The Poisson spectrum of the symmetric algebra of the Virasoro algebra

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(Actually mostly about functions on *Vir* and their “coadjoint orbits”, with applications to the Poisson ideal structure of $S(Vir)$ and conjectures about ideals in $U(Vir)$.)

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Outline

Definition

Let $\partial = \frac{d}{dt}$. The Witt algebra of vector fields on the punctured complex line is $W = \mathbb{C}[t, t^{-1}]\partial$, with bracket

$$[f\partial, g\partial] = (fg' - f'g)\partial.$$

The Virasoro algebra is $Vir = \mathbb{C}[t, t^{-1}]\partial \oplus \mathbb{C}z$, with bracket

$$[f\partial, g\partial] = (fg' - f'g)\partial + \mathrm{Res}_0(f'g'' - f''g')z, \qquad z \text{ central}$$

$W = Vir/(z)$, and Vir is the unique central extension of W. Important in physics, representation theory, ...

We're interested in their universal enveloping algebras, where recall

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle uv - vu = [u, v] | u, v \in \mathfrak{g}\rangle.$$

$U(\mathfrak{g})$ is associative and has the same representation theory as $\mathfrak{g}$.

Ring theory of $U(W)$, $U(Vir)$:

- Both have infinite Gelfand-Kirillov (GK-) dimension (subexponential growth)
  - Basis for $U(W) \leftrightarrow$ generalised partitions $(n_1 \leq n_2 \leq \cdots \leq n_k)$
- Highly noncommutative: for all n, $U(W) \twoheadrightarrow A_n$, the nth Weyl algebra.

Theorem (S.-Walton 2013)

$U(W)$, $U(Vir)$ are not left or right noetherian.

Proof.

There's a ring homomorphism

$$\pi_0 : U(W) \to \mathbb{C}[t, t^{-1}, \partial] \quad f\partial \mapsto f\partial \quad (\text{here } \partial t = t\partial + 1)$$

Fact: $\ker \pi_0$ is not finitely generated as a left or right ideal. □

However, they seem to have few two-sided ideals:

- Example: $\ker \pi_0$ is principal as 2-sided ideal. (Conley-Martin 2007)
- And we have:

Theorem (Iyudu-S., 2019)

Let $\lambda \in \mathbb{C}$. Then $U(Vir)/(z-\lambda)$ has just-infinite GK-dimension: any proper factor has polynomial growth.

- "growth" here (GK-dimension) measures $\dim V^n$ for a finite-dimensional subspace of an algebra R
- a commutative domain R has polynomial growth iff $\operatorname{trdeg} Q(R) < \infty$.
- $U(Vir)/(z-\lambda)$ grows like $\mathcal{P}(n) \Rightarrow$ faster than any polynomial.

Goal: Understand the 2-sided ideal structure of $U(Vir)$, $U(W)$.

Question (*)

Do $U(Vir)$, $U(W)$ have ACC on 2-sided ideals?

Question

What are the primitive ideals? (annihilators of simple representations)

These are too hard!

Instead, consider the symmetric algebras $S(W)$, $S(Vir)$.

Recall: if $x_1, x_2, \ldots$ is a basis for a Lie algebra $\mathfrak{g}$ then

$$S(\mathfrak{g}) = \mathbb{C}[x_1, x_2, \ldots].$$

It's also a Poisson algebra with

$$\{v, w\} = [v, w] \quad \text{for any } v, w \in \mathfrak{g}$$

$\{\ ,\ \}$ here is a Lie bracket on $S(\mathfrak{g})$ such that any $\{s, -\}$ is a derivation.

An ideal $I \lhd S(\mathfrak{g})$ is Poisson if $\{S(\mathfrak{g}), I\} \subseteq I$. We write this as

$$I \lhd_P S(\mathfrak{g}).$$

Also recall: $U(\mathfrak{g})$ has a filtration with $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$,

$$J \lhd U(\mathfrak{g}) \rightsquigarrow \operatorname{gr} J \lhd_P S(\mathfrak{g}).$$

Goal': Understand the Poisson ideal structure of $S(Vir)$, $S(W)$.

Question

Do $S(Vir)$, $S(W)$ have ACC on Poisson ideals?

(if so, implies Question (*))

Two facts:

Theorem (Iyudu-S., 2019)

Let $\lambda \in \mathbb{C}$. Then any proper Poisson factor of $S(Vir)/(z-\lambda)$ has polynomial growth (=finite trdeg).

Theorem (León-Sánchez-S., 2020)

$S(W)$, $S(Vir)$ have ACC on radical Poisson ideals.

Particular question: what are the Poisson primitive ideals of $S(Vir)$?

Given $\mathfrak{g}$, $\chi \in \mathfrak{g}^*$, let $\mathfrak{m}_\chi$ be the corresponding maximal ideal of $S(\mathfrak{g})$. (That is, $\mathfrak{m}_\chi$ = kernel of homomorphism $\chi : S(\mathfrak{g}) \to \mathbb{C}$.)

The Poisson core of χ is

$$\begin{aligned} \mathrm{Core}(\chi) &= \text{maximal Poisson ideal contained in } \mathfrak{m}_\chi \\ &= \sum \{ I \triangleleft_P S(\mathfrak{g}) \mid I \subseteq \mathfrak{m}_\chi \} \end{aligned}$$

Definition

$I \triangleleft_P S(\mathfrak{g})$ is Poisson primitive if there is $\chi \in \mathfrak{g}^*$ so that $I = \mathrm{Core}(\chi)$.

- Expect these to be shadows of primitive ideals in $U(Vir)$
- If $\dim_{\mathbb{C}} \mathfrak{g} < \infty$ and G is the adjoint group, then $\mathrm{Core}(\chi) = I(G \cdot \chi)$.

Question

What are the Poisson primitive ideals of $S(W)$?

This splits into two sub-problems:

Question

For which $\chi \in W^*$ is $\mathrm{Core}(\chi) \neq 0$?

Question

For these χ can we describe $\mathrm{Core}(\chi)$ more explicitly?

Intuition: Suppose that we had an adjoint group G for W; let $\mathbb{O}(\chi) = G \cdot \chi \subseteq W^*$.

If $\mathsf{Core}(\chi) = 0$ then $\mathbb{O}(\chi) :=$ "coadjoint orbit" of χ is dense in W^*. If $\mathsf{Core}(\chi) \neq 0$, then $\dim \mathbb{O}(\chi) = \mathsf{GKdim}\, S(W)/\mathsf{Core}(\chi) < \infty$, so we can hope to do some algebraic geometry.

More rigorous definition:

$$\mathbb{O}(\chi) := \{\nu \in W^* \mid \mathsf{Core}(\nu) = \mathsf{Core}(\chi)\}$$

is called the pseudo-orbit of χ.

If $\mathfrak{g} = \mathsf{Lie}(G)$ with $\dim \mathfrak{g} < \infty$ then $\mathbb{O}(\chi) = G \cdot \chi$.

Fact: we can compute $\mathsf{Core}(\chi)$ from $\mathbb{O}(\chi)$:

$$\mathsf{Core}(\chi) = \bigcap \{\nu \in \mathbb{O}(\chi) \mid \mathfrak{m}_\nu\}.$$

Example

Let $x, \alpha, \beta \in \mathbb{C}$, with $x \neq 0, (\alpha, \beta) \neq (0, 0)$. Define $\chi_{x;\alpha,\beta} \in W^*$ by

$$\chi_{x;\alpha,\beta}(f\partial) = \alpha f(x) + \beta f'(x).$$

Notice: $\chi_{x;\alpha,\beta}$ vanishes on $(t-x)^2\mathbb{C}[t, t^{-1}]\partial$, and we'll see that $\mathrm{Core}(\chi_{x;\alpha,\beta}) \neq 0$.

Notation

if $g \in \mathbb{C}[t, t^{-1}]$ *set* $W(g) = g\mathbb{C}[t, t^{-1}]\partial$.

Define a Poisson bracket on $\mathbb{C}[t, t^{-1}, y]$ by $\{t, y\} = 1$ (localised Poisson-Weyl algebra). Define

$$p_\beta : S(W) \to \mathbb{C}[t, t^{-1}, y] \qquad f\partial \mapsto fy + \beta f'$$

This is a Poisson map, and by construction $\ker p_\beta \subseteq \mathfrak{m}_{\chi_{x;\alpha,\beta}}$.

(equivalently, $\chi_{x;\alpha,\beta} : S(W) \to \mathbb{C}$ factors through p_β.)

Proposition (PS)

$$\mathrm{Core}(\chi_{x;\alpha,\beta}) = \ker p_\beta.$$

Further,

$$\mathbb{O}(\chi_{x;\alpha,\beta}) = \{\chi_{*;*,\beta}\}.$$

Recall: a Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by:

$$\begin{aligned}(v \cdot \chi)(u) &= \chi([v, u]) \\ &\stackrel{\text{def}}{=} B_\chi(v, u)\end{aligned}$$

The <u>isotropy subalgebra</u> of χ is

$$\begin{aligned}\mathfrak{g}^\chi &= \{v \in \mathfrak{g} | v \cdot \chi = 0\} \\ &= \{v \in \mathfrak{g} | B_\chi(v, -) = 0\} \\ &= \ker B_\chi.\end{aligned}$$

In the example, with $\chi = \chi_{x;\alpha,\beta}$, note $W((t-x)^3) \subset W^\chi$, because

$$[W((t-x)^3), W] \subseteq W((t-x)^2), \quad \text{and} \quad \chi|_{W((t-x)^2)} = 0.$$

(In general if $\chi|_{W(h)} = 0$, then $W(h^2) \subseteq W^\chi$.)

Definition

A local function on W is a linear combination of functions of the form

$$f\partial \mapsto \alpha_0 f(x) + \alpha_1 f'(x) + \alpha_2 f''(x) + \cdots + \alpha_n f^{(n)}(x).$$

(Example: $\chi_{x;\alpha,\beta}$)

Theorem (PS)

For $\chi \in W^*$ the following conditions are equivalent
(1) χ is local,
(2) $\mathrm{Core}(\chi) \neq 0$,
(3) $\dim W/W^\chi < \infty$,
(4) there exists $0 \neq h \in \mathbb{C}[t]$ such that $\chi|_{W(h)} = 0$.

(We've seen (1) (2) (3) (4) for $\chi_{x;\alpha,\beta}$.)

(1) $\iff$ (4) is basically the Chinese Remainder Theorem and noticing that

$$\chi(f\partial) = \alpha_0 f(x) + \alpha_1 f'(x) + \alpha_2 f''(x) + \cdots + \alpha_n f^{(n)}(x)$$

$\iff$ χ vanishes on $W((t-x)^{n+1})$.

(4) $\Rightarrow$ (3) was two slides ago: " in general ..."

For (3) $\Rightarrow$ (4) note that if $0 \neq h\partial \in W^\chi$ then

$$0 = \chi([h\partial, hr\partial]) = \chi(h^2 r' \partial) \quad \text{for all } r \in \mathbb{C}[t, t^{-1}].$$

This doesn't quite show $\chi|_{W(h^2)} = 0$, but we've only used $W^\chi \neq 0$.

(3) $\Rightarrow$ (2): If $\dim W/W^\chi = n$ then

$$\begin{vmatrix} \chi([w_0, w_0]) & \dots & \chi([w_0, w_n]) \\ \dots & \dots & \dots \\ \chi([w_n, w_0]) & \dots & \chi([w_n, w_n]) \end{vmatrix} = 0$$

for all $w_0, \dots, w_n \in W$.

Let $I(n)$ be the ideal of $S(W)$ generated by

$$\begin{vmatrix} [w_0, w_0] & \dots & [w_0, w_n] \\ \dots & \dots & \dots \\ [w_n, w_0] & \dots & [w_n, w_n] \end{vmatrix}$$

for all $w_0, \dots, w_n \in W$.

Then $I(n) \subseteq \mathfrak{m}_\chi$.

Fact: $I(n)$ is Poisson, so $\mathrm{Core}(\chi) \neq 0$.

For (2) $\Rightarrow$ (3) do a similar determinental calculation in $Q(S(W)/\operatorname{Core}(\chi))$, which has finite transcendence degree by the Poisson Iyudu-S. theorem. (Note that $\operatorname{Core}(\chi)$ is prime.)

In fact, in general we have

Proposition

For any $\mathfrak{g}$ and $\chi \in \mathfrak{g}^*$, $\operatorname{trdeg} Q(S(\mathfrak{g})/\operatorname{Core}(\chi)) \geq \dim \mathfrak{g}/\mathfrak{g}^\chi$.

Restate the theorem:

Theorem

For $\chi \in W^*$, $\operatorname{Core}(\chi) \neq 0 \iff \chi$ is local.

Corollary

For “most” $\chi \in W^*$, have $\operatorname{Core}(\chi) = 0$.

Let $\chi \in W^*$ be local. What's $\mathrm{Core}(\chi)$? Equivalently, what's $\mathbb{O}(\chi)$?

Theorem

Let $Loc^n = \{\chi_{x;\alpha_0,\ldots,\alpha_n} | x \in \mathbb{C}^\times, \alpha_n \neq 0\}$.

If n is even, then $Loc^n = \mathbb{O}(\chi_{1;0,\ldots,0,1})$ *is a single pseudo-orbit in* W^*.

If n is odd, then Loc^n *fibers into a pencil of hypersurfaces (parameterised by* $\mathbb{A}^1$*), each of which is a pseudo-orbit.*

Theorem

Let χ^I *and* χ^{II} *be local functions on W represented by sums*

$$\chi^I = \sum_{i=1}^{\ell} \chi_i^I, \quad \chi^{II} = \sum_{i=1}^{k} \chi_i^{II}, \qquad \chi_i^{I,II} \in Loc.$$

Then $\mathbb{O}(\chi^I) = \mathbb{O}(\chi^{II}) \iff k = \ell$ *and* $\mathbb{O}(\chi_i^I) = \mathbb{O}(\chi_i^{II})$ *for all i.*

Corollary

The set

$$Loc^{n_1} + Loc^{n_2} + \cdots + Loc^{n_\ell}$$

(where we assume the component one-point functions are based at distinct points) fibers into a family of pseudo-orbits parameterised by some $\mathbb{A}^k$*, where* $k \leq \ell$*.*

(Here k counts the number of odd n_i.)

This lets us describe arbitrary prime Poisson ideals of $S(W)$: such an ideal can be defined from an algebraic subvariety of one of these affine spaces.

It's tempting to think of Poisson primitive ideals like “closed points” in the Poisson prime spectrum of $S(W)$, but …

Theorem

Let Q be any prime Poisson ideal of $S(W)$. There is $\nu \in W^*$ so that $Q \supseteq \mathrm{Core}(\nu)$.

(In geometric language, $V(Q) \subseteq \overline{\mathbb{O}(\nu)}$.)

Proof.

Choose $m_i \geq n_i$ even; then

$$Loc^{m_1} + Loc^{m_2} + \cdots + Loc^{m_\ell} = \mathbb{O}(\nu).$$

□

Corollary

$S(W)$ has no nonzero prime Poisson ideals of finite height.

Application: subalgebras of W of finite codimension.

Theorem

For $\chi \in W^*$, $\mathrm{Core}(\chi) \neq 0 \iff \chi$ is local $\iff$ (3) $\iff$ (4).

Corollary

Let $\mathfrak{g} \subseteq W$ be a Lie subalgebra of finite codimension. Then $\exists f \neq 0$ so that $\mathfrak{g} \supseteq W(f)$.

Proof.

Let $\{\chi_1, \dots, \chi_n\}$ be a basis of $(W/\mathfrak{g})^* \subset W^*$.

Then $B_{\chi_i}(\mathfrak{g}, \mathfrak{g}) = 0$, so $\mathrm{rank}\, B_{\chi_i} \leq 2 \dim W/\mathfrak{g} < \infty$

$\Rightarrow \chi_i$ is local. So there is h_i with $\chi_i|_{W(h_i)} = 0$.

So $\mathfrak{g} = \{w \in W \mid \chi_1(w) = \cdots = \chi_n(w) = 0\} \supseteq W(h_1 \cdots h_n)$. □

Corollary

If $\mathfrak{g} \subseteq Vir$ has finite codimension, then $z \in [\mathfrak{g}, \mathfrak{g}]$.

(previously known only for $\dim Vir/\mathfrak{g} = 1$)

Proof.

The image of $\mathfrak{g}$ in $W = Vir/z$ contains some $W(f)$.

$\Rightarrow$ there is $e_p = ft^p\partial + \lambda_p z \in \mathfrak{g}$ for all $p \in \mathbb{Z}$.

$$[e_p, e_q] = (q-p)f^2 t^{p+q-1}\partial + \_\_z$$

$$\frac{[e_p, e_q]}{q-p} = f^2 t^{p+q-1}\partial + \_\_z$$

$\Rightarrow z \in [\mathfrak{g}, \mathfrak{g}]$. □

Theorem

Let $\chi \in Vir^*$. *Then* $\mathrm{Core}(\chi) \neq (z - \chi(z)) \iff \chi(z) = 0$ *and* χ *defines a local function on* $Vir/(z) = W$.

Proof ($\Rightarrow$).

(Iyudu-S.) $\mathrm{Core}(\chi) \neq (z - \chi(z)) \Rightarrow \mathrm{GKdim}\, S(Vir)/\mathrm{Core}(\chi) < \infty$.

By the Proposition, $\dim Vir/Vir^\chi < \infty$.

From the last Corollary, $z \in [Vir^\chi, Vir^\chi]$.

By definition $\chi([Vir^\chi, Vir^\chi]) = 0$ so $\chi(z) = 0$. □

Corollary

The Poisson primitive ideals of $S(Vir)$ *are:*

- $(z - \lambda)$ *for* $\lambda \in \mathbb{C}$
- $(z) + \mathrm{Core}(\chi)$ *for* χ *a local function on* W.

Corollary

Let $0 \neq \lambda \in \mathbb{C}$*. Then* $S(Vir)/(z-\lambda)$ *is Poisson simple.*

Proof.

If not, there is $I \triangleleft_P S(Vir)$ which strictly contains $(z-\lambda)$.

Choose $\chi \in Vir^*$ so that $\mathfrak{m}_\chi \supseteq I$ (use generalised Nullstellensatz).

Then $I \subseteq \mathrm{Core}(\chi) \neq (z-\lambda)$ and $\chi(z) = \lambda$.

By the previous theorem $\lambda = 0$. □

Question

Is there a correspondence

$$\text{Poisson primitives of } S(Vir) \leftrightarrow \text{primitive ideals of } U(Vir)?$$

Question

If $0 \neq \lambda \in \mathbb{C}$, *is* $U(Vir)/(z - \lambda)$ *simple?*

Question

If $0 \neq \lambda \in \mathbb{C}$, *does Vir have any polynomial growth irreps of central character* λ*?*

Thank you!