Enveloping algebras of infinite-dimensional Lie algebras

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Outline

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Outline

Work over $\mathbb{C}$ unless stated otherwise.

Notation:

- $\mathfrak{l}$ = arbitrary Lie algebra
- L = infinite-dimensional Lie algebra
- $\mathfrak{g}$ = finite-dimensional simple Lie algebra

$$\mathfrak{l} \rightsquigarrow U(\mathfrak{l}) = \frac{T(\mathfrak{l})}{(xy - yx = [x, y] \text{ for } x, y \in \mathfrak{l}\ )}$$

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$U(\mathfrak{l})$ with $\dim \mathfrak{l} < \infty$: some of the most well-studied examples in ring theory.

$U(L)$: much more mysterious!

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$U(L)$: much more mysterious!

Question

What is the ring theory of $U(L)$ for $\dim L = \infty$?

Outline

(i) Algebras of derivations

- W = Witt algebra = $\operatorname{Der} \mathbb{C}[t, t^{-1}] = \mathbb{C}[t, t^{-1}]\partial$. (Here $\partial = \frac{d}{dt}$.)
- Vir = Virasoro algebra $=_{\text{vsp}} W \oplus \mathbb{C}z$

$$z \text{ central}, \quad [f\partial, g\partial] = (fg' - f'g)\partial + \operatorname{Res}_0(f'g'' - f''g')z$$

- $\operatorname{Der} C$ for any commutative (associative, unital) algebra C

(ii) Kac-Moody algebras

$A \in M_{n \times n}(\mathbb{Z}) \rightsquigarrow L(A)$, presented by (generalised) Serre relations.

Three types, depending on A:

- finite-dimensional simple
- affine
- indefinite type

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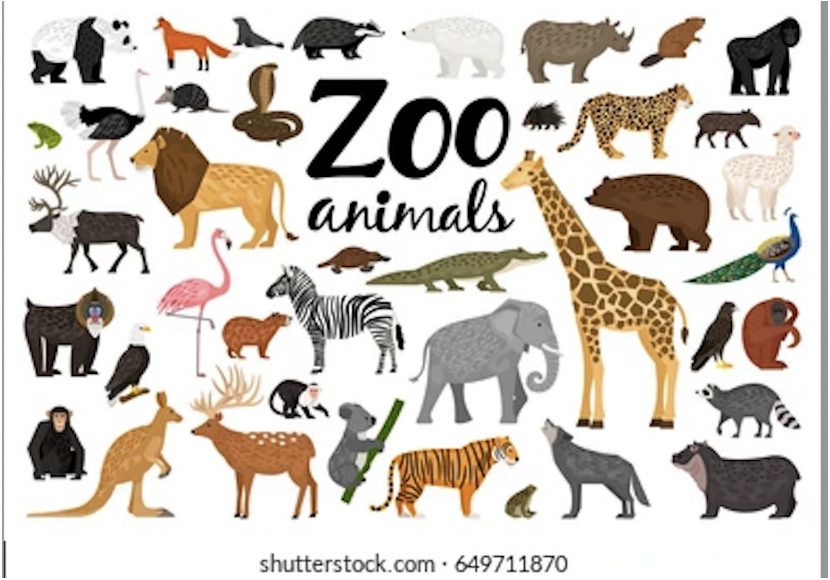
Affine algebras look like (as vector space!)

$$\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}z =: \widehat{\mathfrak{g}} \text{ for some (finite dimensional simple) } \mathfrak{g}$$

$\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is the loop algebra of $\mathfrak{g}$:

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg$$

(iii)-(∞):



Theorem (O. Mathieu '92)

Let L be $\mathbb{Z}$-graded simple, infinite-dimensional, and have polynomial growth. Then L is one of:

- $\mathfrak{g}[t, t^{-1}]$ *(or a twisted form)*
- $\operatorname{Der} \mathbb{C}[t_1, \ldots, t_n]$ *(or one of three subalgebras)*
- W*, the Witt algebra*

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$U(\mathfrak{l}) \cong_{vsp} S(\mathfrak{l}) = \operatorname{gr} U(\mathfrak{l})$

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Corollary

$\dim \mathfrak{l} < \infty \Rightarrow U(\mathfrak{l})$ *is noetherian*

Proof: noncommutative rings are nicer than commutative rings!

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Non-example: L abelian $\Rightarrow U(L) = S(L)$ is a polynomial ring in infinitely many variables, not noetherian.

$U(W)$ is more interesting.

- Very noncommutative: surjects onto every Weyl algebra
- so it's easier for 1-sided ideals to be big.

Question (Dean-Small '90)

Is U(W) noetherian?

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Theorem (S.-Walton '13)

1. *$U(W)$ is not noetherian*
2. *If L is $\mathbb{Z}$-graded simple, infinite-dimensional, polynomial growth then $U(L)$ is not noetherian.*

To prove the theorem, suffices to prove that $U(W_+)$ is not noetherian, where $W_+ = t^2\mathbb{C}[t]\partial$ is the positive Witt algebra.

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The second proof considers the obvious map

$$\phi : U(W_+) \to A_1 = \mathbb{C}[t, \partial], \qquad f\partial \mapsto f\partial$$

(In A_1, the first Weyl algebra, we have $\partial t = t\partial + 1$.)

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Claim: $\ker \phi$ is not finitely generated as a left or right ideal.

Lift ϕ to:

$$\begin{array}{ccc} U(W_+) & \xrightarrow{\Phi} & \mathbb{C}[s,t,\partial] \\ & \searrow^{\phi} & \downarrow{\scriptstyle s\mapsto 0} \\ & & \mathbb{C}[t,\partial] \end{array}$$

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$\operatorname{Im}\Phi$ is easier to understand, and can show $\Phi(\ker\phi)$ is not finitely generated as a left or right ideal of $\operatorname{Im}\Phi$.

Thus $\operatorname{Im}\Phi$ is not left or right noetherian, so neither is $U(W_+)$.

Conjecture (S.-Walton '13)

$U(\mathfrak{l})$ is noetherian if and only if $\dim \mathfrak{l} < \infty$.

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Theorem

$U(L)$ is not noetherian if L is:

- *(S.-Walton) W_+ or an algebra on Mathieu's list*
- *(Buzaglo) $\operatorname{Der} C$ for C a finitely generated commutative domain*
- *an infinite-dimensional Kac-Moody algebra*
- *any other specific example*

Theorem (Topley '18)

Let char $k > 0$*, and let L be a* $\mathbb{Z}$*-graded Lie algebra of linear growth defined over* k*. Then* $U(L)$ *is not noetherian.*

Proof.

Show that $U(L)$ has a very large and non-noetherian centre. □

(In contrast, for algebras on Mathieu's list $Z(U(L)) = \mathbb{C}$.)

Outline

$$\begin{array}{ccc} U(W_+) & \xrightarrow{\Phi} & \mathbb{C}[s,t,\partial] \\ & \searrow^{\phi} & \big\downarrow{\scriptstyle s\mapsto 0} \\ & & \mathbb{C}[t,\partial] \end{array}$$

Observe that $\operatorname{Im}\Phi$, $\operatorname{Im}\phi$ are much smaller than $U(W_+)$.

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Definition

An (associative) $\mathbb{C}$-algebra R has polynomial or finite growth if $\exists d \in \mathbb{N}$ so that for all finite dimensional $V \subseteq R$

$$\dim V^n < n^d \quad \textit{for } n \gg 0$$

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$U(\mathfrak{l})$ has finite growth $\iff \dim \mathfrak{l} < \infty$.

So $U(W_+)$ has infinite growth, and $\operatorname{Im}\phi$, $\operatorname{Im}\Phi$ have finite growth.

Definition

R has just-infinite growth if R has infinite growth but R/I has finite growth for all $0 \neq I \triangleleft R$.

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Conjecture (Petukhov-S. '17)

$U(W_+)$ has just-infinite growth.

(Suggested by computer experiments of I. Stanciu.)

Theorem (Iyudu-S. '19)

$U(W_+)$ has just-infinite growth. So does $U(Vir)/(z-\lambda)$ for any $\lambda \in \mathbb{C}$.

(including $U(W) = U(Vir)/(z)$.)

Method: combinatorics to prove that two-sided ideals of $U(W_+)$ are very big and "have almost all leading terms".

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Question

For which L does $U(L)$ have just-infinite growth?

Outline

Conjecture (Petukhov-S. '17)

$U(W)$, $U(W_+)$, and $U(Vir)$ satisfy ACC on two-sided ideals.

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Evidence: Two-sided ideals in these algebras are big, so maybe they are sparse?

(S.-Walton '15) $\ker \phi$, $\ker \Phi$ are principal, and $\operatorname{Im} \phi$, $\operatorname{Im} \Phi$ satisfy ACC on two-sided ideals.

However (Petukhov-S. '22) $U(\mathfrak{g}[t, t^{-1}])$ does not have ACC on two-sided ideals.

Theorem (Biswal-S. '22)

If $\lambda \neq 0$ then $U_\lambda := U(\widehat{\mathfrak{g}})/(z - \lambda)$ is simple.

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If $\lambda \neq 0$ *then* $U_\lambda := U(\widehat{\mathfrak{g}})/(z - \lambda)$ *is simple.*

Proof ($\mathfrak{g} = \mathfrak{sl}_2$).

$\{z\} \cup \{ht^i\}$ generate an infinite-dimensional Heisenberg algebra $\mathfrak{h}_\infty \subset \widehat{\mathfrak{sl}_2}$. So

$$A_\infty := \frac{U(\mathfrak{h}_\infty)}{(z - \lambda)} \subset U_\lambda.$$

A_∞ is the infinite Weyl algebra. It has infinite growth and is simple.

Let $0 \neq I \lhd U_\lambda$, so

$$\frac{A_\infty}{I \cap A_\infty} \hookrightarrow \frac{U_\lambda}{I}.$$

By the previous theorem, both these algebras have finite growth.

So $I \cap A_\infty \neq 0$. As A_∞ is simple we have $1 \in I$. □

Recall the philosophy that $U(\mathit{Vir})$ is more noncommutative, so nicer, than $U(\widehat{\mathfrak{g}})$.

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Question

Is $U(Vir)/(z-\lambda)$ simple for $\lambda \neq 0$?

Outline

Corollary

Let $M \in \operatorname{Rep} \widehat{\mathfrak{g}}$ *have central character* $\lambda \neq 0$. *Then*

$$\operatorname{Ann}_{U(\widehat{\mathfrak{g}})}(M) = (z - \lambda).$$

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Proof.

$\operatorname{Ann}_{U(\widehat{\mathfrak{g}})}(M) \supseteq (z - \lambda)$, which is a maximal ideal. □

Annihilators previously known only for Verma modules (Chari '85).

Question

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Moral proof that the answer is "no": let M be such a representation, with central character $\lambda \neq 0$. Now conjecturally $U(Vir)/(z-\lambda)$ is simple, so $\mathrm{Ann}_{U(Vir)}(M) = (z-\lambda)$.

In other words, $U(Vir)/(z-\lambda)$, which has infinite growth, acts faithfully on M. This cannot be possible, as $U(Vir)/(z-\lambda)$ is so much larger than M.

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In other words, $U(Vir)/(z - \lambda)$, which has infinite growth, acts faithfully on M. This cannot be possible, as $U(Vir)/(z - \lambda)$ is so much larger than M.

Note that $U(\widehat{\mathfrak{g}})$ does not have polynomial growth irreps with nontrivial central character, and this is basically the proof: such an irrep would be a polynomial growth irrep of A_∞, which are known not to exist by Bernstein's inequality.

Outline

Let $\mathfrak{l}$ be any Lie algebra, with basis $\{x_i\}_{i \in I}$. Recall the symmetric algebra of $\mathfrak{l}$ is $S(\mathfrak{l}) = \mathbb{C}[x_i]_{i \in I}$.

$S(\mathfrak{l})$ is a Poisson algebra: it has a Lie bracket $\{-,-\}$ which is a derivation in each variable and satisfies $\{x_i, x_j\} = [x_i, x_j]$.

An ideal I of $S(\mathfrak{l})$ is a Poisson ideal if it's also a Lie ideal: $\{I, S(\mathfrak{l})\} \subseteq I$. We write $I \triangleleft_P S(\mathfrak{l})$.

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Recall that $S(\mathfrak{l}) = \operatorname{gr} U(\mathfrak{l})$. Thus from $J \lhd U(\mathfrak{l})$ obtain a Poisson ideal $\operatorname{gr} J \lhd_P S(\mathfrak{l})$.

Consequence: if $S(\mathfrak{l})$ has ACC on Poisson ideals then $U(\mathfrak{l})$ has ACC on two-sided ideals.

Theorem (León Sánchez-S. '20)

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Theorem (Iyudu-S. '19)

If I is a nontrivial Poisson ideal of $S(W)$ then $S(W)/I$ has finite growth. (and similarly for $S(Vir)/(z-\lambda)$).

Thus if $0 \neq I \triangleleft_P S(Vir)/(z-\lambda)$ we expect algebraic geometry on $V(I)$, which is finite-dimensional and has finitely many irreducible components.

Let S be any (commutative) Poisson algebra. A Poisson ideal $Q \lhd_P S$ is Poisson primitive if there is a maximal ideal $\mathfrak{m}$ of S so that Q is the Poisson core of $\mathfrak{m}$: the maximal Poisson ideal contained in $\mathfrak{m}$.

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Theorem (Petukhov-S. '21)

1. *Classify the Poisson primitive ideals of $S(Vir)$.*
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Muchas gracias!