The Witt algebra, Lie algebras and enveloping algebras

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Outline

The Witt algebra

Definition

Let $\partial = \frac{d}{dx}$. The *Witt algebra* is $W = \mathbb{C}[x, x^{-1}]\partial$.

Here $\mathbb{C}[x, x^{-1}]$ is the ring of *Laurent polynomials* in x with complex coefficients (e.g. $x^2 + 1$ or $x + 3 + x^{-1}$).

W is the *algebra of derivations* of $\mathbb{C}[x, x^{-1}]$, where a *derivation* is a linear map $d : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$ that obeys the Leibniz rule:

$$d(fg) = d(f)g + fd(g).$$

In what sense is W an algebra? Usually: "algebra" means "associative algebra", i.e. a vector space which is also a ring.

How can we combine two elements of W to get a third?

The product rule means that if $f, g \in \mathbb{C}[x, x^{-1}]$, then

$$\partial(gf) = gf' + g'f = (g\partial + g')(f),$$

and so we write

$$\partial g = g\partial + g'.$$

If $\partial g = g\partial + g'$ then

$$f\partial g\partial = fg\partial^2 + fg'\partial \notin W.$$

On the other hand,

$$f\partial g\partial - g\partial f\partial = (fg\partial^2 + fg'\partial) - (gf\partial^2 + gf'\partial) = (fg' - gf')\partial \in W.$$

Define the *bracket*

$$[f\partial, g\partial] = (fg' - gf')\partial \quad \text{on } W.$$

Properties of $[-,-]$:

- Not associative, not commutative, no identity element!
- ⋆ $\mathbb{C}$-linear in both factors.
- ⋆ Alternating: $[f\partial, f\partial] = 0$.
- ⋆ Jacobi identity:

$$[f\partial, [g\partial, h\partial]] + [g\partial, [h\partial, f\partial]] + [h\partial, [f\partial, g\partial]] = 0.$$

Lie algebras

A *Lie algebra* over a field $\mathbb{k}$ is a $\mathbb{k}$-vector space $\mathfrak{g}$ together with a *Lie bracket* $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

- $\mathbb{k}$-linearity in both factors.
- Alternativity: $[x, x] = 0$.
- Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

So the Witt algebra with the bracket we defined is a Lie algebra.

Note that by expanding $[x + y, x + y] = 0$ we automatically get $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

Basic examples

Lie algebras are ubiquitous in maths (and physics). Examples:

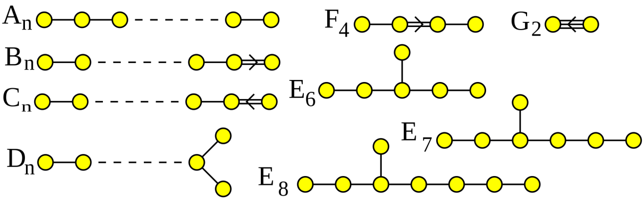
- Any associative algebra A with Lie bracket $[a, b] = ab - ba$ (*commutator bracket*) is a Lie algebra.
- $\mathfrak{gl}_n(\mathbb{C}) = \{n \times n \text{ matrices}\}$ with commutator bracket.
- Slightly more abstractly, if V is a vector space, then $\mathfrak{gl}(V) = \mathrm{End}(V)$ with commutator bracket is a Lie algebra.
- Any vector space V, with $[-, -] = 0$ (*abelian* Lie algebra).
- $\mathbb{R}^3$ with $[x, y] = x \times y$, the cross product of x and y.

Examples – "In the wild"

- If A is an associative algebra, then the vector space of derivations of A with the commutator bracket, denoted $\mathrm{Der}(A)$, is a Lie algebra.
- If X is a smooth manifold, then the space of vector fields Θ_X on X with the *commutator of vector fields* is a Lie algebra.
- If G is a *Lie group* (manifold which is also a group, like S^1), then T_eG is automatically a Lie algebra, possibly over $\mathbb{R}$. The Lie bracket echoes the structure of the group, and the group (near e) can be reconstructed from the Lie algebra! This allows us to study the structure and classification of Lie groups in terms of Lie algebras.

Examples – Classical Lie algebras

- $\mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}$.
- $\mathfrak{so}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^T = 0\}$.
- $\mathfrak{sp}_{2n}(\mathbb{C}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid J_nX + X^TJ_n = 0, J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$.



Picture due to Tom Ruen - `https://commons.wikimedia.org/w/index.php?curid=12354231`

We can equivalently define the Witt algebra more abstractly as an infinite dimensional complex Lie algebra with basis $\{e_n \mid n \in \mathbb{Z}\}$ and Lie bracket

$$[e_n, e_m] = (n - m)e_{n+m}.$$

We can see this by setting $e_n = -x^{n+1}\partial \in W$.

The Witt algebra is an interesting object to study for several reasons:

- Its construction is natural.
- The *Virasoro algebra* $Vir = W \oplus \mathbb{C}z$ is the unique nontrivial central extension of W. It is an important object in modern physics, playing a major role in 2-dimensional conformal field theory.
- Provides a good candidate to test conjectures about infinite-dimensional Lie algebras.

Enveloping algebras

We can turn a Lie algebra into an associative ring.

Definition

Let $\mathfrak{g}$ be a Lie algebra. The *universal enveloping algebra* of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/(xy - yx = [x, y] \mid x, y \in \mathfrak{g}).$$

Here, $T(\mathfrak{g})$ is the *tensor algebra* of $\mathfrak{g}$. It is defined as:

$$T(\mathfrak{g}) = \Bbbk \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$$

with multiplication given by tensoring two elements.

For a finite-dimensional Lie algebra $\mathfrak{g}$ with basis $\{x_1, \dots, x_n\}$, we can equivalently view $T(\mathfrak{g})$ as the algebra of polynomials in n non-commuting variables $\Bbbk\langle X_1, \dots, X_n\rangle$.

Definition

A *Lie algebra homomorphism* is a linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

for all $x, y \in \mathfrak{g}_1$.

Proposition (Universal property of the enveloping algebra)

Let A be an associative algebra, viewed as a Lie algebra with the commutator bracket. Any Lie algebra homomorphism $\tau : \mathfrak{g} \to A$ can be uniquely extended to an associative algebra homomorphism $\hat{\tau} : U(\mathfrak{g}) \to A$.

Example

The *Heisenberg algebra* $\mathfrak{h}$ is a 3-dimensional Lie algebra with basis $\{x, y, z\}$ and Lie bracket

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0.$$

These relations are motivated by the canonical commutation relations in quantum mechanics:

$$[\hat{x}, \hat{p}] = i\hbar I, \quad [\hat{x}, i\hbar I] = 0, \quad [\hat{p}, i\hbar I] = 0,$$

where $\hat{x}$ is the position operator, $\hat{p}$ is the momentum operator, and $\hbar$ is Planck's constant.

Example (continued)

The *first Weyl algebra* $A_1(\mathbb{C})$ is the ring of differential operators with polynomial coefficients:

$$A_1(\mathbb{C}) = \mathbb{C}[x, \partial] \cong \mathbb{C}\langle X, Y\rangle/(YX - XY - 1).$$

It is a quotient of the enveloping algebra $U(\mathfrak{h})$:

$$A_1(\mathbb{C}) \cong U(\mathfrak{h})/(z - 1).$$

There is also a homomorphism $\Psi : U(W) \to \mathbb{C}[x, x^{-1}, \partial]$ from $U(W)$ to the *localised Weyl algebra* induced by the inclusion $W \subset \mathbb{C}[x, x^{-1}, \partial]$.

Example

A basis for $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $U(\mathfrak{sl}_2)$ consists of noncommutative polynomials in e, f, h, subject to the rules:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f.$$

Theorem (Poincaré–Birkhoff–Witt)

Let $\mathcal{B}$ be a totally ordered basis of a Lie algebra $\mathfrak{g}$. Then

$$\{x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \mid x_k \in \mathcal{B}, x_1 < x_2 < \dots < x_n\}$$

is a basis for $U(\mathfrak{g})$.

Example

Monomials $e^i f^j h^k$ give a basis for $U(\mathfrak{sl}_2)$.

Likewise, in $U(W)$, the monomials $e_{n_1}^{i_1} e_{n_2}^{i_2} \dots e_{n_k}^{i_k}$ with $n_1 < n_2 < \dots < n_k$ form a basis for $U(W)$.

Definition

A *Lie algebra representation* is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space.

Example

The representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}(x)(y) = [x, y]$ is called the *adjoint representation* of $\mathfrak{g}$.

Note that representations of $\mathfrak{g}$ correspond precisely to $U(\mathfrak{g})$-modules:

If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then V is a left $U(\mathfrak{g})$-module with action $x \cdot v = \varphi(x)(v)$ for all $x \in \mathfrak{g}, v \in V$.

Conversely, if V is a $U(\mathfrak{g})$-module, then there is an algebra homomorphism $\psi : U(\mathfrak{g}) \to \mathrm{End}(V)$. The restriction $\varphi = \psi\big|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$.

- Enveloping algebras are used to study the representation theory of Lie algebras and Lie groups.
- Enveloping algebras are a good source of interesting noncommutative rings.
- Many questions about enveloping algebras (particularly those of infinite-dimensional Lie algebras) are long-standing.

Enveloping algebras of finite-dimensional Lie algebras:

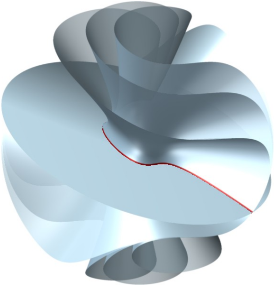
- ⋆ Very well understood.
- ⋆ Deep links with geometry.
- ⋆ Fundamental examples of well-behaved noncommutative rings.

Fact: If $\dim \mathfrak{g} = d < \infty$, then $U(\mathfrak{g})$ has all the nice properties of the polynomial ring $\Bbbk[x_1, \ldots, x_d]$, but is more interesting:

- $U(\mathfrak{g})$ is (left and right) *noetherian*: left and right ideals are finitely generated.
- $U(\mathfrak{g})$ has *polynomial growth*: if $V \subset U(\mathfrak{g})$ is finite dimensional with $1 \in V$, then $\dim V^n \sim n^d$.
- ⋆ 2-sided ideals of $U(\mathfrak{g})$ are much harder to understand than those of $\Bbbk[x_1, \ldots, x_d]$.

Prime ideals of $\mathbb{C}[x_1, x_2, x_3]$ correspond to *subvarieties* of $\mathbb{C}^3$:

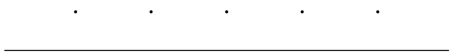
- $\mathbb{C}^3$
- surfaces in $\mathbb{C}^3$
- curves
- points



Picture due to Brent Pym

Prime ideals of $U(\mathfrak{sl}_2)$ are:

- ⋆ (0)
- ⋆ I_λ for $\lambda \in \mathbb{C}$
- ⋆ J_n for $n \in \mathbb{N}$



J_n

I_λ

(0)

Big enveloping algebras

Definition

A *big enveloping algebra* is the enveloping algebra of an infinite-dimensional Lie algebra.

Fundamental question: Are big enveloping algebras ever nice?

Theorem (M. Smith, 1976)

$\dim \mathfrak{g} < \infty \iff U(\mathfrak{g})$ *has polynomial growth.*

Example

If $1 \in V \subset U(W)$, then $\dim V^n \sim e^{\sqrt{n}}$ ("subexponential growth").

Question (Amayo–Stewart, 1974)

Can big enveloping algebras ever be noetherian?

Note: if $\mathfrak{g}$ is abelian then $U(\mathfrak{g}) = S(\mathfrak{g})$, which is noetherian if and only if $\dim \mathfrak{g} < \infty$.

Question (Dean–Small, 1990)

Is $U(W)$ noetherian?

Theorem (Sierra–Walton, 2013)

No. $U(W)$ is neither left nor right noetherian.

Proof outline.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}[a,b])$ such that $\sigma(f(a,b)) = f(a+1,b)$. Define $T = \mathbb{C}[a,b][t^{\pm 1};\sigma]$, i.e. as a vector space

$$T = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}[a,b]t^n,$$

with $t \cdot f(a,b) = \sigma(f(a,b))t = f(a+1,b)t$.
Define a homomorphism $\Phi : U(W) \to T$ as follows:

$$\Phi(e_n) = (a+bn)t^n.$$

Let $B = \Phi(U(W))$.

Proof outline (continued).

Define
$$p_n = e_n e_{3n} - e_{2n}^2 - n e_{4n} \in U(W).$$
Fix $n \in \mathbb{Z} \setminus \{0\}$ and let I be the two-sided ideal of B generated by $\Phi(p_n)$. We can show that I is independent of the choice of n, and is not finitely generated as a left or right ideal of B.

Hence, $B = \Phi(U(W))$ is not left or right noetherian and therefore $U(W)$ is not left or right noetherian. $\square$

Remark

Recall the homomorphism $\Psi : U(W) \to \mathbb{C}[x, x^{-1}, \partial]$. For some n, the element p_n generates $\ker(\Psi)$ as a two-sided ideal of $U(W)$, so I is actually the image of $\ker(\Psi)$ under the homomorphism Φ.

Corollary (Sierra–Walton, 2013)

- *$U(Vir)$ is not left or right noetherian.*
- *If $\mathfrak{g}$ is a simple, $\mathbb{Z}$-graded Lie algebra with polynomial growth then $U(\mathfrak{g})$ is not left or right noetherian.*

In fact there is no known example of an infinite-dimensional Lie algebra with a left or right noetherian enveloping algebra.

Conjectures

Conjecture (Dixmier? Sierra–Walton, 2013)

Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is left and right noetherian if and only if $\mathfrak{g}$ is finite-dimensional.

This is hard! Let's look at W and make some conjectures about $U(W)$.

Conjecture (Growth conjecture, Petukhov–Sierra, 2017)

Any proper factor of $U(W)$ has polynomial growth ("ideals are big").

Suggested by computer experiments of I. Stanciu in 2016-17.

If ideals are big, then there probably aren't very many of them.

Conjecture (Noetherianity conjecture, Petukhov–Sierra, 2017)

Two-sided ideals of $U(W)$ are finitely-generated ("ideals are sparse").

Theorem (Growth ~~conjecture~~ theorem, Iyudu–Sierra, 2018)

The growth conjecture holds: any proper factor of $U(W)$ has polynomial growth.

Suggested by computer experiments of I. Stanciu in 2016-17.

If ideals are big, then there probably aren't very many of them.

Conjecture (Noetherianity conjecture, Petukhov–Sierra, 2017)

Two-sided ideals of $U(W)$ are finitely-generated ("ideals are sparse").

Thank you for listening!