Universal enveloping algebras of Krichever-Novikov algebras

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Motivation

Conjecture (Sierra–Walton, 2013)

$U(\mathfrak{g})$ is noetherian $\iff$ $\dim \mathfrak{g} < \infty$.

For the rest of the story go to Sue Sierra's talk!

Note: "Noetherian" $=$ Left and right noetherian.

Subalgebras of finite codimension

Lemma (Sierra–Walton, 2013)

If $\mathfrak{h} \subseteq \mathfrak{g}$ and $U(\mathfrak{g})$ is noetherian then $U(\mathfrak{h})$ is noetherian.

What about the other direction?

Proposition (B., 2021)

If $\mathfrak{h}$ has finite codimension in $\mathfrak{g}$ then $U(\mathfrak{h})$ is noetherian iff $U(\mathfrak{g})$ is noetherian.

"Proof".

$U(\mathfrak{g})$ looks like

$$U(\mathfrak{g}) \approx U(\mathfrak{h})[x_1, \dots, x_n],$$

so if $U(\mathfrak{h})$ is noetherian then $U(\mathfrak{g})$ is noetherian by Hilbert's basis theorem. □

Krichever-Novikov algebras

Definition

Let C be a nonsingular affine curve. The Krichever-Novikov (KN) algebra on C, denoted $\mathcal{L}(C)$, is the Lie algebra of algebraic vector fields on C.

In other words, $\mathcal{L}(C) = \mathrm{Der}(\mathbb{C}[C])$.

Example

1. $W := \mathrm{Der}(\mathbb{C}[t, t^{-1}]) = \mathcal{L}(\mathbb{A}^1 \setminus \{0\})$ is the Witt algebra.
   $\mathbb{W}_1 := \mathrm{Der}(\mathbb{C}[t]) = \mathcal{L}(\mathbb{A}^1)$ is a Lie algebra of Cartan type.
   Easy to describe explicitly: $W = \mathbb{C}[t, t^{-1}]\frac{d}{dt}$, $\mathbb{W}_1 = \mathbb{C}[t]\frac{d}{dt}$.
2. Define an elliptic curve $E: \quad y^2 = x(x^2 - 1)$.
   Much harder to describe $\mathcal{L}(E)$ explicitly!

Why Krichever-Novikov algebras?

Theorem (Sierra–Walton, 2013)

$U(W)$ and $U(\mathbb{W}_1)$ are not noetherian.

Question

Are enveloping algebras of KN algebras noetherian in general?

Theorem (B., 2021)

No. Enveloping algebras of KN algebras are never noetherian.

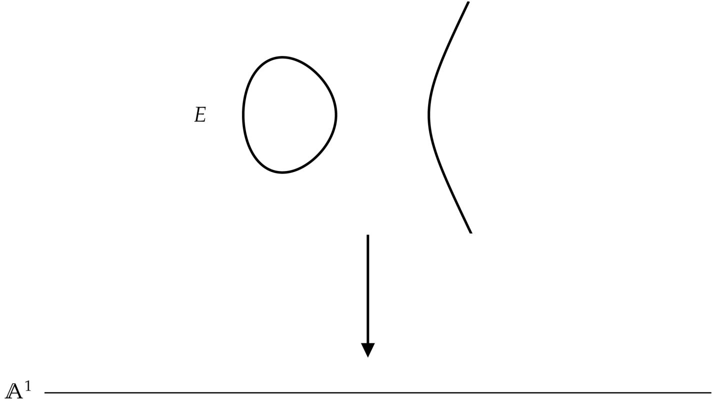
Problem: We don't know what KN algebras look like in general!

Solution: Suffices to find enough vector fields to get a subalgebra $\mathfrak{g} \subseteq \mathcal{L}(C)$ such that $U(\mathfrak{g})$ is not noetherian.

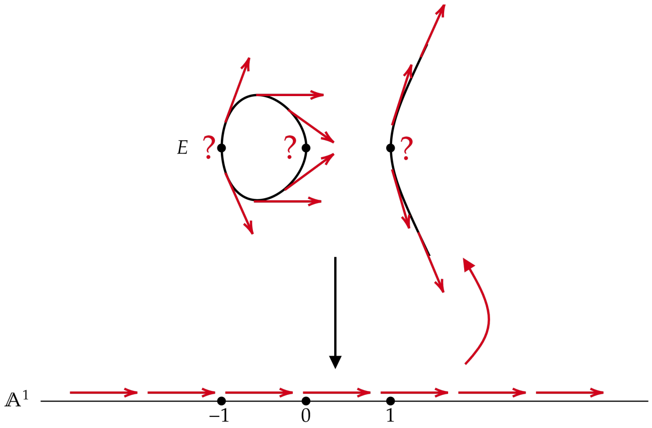
How do we do this?

Pulling back vector fields

Idea: Dominant map $C \to \mathbb{A}^1$ allows us to pull back vector fields on $\mathbb{A}^1$ to vector fields on C.

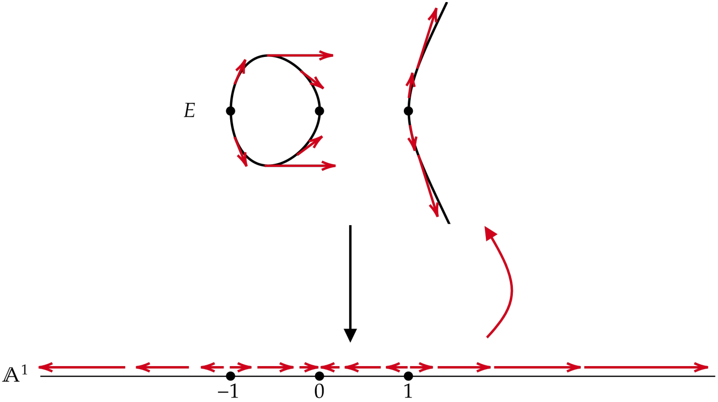


Example: $\frac{d}{dt}$



Can't pull this vector field back to E! Problems at $-1, 0, 1$.

Example: $t(t^2 - 1)\frac{d}{dt}$



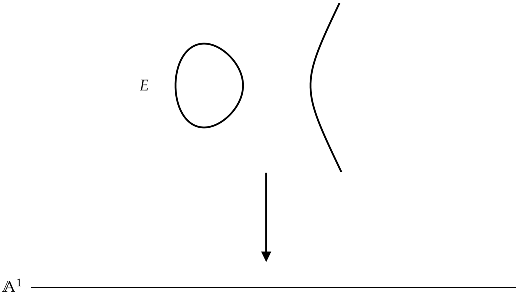
Vector field vanishes at problematic points, so we can pull it back!

Proposition (B., 2021)

Let X and Y be affine curves, and let $\pi\colon X \to Y$ be a dominant morphism. Then there is an injective Lie algebra homomorphism

$$\varphi\colon \mathcal{L}_\pi(Y) \to \mathcal{L}(X),$$

where $\mathcal{L}_\pi(Y) \subseteq \mathcal{L}(Y)$ consists of vector fields which vanish at the ramification points of π.



Ramification points are $\{0, \pm 1\}$, so we get an injective map

$$\mathbb{W}_1(t(t^2-1)) \hookrightarrow \mathcal{L}(E)$$
$$t(t^2-1)f(t)\frac{d}{dt} \mapsto x(x^2-1)f(x)\frac{d}{dx},$$

where $\mathbb{W}_1(t(t^2-1)) := \{f(t)\frac{d}{dt} \in \mathbb{W}_1 \mid t(t^2-1) \text{ divides } f(t)\}$.

Proof of theorem

Theorem

Let C be an affine curve. Then $U(\mathcal{L}(C))$ is not noetherian.

Proof.

Let $\pi : C \to \mathbb{A}^1$ be dominant. Then there is an injective map

$$\mathcal{L}_\pi(\mathbb{A}^1) \hookrightarrow \mathcal{L}(C).$$

Let $\{x_1, \ldots, x_n\} \subseteq \mathbb{A}^1$ be the ramification points of π, and let

$$f(t) = (t - x_1) \ldots (t - x_n).$$

Then $\mathcal{L}_\pi(\mathbb{A}^1) = \mathbb{W}_1(f)$ has finite codimension in $\mathbb{W}_1$, so $U(\mathcal{L}_\pi(\mathbb{A}^1))$ is not noetherian. Therefore, $U(\mathcal{L}(C))$ is not noetherian. $\square$

What next?

Direction #1: Subalgebras of $\mathbb{W}_1$.

Conjecture

If $\mathfrak{g}$ is an infinite-dimensional subalgebra of $\mathbb{W}_1$, then is $U(\mathfrak{g})$ is not noetherian.

A lot of progress made in this direction, including a partial classification of subalgebras of $\mathbb{W}_1$.

Direction #2: Higher-dimensional varieties.

Turns out that the higher-dimensional case is much easier than the curve case!

Theorem (B., 2022)

Let X be an affine variety and let $\mathfrak{g} = \mathrm{Der}(\mathbb{C}[X])$. Then $U(\mathfrak{g})$ is not noetherian.

What next?

Direction #3: ACC on two-sided ideals.

Question

Let $\mathfrak{g}$ be one of the Lie algebras from before. Does $U(\mathfrak{g})$ satisfy ACC on two-sided ideals?