The almost-grading in Krichever-Novikov algebras

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Outline

Notation

Recall our setup:

- A compact Riemann surface Σ of genus g.
- A finite subset $A \subseteq \Sigma$.
- Function algebra: $\mathcal{A}$ = meromorphic functions on Σ that are holomorphic outside A.
- KN algebra: $\mathcal{L}$ = meromorphic vector fields on Σ that are holomorphic outside A.
- Current algebra: $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ (where $\dim \mathfrak{g} < \infty$).
- Differential operator algebra: $\mathcal{D}^1 = \mathcal{A} \rtimes \mathcal{L}$.
- Differential operator algebra with coefficients in $\overline{\mathfrak{g}}$: $\mathcal{D}^1_{\overline{\mathfrak{g}}} = \overline{\mathfrak{g}} \rtimes \mathcal{L}$.

Meromorphic differentials

Let $\mathcal{K}$ be the canonical line bundle of Σ.

A global meromorphic section f of $\mathcal{K}$ (a meromorphic differential) can be described locally in coordinates $(U_i, z_i)_{i \in J}$ and local meromorphic functions $(f_i)_{i \in J}$:

$$f = (f_i dz_i)_{i \in J}.$$

On the intersection $U_i \cap U_j$, we require $f_i dz_i = f_j dz_j$, yielding

$$f_j = f_i \left(\frac{dz_i}{dz_j} \right).$$

Meromorphic forms

Let $\lambda \in \mathbb{Z}$. We define

- $\mathcal{K}^\lambda = \mathcal{K}^{\otimes\lambda}$ for $\lambda \geq 1$,
- $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle,
- $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda \leq -1$.

$\mathcal{K}^*$ denotes the dual line bundle, i.e. the tangent bundle.

Global meromorphic sections of $\mathcal{K}^*$ are meromorphic vector fields

$$f = (f_i \frac{d}{dz_i})_{i \in J},$$

where

$$f_i \frac{d}{dz_i} = f_j \frac{d}{dz_j} \quad \text{on } U_i \cap U_j.$$

Meromorphic forms

We set

$$\mathcal{F}^\lambda = \{f \text{ meromorphic section of } \mathcal{K}^\lambda \mid f \text{ holomorphic on } \Sigma \setminus A\}.$$

These are called meromorphic forms of weight λ. We write

$$\mathcal{F} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{F}^\lambda.$$

Then $\mathcal{L} = \mathcal{F}^{-1}$, $\mathcal{A} = \mathcal{F}^0$, and $\mathcal{D}^1 = \mathcal{F}^{-1} \oplus \mathcal{F}^0$.

Associative and Lie products on $\mathcal{F}$

The bilinear map

$$\cdot : \mathcal{F}^{\lambda} \times \mathcal{F}^{\nu} \to \mathcal{F}^{\lambda+\nu}$$
$$(s\ dz^{\lambda}, t\ dz^{\nu}) \mapsto st\ dz^{\lambda+\nu}$$

defines an associative product on $\mathcal{F}$, while the map

$$[-,-] : \mathcal{F}^{\lambda} \times \mathcal{F}^{\nu} \to \mathcal{F}^{\lambda+\nu+1}$$
$$(e\ dz^{\lambda}, f\ dz^{\nu}) \mapsto \left((-\lambda)e\frac{df}{dz} + \nu f\frac{de}{dz}\right) dz^{\lambda+\nu+1}$$

defines a Lie bracket on $\mathcal{F}$. In fact, $\mathcal{F}$ is a Poisson algebra.

$\mathcal{L}$, $\mathcal{A}$ and $\mathcal{D}^1$ are subalgebras of $\mathcal{F}$.

Graded algebras

Definition

A Lie algebra $\mathcal{L}$ is graded if

- $\mathcal{L}$ is a vector space direct sum $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$.
- $[\mathcal{L}_n, \mathcal{L}_m] \subseteq \mathcal{L}_{n+m}$.

Gradings allow us to introduce

- Triangular decompositions,
- Highest weight representations,
- Fock space representations,
- Semi-infinite wedge forms,
- And more!

Furthermore, if $\dim \mathcal{L}_n < \infty$, some tools from finite-dimensional Lie theory can be applied.

In- and out-points

We assume that $N := |A| \geq 2$. In order to construct an almost-grading, we will need a splitting of A:

$$A = I \sqcup O, \quad (I, O \neq \varnothing),$$

(I = "in-points", O = "out-points").

Sometimes, we view I and O as ordered tuples

$$I = (P_1, \ldots, P_K), \quad O = (Q_1, \ldots, Q_M).$$

Note: The almost-grading will depend on the choice of splitting (if $N = 2$ this is essentially unique).

Classical case

Classical case: $\Sigma = S^2 = \mathbb{P}^1$, $|A| = 2$.

By a holomorphic automorphism, we can always choose $I = \{0\}$, $O = \{\infty\}$.

Goal: Make contact with the classical case so we know what to generalise to arbitrary genus.

We have

- $\mathcal{A} = \mathbb{C}[z, z^{-1}]$.
- $\mathcal{L} = \mathbb{C}[z, z^{-1}]\frac{d}{dz}$, the Witt algebra.
- $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$, the loop algebra of $\mathfrak{g}$.

The Witt algebra

The Witt algebra $\mathcal{L} = \mathbb{C}[z, z^{-1}]\frac{d}{dz}$ is the classical KN algebra. It has basis

$$e_n = z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z},$$

and Lie bracket given by

$$[e_n, e_m] = (m-n)e_{n+m}.$$

It is a graded Lie algebra with $\deg e_n = n$ (i.e. $\mathcal{L}_n = \mathbb{C} \cdot e_n$).

The element e_0 plays a special role:

$$[e_0, e_n] = ne_n.$$

$\implies$ The homogeneous spaces $\mathcal{L}_n$ are the eigenspaces of e_0.

Loop algebras

The loop algebra $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is the classical current algebra of $\mathfrak{g}$. It is spanned by

$$x \otimes z^n, \quad x \in \mathfrak{g}, n \in \mathbb{Z},$$

and has Lie bracket given by

$$[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m}.$$

It is a graded Lie algebra, with the grading induced by the grading on $\mathcal{A} = \mathbb{C}[z, z^{-1}]$:

$$\deg(x \otimes z^n) = n,$$

i.e. $\overline{\mathfrak{g}}_n = \mathfrak{g} \otimes z^n$.

The classical differential operator algebra

The Lie algebra $\mathcal{D}^1 = \mathbb{C}[z, z^{-1}] \rtimes \mathbb{C}[z, z^{-1}]\frac{d}{dz}$ is the classical differential operator algebra. It has basis

$$e_n = z^{n+1}\frac{d}{dz}, A_n = z^n, \quad n \in \mathbb{Z},$$

and Lie bracket

$$\begin{aligned} [e_n, e_m] &= (m-n)e_{n+m}, \\ [A_n, A_m] &= 0, \\ [e_n, A_m] &= mA_{n+m}. \end{aligned}$$

Once again, it is a graded Lie algebra, with

$$\deg e_n = \deg A_n = n.$$

Conclusion: Everything is graded in the classical case.

Almost-graded algebras

In the general case, there is no nontrivial grading anymore.

Definition

A Lie algebra $\mathcal{L}$ is almost-graded if

- $\mathcal{L}$ is a vector space direct sum $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$.
- There are constants $L_1, L_2 \in \mathbb{Z}$ such that

$$[\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h.$$

- $\dim \mathcal{L}_n < \infty$.

If $\dim \mathcal{L}_n = \infty$ for some n the Lie algebra is called weakly almost-graded.

Separating cycles

Let C_i be positively oriented circles around the points P_i in I, and C_j^* positively oriented circles around the points Q_j in O.

Definition

A cycle C_S is called a separating cycle if it is smooth, positively oriented, of multiplicity one, and if it separates the in- from the out-points.

We will now integrate meromorphic differentials on Σ over separating cycles.

Integrating over the separating cycle

Given C_S, we define a linear map

$$\begin{aligned} \mathcal{F}^1 &\to \mathbb{C} \\ \omega &\mapsto \frac{1}{2\pi i}\int_{C_S}\omega. \end{aligned}$$

This can be described as integration over the special cycles C_i or C_j^*:

$$\omega \mapsto \frac{1}{2\pi i}\int_{C_S}\omega = \sum_{i=1}^{K}\mathrm{res}_{P_i}(\omega) = -\sum_{j=1}^{M}\mathrm{res}_{Q_j}(\omega).$$

Krichever-Novikov pairing

Definition

The bilinear pairing

$$\mathcal{F}^{\lambda} \times \mathcal{F}^{1-\lambda} \to \mathbb{C}$$
$$(f, g) \mapsto \langle f, g \rangle := \frac{1}{2\pi i} \int_{C_S} f \cdot g.$$

is called the Krichever-Novikov (KN) pairing.

Note: The pairing depends not only on A but also the splitting $A = I \sqcup O$.

We will see that this pairing is nondegenerate.

The homogeneous subspaces

For $\mathcal{F}^\lambda$ we will introduce subspaces $\mathcal{F}_n^\lambda$ ($n \in \mathbb{Z}$) of dimension K by exhibiting certain elements

$$f_{n,p}^\lambda \in \mathcal{F}^\lambda, \quad p = 1, \dots, K,$$

which form a basis of $\mathcal{F}_n^\lambda$. Elements of $\mathcal{F}_n^\lambda$ are called <u>elements of degree n</u>.

As a result, we get the following homogeneous spaces:

$$\mathcal{L}_n = \mathrm{span}_{\mathbb{C}}\{e_{n,p} \mid p = 1, \dots, K\},$$

$$\mathcal{A}_n = \mathrm{span}_{\mathbb{C}}\{A_{n,p} \mid p = 1, \dots, K\},$$

where $e_{n,p} = f_{n,p}^{-1}$ and $A_{n,p} = f_{n,p}^0$.

The elements $f_{n,p}^{\lambda}$

We will collect properties of the elements $f_{n,p}^{\lambda}$, but we will not show their existence (this should be covered in the next talk!).

The zero-order of $f_{n,p}^{\lambda}$ at the points $P_i \in I$ is

$$\mathrm{ord}_{P_i}(f_{n,p}^{\lambda}) = (n+1-\lambda) - \delta_{ip}, \quad i = 1, \ldots, K.$$

The prescriptions at O are made so that $f_{n,p}^{\lambda}$ is unique up to multiplication by a constant, but will not be discussed in this talk. Fixing a system of coordinates z_i centered at P_i, and requiring

$$f_{n,p}^{\lambda}(z_p) = z_p^{n-\lambda}(1 + O(z_p))(dz_p)^{\lambda},$$

the element $f_{n,p}^{\lambda}$ is uniquely fixed.

The model case

We will only consider the following model case:

(1) $O = \{Q\}$ is a one-element set, and

(2a) either the genus $g = 0$,

(2b) or $g \geq 2$, $\lambda \neq 0, 1$, and the points in A are in generic position.

In this case, we require

$$\mathrm{ord}_Q(f_{n,p}^{\lambda}) = -K(n + 1 - \lambda) + (2\lambda - 1)(g - 1).$$

Theorem

The basis elements of $\mathcal{F}^\lambda$ and $\mathcal{F}^{1-\lambda}$ are dual to each other, i.e.

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{-m,r} \rangle = \delta_{pr}\delta_{nm}, \quad n, m \in \mathbb{Z}, r, p = 1, \ldots, K.$$

Corollary

Let $f \in \mathcal{F}^\lambda$ and write f as a sum

$$f = \sum_{n\in\mathbb{Z}} \sum_{p=1}^{K} \alpha_{n,p} f^\lambda_{n,p}, \quad \alpha_{n,p} \in \mathbb{C}.$$

We can compute the coefficients $\alpha_{n,p}$ as

$$\alpha_{n,p} = \langle f, f^{1-\lambda}_{-n,p} \rangle = \frac{1}{2\pi i} \int_{C_S} f \cdot f^{1-\lambda}_{-n,p}.$$

The almost-graded structure of KN algebras

Proposition

We have

$$f_{n,p}^{\lambda} \cdot f_{m,r}^{\nu} = f_{n+m,r}^{\lambda+\nu}\delta_{pr} + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a_{h,s} f_{h,s}^{\lambda+\nu},$$

$$[f_{n,p}^{\lambda}, f_{m,r}^{\nu}] = (-\lambda m + \nu n) f_{n+m,r}^{\lambda+\nu+1}\delta_{pr} + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b_{h,s} f_{h,s}^{\lambda+\nu+1},$$

for some $a_{h,s}, b_{h,s} \in \mathbb{C}$, where $R_1 = \lfloor \frac{g-2}{K} \rfloor + 2$ and $R_2 = \lfloor \frac{3g-3}{K} \rfloor + 3$.

If $K = 1$, then in the model case we have $R_1 = g, R_2 = 3g$.

Sketch of proof for Lie bracket

Consider

$$(-\lambda)f^{\lambda}_{n,p}\frac{df^{\nu}_{m,r}}{dz}+\nu\frac{df^{\lambda}_{n,p}}{dz}f^{\nu}_{m,r}.$$

We want to find the coefficient of $f^{\lambda+\nu+1}_{k,s}$ in the expansion of this element. Calculating this at the point P_i, the lowest order term is

$$(-\lambda(m-\nu+(1-\delta_{ip}))+\nu(n-\lambda+(1-\delta_{ir})))z_i^{n+m-(\lambda+\nu)+(1-\delta_{ip})+(1-\delta_{ir})-1}.$$

Multiplying by the dual element $f^{-(\lambda+\nu)}_{-k,s}$, the order becomes

$$(n+m-k-1)+(1-\delta_{ip})+(1-\delta_{ir})+(1-\delta_{is}).$$

$\Longrightarrow$ A residue is only possible if $k\geq n+m$. For $k=n+m$, there is a residue only when $i=r=p=s$, and in this case the coefficient is $-\lambda m+\nu n$.

Sketch of proof (cont.)

Now we need to consider the order at the point Q. We repeat the method from the previous slide with Q to find the coefficient of $f_{k,s}^{\lambda+\nu+1}$. After multiplying by $f_{-k,s}^{-(\lambda+\nu)}$, the zero-order at Q becomes

$$K(k-(n+m)) - 3K - 3g + 3 - 1.$$

$\Longrightarrow$ A residue is only possible if

$$k \leq n + m + \frac{3g-3}{K} + 3.$$

$\Longrightarrow$ $R_2 = \lfloor \frac{3g-3}{K} \rfloor + 3.$

Theorem

Both the multiplicative and the Lie structures of $\mathcal{F}$ are almost-graded:

$$\mathcal{F}_n^\lambda \cdot \mathcal{F}_m^\mu \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{F}_h^{\lambda+\mu}, \quad [\mathcal{F}_n^\lambda, \mathcal{F}_m^\mu] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{F}_h^{\lambda+\mu+1}.$$

In particular, $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{D}^1$ are all almost-graded. For example,

$$[\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h.$$

The current algebra $\overline{\mathfrak{g}}$ also inherits an almost-grading from $\mathcal{A}$. Furthermore, the homogeneous spaces are all finite-dimensional:

$$\dim \mathcal{L}_n = \dim \mathcal{A}_n = K, \quad \dim \mathcal{D}_n^1 = 2K, \quad \dim \overline{\mathfrak{g}}_n = K \dim \mathfrak{g}.$$

Triangular decomposition

Let $\mathcal{U}$ be one of the algebras $\mathcal{L}$, $\mathcal{A}$, $\mathcal{D}^1$, or $\overline{\mathfrak{g}}$. From the almost-grading, we obtain a triangular decomposition of the algebras

$$\mathcal{U} = \mathcal{U}_{[+]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[-]},$$

where

$$\mathcal{U}_{[+]} = \bigoplus_{n>0} \mathcal{U}_n, \quad \mathcal{U}_{[0]} = \bigoplus_{n=-R_i}^{n=0} \mathcal{U}_n, \quad \mathcal{U}_{[-]} = \bigoplus_{n<-R_i} \mathcal{U}_n.$$

The $[+]$ and $[-]$ subspaces are subalgebras, while in general the $[0]$ spaces are not. The space $\mathcal{U}_{[0]}$ is called the critical strip.

Filtrations

We introduce a descending filtration on $\mathcal{F}^\lambda$

$$\mathcal{F}^\lambda_{(n)} = \bigoplus_{m \geq n} \mathcal{F}^\lambda_m,$$

$$\cdots \supseteq \mathcal{F}^\lambda_{(n-1)} \supseteq \mathcal{F}^\lambda_{(n)} \supseteq \mathcal{F}^\lambda_{(n+1)} \supseteq \cdots$$

Proposition

The filtration is compatible with our algebraic structures:

$$\mathcal{A}_{(n)} \cdot \mathcal{A}_{(m)} \subseteq \mathcal{A}_{(n+m)}, \quad [\mathcal{L}_{(n)}, \mathcal{L}_{(m)}] \subseteq \mathcal{L}_{(n+m)}.$$

Associated graded algebras

Consider

$$\operatorname{gr}\mathcal{A}_{(\bullet)} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_{(n)}/\mathcal{A}_{(n+1)}, \quad \operatorname{gr}\mathcal{L}_{(\bullet)} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_{(n)}/\mathcal{L}_{(n+1)}.$$

Note that $\mathcal{A}_{(n)}/\mathcal{A}_{(n+1)} \cong \mathcal{A}_n$ and $\mathcal{L}_{(n)}/\mathcal{L}_{(n+1)} \cong \mathcal{L}_n$.

Proposition

We have

$$A_{n,p} \cdot A_{m,r} = \delta_{pr} A_{n+m,r} \mod \mathcal{A}_{(n+m+1)},$$
$$[e_{n,p}, e_{m,r}] = (m-n)\delta_{pr} e_{n+m,r} \mod \mathcal{L}_{(n+m+1)}.$$

One-point case

Consider the case where $A = \{P\}$. There is no natural choice of almost-grading: we need a reference point $Q \neq P$.

Letting $A' = \{P, Q\}$, we obtain an almost-grading on $\mathcal{F}(\Sigma, A')$. Our original algebra

$$\mathcal{F} = \mathcal{F}(\Sigma, A) = \{f \in \mathcal{F}(\Sigma, A') \mid f \text{ is holomorphic at } Q\}$$

inherits the almost-grading.

Example of one-point case

Example ($\Sigma = \mathbb{P}^1$, $A = \{\infty\}$)

We have $\mathcal{L} = \mathbb{C}[z]\frac{d}{dz}$. Choosing $Q = 0$ as our reference point, our homogeneous elements are e_n, giving the decomposition

$$\mathcal{L} = \bigoplus_{n=-1}^{\infty} \mathcal{L}_n, \quad \mathcal{L}_n = \mathbb{C} \cdot e_n.$$

Instead choosing $Q = a \neq 0$, the elements $e_n(a) := (z-a)^{n+1}\frac{d}{dz}$ become homogeneous, giving the decomposition

$$\mathcal{L} = \bigoplus_{n=-1}^{\infty} \mathcal{L}'_n, \quad \mathcal{L}'_n = \mathbb{C} \cdot e_n(a).$$

Summary up to this point

KN algebras are interesting not only algebraically but also geometrically. They are generalisations of the Witt and Virasoro algebras.

- Easy part: introduction of meromorphic objects.
- Hard part: almost-grading.