Moduli spaces in noncommutative ring theory

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Outline

In this talk, all algebras are graded and homomorphisms between graded algebras are degree-preserving.

We make the following definition for brevity:

Definition

A $\mathbb{k}$-algebra A is finitely graded if it is $\mathbb{N}$-graded, connected (i.e. $A_0 = \mathbb{k}$) and finitely generated as a $\mathbb{k}$-algebra.

Note: Finitely graded algebras are of the form $\mathbb{k}\langle x_1, \ldots, x_n \rangle / I$ for some graded ideal I.

Normal elements

Definition

An element x of a ring A is normal if $xA = Ax$.

Example

The algebra

$$J = \frac{\Bbbk\langle x, y\rangle}{(yx - xy - x^2)}$$

is called the Jordan plane. Clearly x is a normal element.

Lemma

Let A be a finitely graded $\mathbb{k}$-algebra, and let $x \in A_d$ be a homogeneous normal element for some $d \geq 1$.

1. *If x is regular (i.e. not a zero divisor) in A, then if A/xA is a domain then A is a domain.*
2. *If A/xA is left or right noetherian, then A is left or right noetherian.*

Corollary

The Jordan plane J is a noetherian domain.

Proof.

$J/xJ \cong \mathbb{k}[y]$ and x is regular. □

The Sklyanin algebra

Example

The algebra

$$S = \frac{\Bbbk\langle x, y, z\rangle}{(ayx + bxy + cz^2,\ axz + bzx + cy^2,\ azy + byz + cx^2)}$$

for any $a, b, c \in \Bbbk$ is called the Sklyanin algebra.

Question

- Does S have a normal element?
- Is S a domain?
- Is S noetherian?

Solution: Point modules.

Point modules

Definition

Let A be a finitely graded $\Bbbk$-algebra that is generated in degree 1. A left or right point module for A is a graded left or right module M such that

- M is cyclic,
- M is generated in degree 0,
- $\dim M_n = 1$ for all $n \geq 0$.

Convention: All point modules in this talk are right modules.

Motivation for point modules

Let $B = \Bbbk[x_0, \ldots, x_n]$ with $\deg x_i = 1$. For each $p = [a_0 : \ldots : a_n] \in \mathbb{P}^n$ define a homogeneous ideal $I = I(p)$ of B as follows:

$$I_d = \{f \in B_d \mid f(a_0, \ldots, a_n) = 0\}.$$

Then $B/I(p)$ is a point module for B. It turns out that all point modules for B are of the form $B/I(p)$ for some $p \in \mathbb{P}^n$.

Conclusion: $\mathbb{P}^n$ is a moduli space for the isomorphism classes of point modules for $B = \Bbbk[x_0, \ldots, x_n]$.

Motivation for point modules

This generalises to any finitely graded commutative algebra which is generated in degree 1: given any homogeneous ideal J of B, consider $A = B/J$. Then

$$\operatorname{Proj} A = \{p \in \mathbb{P}^n \mid f(p) = 0 \text{ for all homogeneous } f \in J\}$$

is a moduli space for the point modules of A.

Point modules for free algebra

Let $A = \mathbb{k}\langle x_0, \ldots, x_n \rangle$ with $\deg x_i = 1$ for all i. Let

$$M = \mathbb{k}m_0 \oplus \mathbb{k}m_1 \oplus \ldots$$

We want M to be a point module for A. We need

$$m_i x_j = \lambda_{ij} m_{i+1} \text{ for some } \lambda_{ij} \in \mathbb{k}$$

for M to be a graded A-module.

For M to be cyclic, we require that for each i, some x_j takes m_i to some nonzero multiple of m_{i+1}, i.e. for all i there exists some j such that $\lambda_{ij} \neq 0$.

Point modules for free algebra

This gives a sequence $p_i = [\lambda_{i,0} : \ldots : \lambda_{i,n}] \in \mathbb{P}^n$. We can easily check that a sequence $(p_i)_{i \in \mathbb{N}}$ uniquely determines a point module up to isomorphism.

Conclusion: $\prod_{i=0}^{\infty} \mathbb{P}^n$ is a moduli space for the point modules of $A = \mathbb{k}\langle x_0, \ldots, x_n \rangle$.

Just like in the commutative case, the moduli space for point modules of any finitely graded $\mathbb{k}$-algebra generated in degree 1 is some closed subset of $\prod_{i=0}^{\infty} \mathbb{P}^n$.

Applications of point modules

From the moduli space of point modules of an algebra A, we can construct a twisted homogeneous coordinate ring B.

Example

One can show that the point modules for the Sklyanin algebra

$$S = \frac{\mathbb{k}\langle x, y, z\rangle}{(ayx + bxy + cz^2,\ axz + bzx + cy^2,\ azy + byz + cx^2)}$$

are parametrised by the elliptic curve

$$E : (a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3) = 0$$

provided $abc \neq 0$ and $((a^3 + b^3 + c^3)/3abc)^3 \neq 1$.

Example (cont.)

What is B in this case? Using algebraic geometry, one can prove the following:

1. B is a noetherian domain generated in degree 1;
2. There is a canonical surjective ring homomorphism $\varphi : S \to B$;
3. $B \cong \Bbbk\langle x, y, z\rangle/J$, where J is generated by three quadratic relations and one cubic relation;
4. The cubic relation g provides a normal element of S such that $S/gS \cong B$.

Using the lemma about normal elements, we conclude that S is a noetherian domain!

The above method applies much more generally. For example, it can be used to classify all Artin-Schelter regular algebras of dimension 3.

The Witt algebra

Example

The Witt algebra W is the Lie algebra with basis $\{e_n \mid n \in \mathbb{Z}\}$ such that

$$[e_n, e_m] = (m - n)e_{n+m}$$

for all $n, m \in \mathbb{Z}$. Its universal enveloping algebra

$$U(W) = \frac{\Bbbk\langle e_n \mid n \in \mathbb{Z}\rangle}{(e_n e_m - e_m e_n = (m-n)e_{n+m})}$$

is $\mathbb{Z}$-graded with $\deg e_n = n$.

Question (Dean–Small, 1990)

Is $U(W)$ noetherian? **Problem:** $U(W)$ is not $\mathbb{N}$-graded.

The Witt algebra

What if we look at a subalgebra of $U(W)$ that is $\mathbb{N}$-graded?

Example

The positive Witt algebra W_+ is the Lie subalgebra of W spanned by $\{e_n \mid n \geq 1\}$. Its universal enveloping algebra

$$U(W_+) = \frac{\mathbb{k}\langle e_n \mid n \geq 1\rangle}{(e_n e_m - e_m e_n = (m-n)e_{n+m})}$$

is $\mathbb{N}$-graded. In fact, it is finitely graded (generated by e_1 and e_2 as a $\mathbb{k}$-algebra).

Problem: $U(W_+)$ is not generated in degree 1.

Solution: Intermediate series modules.

Definition

Let A be a $\mathbb{Z}$-graded $\Bbbk$-algebra. An intermediate series module for A is a $\mathbb{Z}$-graded left or right module M such that M_n is one-dimensional for all $n \in \mathbb{Z}$.

Remark

We will stick to $\mathbb{Z}$-graded rings for simplicity, but everything that follows works for gradings by any monoid.

Intermediate series modules of $U(W)$

Kaplansky and Santharoubane showed that there are three families of indecomposable intermediate series $U(W)$-modules:

$$V_{(\alpha,\beta)} = \bigoplus_{n\in\mathbb{Z}} \mathbb{k} v_n, \quad v_n e_m = -(\alpha + \beta m + n) v_{n+m}$$

$$A_{(\alpha,\beta)} = \bigoplus_{n\in\mathbb{Z}} \mathbb{k} a_n, \quad a_n e_m = \begin{cases} -n a_{n+m} & \mathrm{n} \neq 0, n+m \neq 0 \\ -(\alpha + \beta m) a_m & \mathrm{n} = 0 \\ 0 & \mathrm{n} + \mathrm{m} = 0 \end{cases}$$

$$B_{(\alpha,\beta)} = \bigoplus_{n\in\mathbb{Z}} \mathbb{k} b_n, \quad b_n e_m = \begin{cases} -(n+m) b_{n+m} & \mathrm{n} \neq 0, n+m \neq 0 \\ 0 & \mathrm{n} = 0 \\ -(\alpha + \beta m) a_m & \mathrm{n} + \mathrm{m} = 0 \end{cases}$$

where $(\alpha, \beta) \in \mathbb{A}^2$.

Intermediate series modules of $U(W)$

Note that $A_{(\alpha,\beta)}, B_{(\alpha,\beta)}$ are only defined where $(\alpha,\beta) \neq (0,0)$ and depend up to isomorphism only on $[\alpha : \beta] \in \mathbb{P}^1$. They are therefore more appropriately denoted by $A_{[\alpha:\beta]}$ and $B_{[\alpha:\beta]}$.

So the three families are parametrised by two copies of $\mathbb{P}^1$ and one copy of $\mathbb{A}^2$.

Question

If we know a moduli space of intermediate series modules, do we get a homomorphism to a "nice" ring?

Answer: Yes!

Applications of intermediate series modules

Definition

Let R be a ring, and let $\sigma : R \to R$ be an automorphism of R. The skew Laurent polynomial ring $R[t^{\pm 1}; \sigma]$ is the $\mathbb{Z}$-graded ring

$$R[t^{\pm 1}; \sigma] = \bigoplus_{n \in \mathbb{Z}} Rt^n$$

with multiplication defined by the rule $t^n r = \sigma^n(r) t^n$ for all $r \in R, n \in \mathbb{Z}$.

Applications of intermediate series modules

Definition

Let A be a $\mathbb{Z}$-graded ring, and let X be a reduced affine scheme that parametrises a set of intermediate series right A-modules

$$\{M^x = \bigoplus_{i \in \mathbb{Z}} \Bbbk m_i^x \mid x \in X\}.$$

This family is <u>shift invariant</u> if there is an automorphism $\sigma : X \to X$ such that $M^x[n] \cong M^{\sigma^n(x)}$, where the isomorphism maps $m_{i+n}^x \mapsto m_i^{\sigma^n(x)}$.

Theorem (Sierra–Špenko, 2017)

Let A be a $\mathbb{Z}$-graded ring, and let $\{M^x \mid x \in X\}$ be a shift-invariant family of intermediate series right A-modules with automorphism $\sigma : X \to X$. Suppose there is a $\Bbbk$-linear function $\varphi : A \to \Bbbk[X]$ such that

$$m_0^x a = \varphi(a)(x) m_n^x$$

for all $n \in \mathbb{Z}$, $a \in A_n$. Then the map

$$\Phi : A \to \Bbbk[X][t^{\pm 1}; \sigma^*], \quad a \mapsto \varphi(a) t^n \text{ for } a \in A_n$$

is a graded homomorphism of algebras.

Proof.

Just check that Φ is a homomorphism! □

Corollary

Let ρ be the automorphism of $\Bbbk[\mathbb{A}^2] = \Bbbk[x, y]$ defined by $\rho(p(x, y)) = p(x + 1, y)$, and let $T = \Bbbk[x, y][t^{\pm 1}; \rho]$. There is a homomorphism $\Phi : U(W) \to T$ defined by $\Phi(e_n) = -(x + yn)t^n$.

Proof.

Apply the previous theorem with $X = \mathbb{A}^2$ and $M^{(\alpha,\beta)} = V_{(\alpha,\beta)}$. □

Theorem

$U(W)$ is not left or right noetherian.

Proof.

We prove this by showing that $B = \Phi(U(W))$ is not left or right noetherian. For $n \neq 0$, let $p_n = e_n e_{3n} - e_{2n}^2 - n e_{4n}$. By a straightforward computation, we get

$$\Phi(p_n) = n^2 y(1-y)t^{4n}.$$

Fix $m \in \mathbb{Z} \setminus \{0\}$ and let $I = B\Phi(p_m)B$. Another straightforward computation gives

$$\Phi(e_{n-4m})y(1-y)t^{4m} - y(1-y)t^{4m}\Phi(e_{n-4m}) = 4my(1-y)t^n$$

and therefore $y(1-y)t^n \in I$ for all $n \in \mathbb{Z}$.

Theorem

$U(W)$ is not left or right noetherian.

Proof (cont.)

An easy induction proof shows that $I = y(1-y)T$. Assume for a contradiction that I is finitely generated as a left ideal of B. The rest of the proof goes as follows:

- Letting $I_n = y(1-y)T_n$, we see that $I = \bigoplus_{n\in\mathbb{Z}} I_n$ is a graded ideal of B.
- There exist $m_1, \ldots, m_k \in \mathbb{Z}$ such that $I = B(I_{m_1} + \ldots + I_{m_k})$.
- Take $m \neq m_i$, $1 \leq i \leq k$. Then I_m is contained in $(x, y)y(1-y)t^m$.
- But $y(1-y)t^m \notin (x, y)y(1-y)t^m$, a contradiction.

We conclude that B is not left noetherian. $\square$

Conjecture (Dixmier? Sierra–Walton, 2013)

Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is left and right noetherian if and only if $\mathfrak{g}$ is finite-dimensional.

Thank you for listening!