}$.

Properties of ADC rings

Proposition (S.)

S is noetherian if and only if the orbit of P is (critically) dense.

- An ADC ring may be a maximal order (“integrally closed”).
- No Veronese is ever generated in degree 1 unless $\mathfrak{ac} = \mathfrak{d}$.
- Question: other properties that distinguish naïve blowups from ADC rings?
- Proj looks like Proj of a naïve blowup.

BC surfaces generated in degree 1

Theorem (Rogalski-Stafford 06)

R is a birationally commutative projective surface that is generated in degree 1 if and only if (up to replacing R by a Veronese subring), R is either

(1) *a twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ where X is a projective surface (and $\mathcal{L}$ is σ-ample); or*

(2) *a naïve blowup $B(X, \mathcal{L}, \sigma, Z)$ on a projective surface X at a 0-dimensional subscheme Z supported at points with (critically) dense orbits.*

General birationally commutative surfaces

Theorem

R is a birationally commutative projective surface if and only if (up to a finite dimensional vector space and/or a Veronese ring, as always) R is either:

(1) *the twisted homogeneous coordinate ring of a projective surface;*

(2) *an ADC ring on a projective surface;*

(1a) *a geometric idealizer on a projective surface; or*

(2a) *an idealizer in a ring of type* (2).

Furthermore, all defining data is (critically) transverse.

(This also gives a new proof of Rogalski-Stafford.)

Proof techniques

Difficult part: construct the surface X on which R lives.

- Rogalski-Stafford: X is a subscheme of a proscheme (a projective limit of schemes) that “tries to parameterize” point modules over R.
- Our technique: We are given the function field K and σ and so the birational equivalence class of X. Choose any smooth model Y for K and modify.
- Key result (Rogalski 07): There is some Y such that σ induces an automorphism of Y.
- Philosophy: method of successive approximation. Work on Y; if Y is not correct, modify to get closer. This terminates after finitely many steps.
- Constructs X more directly but less functorially.
- Then show other defining data is of the form claimed.

Consequences

- Any birationally commutative surface is contained in a twisted homogeneous coordinate ring and has geometric data canonically associated to it.

Definition

A connected graded $\mathbb{N}$-graded ring R satisfies χ if for any finitely generated left (or right) R-module M and for all $j \geq 0$ $\underline{\mathrm{Ext}}^j_R(k, M)$ is finite-dimensional.

- If R is a birationally commutative surface then R satisfies χ if and only if (some Veronese of) R is a twisted homogeneous coordinate ring. (In fact: R satisfies right (or left) χ_2 is sufficient.)
- All birationally commutative surfaces have cohomological dimension 2.

Future work

- Remove restrictions on σ. What are the birationally commutative surfaces of GK-dimension 5?
  - Eg: $X = E \times E$, $\sigma : (x, y) \mapsto (x + y, y)$.
  - Look at $B(X, \mathcal{L}, \sigma)$ and subrings. Are all surfaces of GK 5 of this form?
  - Rogalski-Stafford's result holds in GK 5 case.
- What about BC surfaces of GK-dimension 4? Do any exist?
  - Rogalski: Here σ is not an automorphism of any model of the function field.
  - No twisted homogeneous coordinate rings are noetherian.

Conjecture

Let R be a birationally commutative noetherian connected $\mathbb{N}$-graded domain of GK-dimension d. Suppose also that σ is geometric *(excludes GK 4 surfaces). Then R falls into one of the families (1), (1a), (2), (2a) and is associated to a projective variety of dimension $\leq d$.*