Krichever-Novikov algebras and deformations of the Witt algebra

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Outline

Deformations of Lie algebras

Vague idea: Deformations are continuous modifications of Lie brackets.

Definition

A deformation of a Lie algebra $(\mathfrak{g}, [-,-])$ is a family of Lie algebra structures on $\mathfrak{g}$,

$$[x,y]_t = [x,y] + t \cdot \phi_1(x,y) + t^2 \cdot \phi_2(x,y) + \ldots, \quad (x,y \in \mathfrak{g})$$

with bilinear maps $\phi_i : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

- $\mathfrak{g}_t = (\mathfrak{g}, [-,-]_t)$ is a Lie algebra,
- $\mathfrak{g}_0 = \mathfrak{g}$.

What is t?

Different choices for t are possible:

- t might be a variable, i.e. we can plug in numbers $\alpha \in \mathbb{C}$.
  $\rightsquigarrow$ Get deformation over $\mathbb{C}[t]$ (geometric deformation) or over convergent power series $\mathbb{C}\{\{t\}\}$ (analytic deformation).
- t might be a formal variable, i.e. we allow infinitely many terms in $[-,-]_t$.
  $\rightsquigarrow$ Get deformation over $\mathbb{C}[[t]]$ (formal deformation).
- t might be an infinitesimal variable, i.e. we take $t^2 = 0$.
  $\rightsquigarrow$ Get deformation over $\mathbb{C}[t]/(t^2)$ (infinitesimal deformation).

Equivalence of deformations

Definition

Two deformations $[-,-]_t$ and $[-,-]'_t$ are equivalent if a linear automorphism

$$\psi_t = \mathrm{id} + t\alpha_1 + t^2\alpha_2 + \dots$$

exists with $\alpha_i : \mathfrak{g} \to \mathfrak{g}$ linear maps such that

$$\psi_t([x,y]'_t) = [\psi_t(x), \psi_t(y)]_t.$$

$(\mathfrak{g}, [-,-])$ is rigid if every deformation $[-,-]_t$ of $[-,-]$ is locally equivalent to the trivial family.

Intuitively, rigid means $\mathfrak{g}$ cannot be deformed.

Rigidity of Lie algebras

"Locally" equivalent means we only consider t "near 0".
$\rightsquigarrow$ Depends on the kind of deformation we are considering.

Formal/infinitesimal: 0 is the only closed point, so every deformation is already local.

Geometric/analytic: "local" means there is an open neighborhood U of 0 such that $[-,-]_t$ restricted to U is equivalent to the trivial family.
$\rightsquigarrow$ In particular, $\mathfrak{g}_\alpha \cong \mathfrak{g}$ for all $\alpha \in U$.

Lie algebra cohomology

Question: When does

$$[x,y]_t = [x,y] + t \cdot \phi_1(x,y) + t^2 \cdot \phi_2(x,y) + \dots$$

define a Lie bracket? We need:

Antisymmetry: $[x,y]_t = -[y,x]_t \implies \phi_i$ are antisymmetric, i.e. $\phi_i \in \mathrm{Hom}(\bigwedge^2 \mathfrak{g}, \mathfrak{g}) = C^2(\mathfrak{g};\mathfrak{g})$.

Jacobi identity: $[x,[y,z]_t]_t + [y,[z,x]_t]_t + [z,[x,y]_t]_t = 0$.
Looking at the coefficient of t, we get $d\phi_1 = 0$, i.e. ϕ_1 is a cocycle.

Coefficients of higher powers of t give us further properties the ϕ_i must satisfy.

Suppose $[-,-]_t$ and $[-,-]_t'$ are equivalent deformations. We have

$$\psi_t([x,y]_t') = [\psi_t(x), \psi_t(y)]_t.$$

Equating coefficients of t, we get

$$(\phi_1 - \phi_1')(x,y) = \alpha_1([x,y]) - [x, \alpha_1(y)] + [y, \alpha_1(x)] = d\alpha_1(x,y).$$

Therefore, $\phi_1 - \phi_1' = d\alpha_1$.

$$\implies [\phi_1] = [\phi_1'] \in H^2(\mathfrak{g}; \mathfrak{g}).$$

Infinitesimal deformations

If we consider infinitesimal deformations (i.e. $t^2 = 0$), there are no higher powers of t.

$\implies$ Any 2-cocycle ϕ defines an infinitesimal deformation

$$[x, y]_t = [x, y] + t\phi(x, y).$$

Furthermore, two infinitesimal deformations $[-, -]_t$ and $[-, -]'_t$ are equivalent if $[\phi] = [\phi'] \in H^2(\mathfrak{g}; \mathfrak{g})$.

Rigidity from cohomology

Facts:

1. $H^2(\mathfrak{g};\mathfrak{g})$ classifies infinitesimal deformations of $\mathfrak{g}$.
2. If $\dim H^2(\mathfrak{g};\mathfrak{g}) < \infty$ then all formal deformations of $\mathfrak{g}$ up to equivalence can be realised in $H^2(\mathfrak{g};\mathfrak{g})$.
3. If $H^2(\mathfrak{g};\mathfrak{g}) = 0$ then $\mathfrak{g}$ is infinitesimally and formally rigid.
4. If $\dim \mathfrak{g} < \infty$ then $H^2(\mathfrak{g};\mathfrak{g}) = 0$ implies that $\mathfrak{g}$ is also geometrically and analytically rigid.

Remark

If $\dim \mathfrak{g} = \infty$ then $H^2(\mathfrak{g};\mathfrak{g}) = 0$ **does not** imply that $\mathfrak{g}$ is geometrically/analytically rigid.

Formal rigidity of Witt and Virasoro algebras

Let $\mathcal{W}$ be the Witt algebra, let $\mathcal{V}$ be the Virasoro algebra, and let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra.

Theorem

We have

$$H^2(\mathcal{W};\mathcal{W}) = 0,$$

$$H^2(\mathcal{V};\mathcal{V}) = 0,$$

$$H^2(\overline{\mathfrak{g}};\overline{\mathfrak{g}}) = 0,$$

where $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Therefore $\mathcal{W}$, $\mathcal{V}$, and $\overline{\mathfrak{g}}$ are infinitesimally and formally rigid.

Goal: Construct nontrivial families of Krichever-Novikov algebras $\{\mathcal{L}_\alpha \mid \alpha \in \mathbb{C}\}$ such that $\mathcal{L}_0 \cong \mathcal{W}$.

Families of KN algebras

We will consider the genus 1 case, i.e. KN algebras over elliptic curves. We will use the description

$$E: \quad Y^2Z = 4(X - e_1Z)(X - e_2Z)(X - e_3Z),$$

with $e_1 + e_2 + e_3 = 0$ and $e_i \neq e_j$ for $i \neq j$. We set

$$B = \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, e_i \neq e_j \text{ for } i \neq j\}.$$

We can extend this family to

$$\widehat{B} = \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}.$$

Resolving $e_3 = -(e_1 + e_2)$ in $\widehat{B}$, we obtain a family over $\mathbb{C}^2$.

Note: The fibers above $\widehat{B} \setminus B$ are singular cubic curves.

For $s \in \mathbb{C}$, consider

$$D_s = \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad D_\infty = \{(0, e_2) \in \mathbb{C}^2\}.$$

Set also

$$D_s^* = D_s \setminus \{(0,0)\}.$$

We have

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}).$$

Above D_1^*, $D_{-1/2}^*$ and D_{-2}^*, we get a nodal cubic

$$E_N : \quad Y^2 Z = 4(X - eZ)^2 (X + 2eZ).$$

At $(0,0)$, we get the cuspidal cubic

$$E_C : \quad Y^2 Z = 4X^3.$$

In all cases (non-singular or singular), the point $\infty = [0 : 1 : 0]$ lies on the curves and is the only intersection with the line at infinity.

$\implies$ If we keep in mind that there are poles at ∞, we lose nothing by passing to an affine chart.

The modular parameter j is constant for the curves above D_s^*:

$$j(s) = 1728 \frac{4(1+s+s^2)^3}{(1-s)^2(2+s)^2(1+2s)^2}, \quad j(\infty) = 1728.$$

In particular, all the curves above D_s^* are isomorphic.

Marked points

For now, we fix e_1, e_2, e_3 such that $e_i \neq e_j$ for $i \neq j$ and let E be the corresponding elliptic curve. Meromorphic functions on E are rational functions on E, given by

$$\mathbb{C}(E) = \mathbb{C}(X)[Y]/(Y^2 - f(X)), \quad f(T) = 4(T - e_1)(T - e_2)(T - e_3).$$

Which points do we choose where poles are allowed?

We will only consider the two-point situation. Our marked points will be $\infty = [0:1:0]$ and the point with affine coordinate $(e_1, 0)$.

Proposition (Deck–Ruffing–Schlichenmaier, 1992)

A basis of the function algebra $\mathcal{A}$ is given by

$$A_{2k} = (X - e_1)^k, \quad A_{2k+1} = \frac{1}{2} Y (X - e_1)^{k-1}, \quad k \in \mathbb{Z}.$$

The Krichever-Novikov algebra

We express the analytic differential dz as

$$dz = \frac{dX}{Y}.$$

This gives a basis for the Krichever-Novikov algebra $\mathcal{L}$.

Proposition

A basis for the Krichever-Novikov algebra $\mathcal{L}$ is given by $V_n = A_{n-1} dz^{-1}$ ($n \in \mathbb{Z}$), i.e.

$$V_{2k} = \frac{1}{2} f(X)(X - e_1)^{k-2} \frac{d}{dX},$$

$$V_{2k+1} = (X - e_1)^k Y \frac{d}{dX}.$$

Setting $\deg V_n = \deg A_n = n$ yields an almost-grading on $\mathcal{A}$ and $\mathcal{L}$.

The families of algebras

Theorem (Schlichenmaier, 1993)

The Lie bracket on $\mathcal{L}$ is given by:

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & (n, m \text{ odd}), \\ (m-n)(V_{n+m} + pV_{n+m-2} + qV_{n+m-4}) & (n, m \text{ even}), \\ (m-n)V_{n+m} + (m-n-1)pV_{n+m-2} & (n \text{ odd}, \\ \quad +(m-n-2)qV_{n+m-4} & m \text{ even}), \end{cases}$$

where $p = 3e_1$ and $q = (e_1 - e_2)(e_1 - e_3)$.

In fact, these relations define a Lie algebra for every $(e_1, e_2) \in \mathbb{C}^2$.

We write $\mathcal{L}^{(e_1,e_2)}$ for the Lie algebra corresponding to (e_1, e_2). Clearly,

$$\mathcal{L}^{(0,0)} \cong \mathcal{W}.$$

Proposition (Fialowski–Schlichenmaier, 2003)

For $(e_1, e_2) \neq (0,0)$, *we have*

$$\mathcal{L}^{(e_1,e_2)} \not\cong \mathcal{W}.$$

On the other hand, $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

In order to get one-dimensional families of Lie algebras, we fix $s \in \mathbb{C} \cup \{\infty\}$ and only consider $(e_1, e_2) \in D_s$.

Lie bracket above D_s

If $s \neq \infty$ then $e_2 = se_1$ and we get

$$q = (e_1 - e_2)(e_1 - e_3) = e_1^2(1-s)(2+s) = \frac{1}{9}(1-s)(2+s)p^2.$$

Letting $a_s = \frac{1}{9}(1-s)(2+s) \in \mathbb{C}$ and $t = p$, the Lie bracket becomes

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & (n, m \text{ odd}), \\ (m-n)(V_{n+m} + tV_{n+m-2} + a_s t^2 V_{n+m-4}) & (n, m \text{ even}), \\ (m-n)V_{n+m} + (m-n-1)tV_{n+m-2} & (n \text{ odd}, \\ \quad +(m-n-2)a_s t^2 V_{n+m-4} & m \text{ even}). \end{cases}$$

Lie bracket above D_∞

For D_∞, we have $e_3 = -e_2$ and $e_1 = 0$. Hence, $p = 0$ and $q = -e_2^2$. Letting $t = q$, the Lie bracket becomes

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & (n, m \text{ odd}), \\ (m-n)(V_{n+m} + t^2 V_{n+m-4}) & (n, m \text{ even}), \\ (m-n)V_{n+m} + (m-n-2)t^2 V_{n+m-4} & (n \text{ odd}, \ m \text{ even}). \end{cases}$$

For a fixed value of s, we therefore obtain is a family of Lie algebras $\mathcal{L}_t$ over the affine line with $\mathcal{L}_0 \cong \mathcal{W}$.

Theorem (Fialowski–Schlichenmaier, 2003)

Despite its infinitesimal and formal rigidity, the Witt algebra $\mathcal{W}$ admits locally nontrivial deformations $\mathcal{L}_t$ over the affine line.

$\rightsquigarrow$ We also get nontrivial deformations of $\mathcal{V}$ and $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

Degenerate cases

Goal: Study what happens above D_1, D_{-2} and $D_{-1/2}$ and understand why $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

For E_N, let e be the coinciding value of the e_i's. Consider desingularisations of E_N and E_C:

$$\psi_N : \mathbb{P}^1 \to E_N, \quad \psi_C : \mathbb{P}^1 \to E_C,$$

where

$$\psi_N([t:s]) = [t^2 s - 2es^3 : 2t(t^2 - 3es) : s^3],$$
$$\psi_C([t:s]) = [t^2 s : 2t^3 : s^3].$$

It's enough to consider the affine part (i.e. set $s = 1$).

Properties of the maps:

1. They are surjective.
2. $\infty = [1:0] \in \mathbb{P}^1$ corresponds to ∞ on both E_N and E_C.
3. The map ψ_C is $1:1$.
4. The map ψ_N is not $1:1$ only at $t = \pm\sqrt{3e}$. Both points project onto the singular point $(e, 0)$.
5. The point $(-2e, 0) \in E_N$ corresponds to $t = 0$.

Consider the pullback of vector fields on E_N and E_C to $\mathbb{P}^1$. Setting $a = \sqrt{3e}$, we have

$$\psi_N^*(Y\frac{d}{dX}) = (t^2 - a^2)\frac{d}{dt},$$
$$\psi_C^*(Y\frac{d}{dX}) = t^2\frac{d}{dt}.$$

Cuspidal cubic

After pullback to $\mathbb{P}^1$, poles are allowed at 0 and ∞.

What happens functions on E_C?

$$\psi_C^*(A_{2k})(t) = \psi_C^*(X^k)(t) = X^k(t^2, 2t^3) = t^{2k},$$
$$\psi_C^*(A_{2k+1})(t) = \psi_C^*(\frac{1}{2}X^{k-1}Y)(t) = (\frac{1}{2}X^{k-1}Y)(t^2, 2t^3) = t^{2k+1}.$$

What about vector fields?
$\rightsquigarrow \psi_C^*(Y\frac{d}{dX}) = t^2\frac{d}{dt}$ just increases the zero order at 0.

$\implies$ We recover the Witt algebra

$$V_n = t^{n+1}\frac{d}{dt}.$$

Nodal cubic above $D_1^* \cup D_{-2}^*$

First consider the case where $e = e_1$ (i.e. we are in D_1^* or D_{-2}^*). After pullback to $\mathbb{P}^1$, poles are allowed at $\pm a$ and ∞.

What happens to functions on E_N?

$$\begin{aligned}
\psi_N^*(A_{2k})(t) &= \psi_N^*((X-e)^k)(t) = (X-e)^k(t^2-2e, 2t(t^2-3e)) \\
&= (t^2-a^2)^k, \\
\psi_N^*(A_{2k+1})(t) &= \psi_N^*(\frac{1}{2}(X-e)^{k-1}Y)(t) \\
&= (\frac{1}{2}(X-e)^{k-1}Y)(t^2-2e, 2t(t^2-3e)) \\
&= t(t^2-a^2)^k.
\end{aligned}$$

Nodal cubic above $D_1^* \cup D_{-2}^*$

What about vector fields?
$\rightsquigarrow \psi_N^*(Y\frac{d}{dX}) = (t^2 - a^2)\frac{d}{dt}$ just increases the zero order at $\pm a$.

$\implies$ We get a KN algebra on $\mathbb{P}^1$ with 3 marked points $\{\pm a, \infty\}$.
It has basis

$$V_{2k} = t(t-a)^k(t+a)^k\frac{d}{dt},$$
$$V_{2k+1} = (t-a)^{k+1}(t+a)^{k+1}\frac{d}{dt}.$$

The Lie bracket is given by

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & (n, m \text{ odd}), \\ (m-n)(V_{n+m} + a^2 V_{n+m-2}) & (n, m \text{ even}), \\ (m-n)V_{n+m} + (m-n-1)a^2 V_{n+m-2} & (n \text{ odd}, \ m \text{ even}). \end{cases}$$

Nodal cubic above $D^*_{-1/2}$

Now consider the case where $e = e_2 = e_3 \neq e_1$ (i.e. we are in $D^*_{-1/2}$). In this case, $e_1 = -2e$. After pullback to $\mathbb{P}^1$, poles are allowed at 0 and ∞.

What happens to functions on E_N?

$$\begin{aligned}
\psi_N^*(A_{2k})(t) &= \psi_N^*((X+2e)^k)(t) \\
&= (X+2e)^k(t^2-2e, 2t(t^2-3e)) \\
&= t^{2k}, \\
\psi_N^*(A_{2k+1})(t) &= \psi_N^*(\frac{1}{2}(X+2e)^{k-1}Y)(t) \\
&= (\frac{1}{2}(X+2e)^{k-1}Y)(t^2-2e, 2t(t^2-3e)) \\
&= t^{2k-1}(t^2-a^2).
\end{aligned}$$

Nodal cubic above $D^*_{-1/2}$

What about vector fields?
$\rightsquigarrow \psi_N^*(Y\frac{d}{dX}) = (t^2 - a^2)\frac{d}{dt}$ introduces zeros at $\pm a$.

$\implies$ We get a subalgebra of the Witt algebra whose elements vanish at $\pm a$. It has basis

$$V_{2k} = t^{2k-3}(t^2 - a^2)^2 \frac{d}{dt},$$
$$V_{2k+1} = t^{2k}(t^2 - a^2)\frac{d}{dt}.$$

This subalgebra of $\mathcal{W}$ can be described as

$$\{f(t)(t^2 - a^2)\frac{d}{dt} \in \mathcal{W} \mid f(a) = f(-a)\}.$$

Nodal cubic above $D^*_{-1/2}$

The Lie bracket is given by

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & (n, m \text{ odd}), \\ (m-n)(V_{n+m} - 2a^2 V_{n+m-2} + a^4 V_{n+m-4}) & (n, m \text{ even}), \\ (m-n)V_{n+m} - 2(m-n-1)a^2 V_{n+m-2} & (n \text{ odd}, \\ \quad +(m-n-2)a^4 V_{n+m-4} & m \text{ even}). \end{cases}$$