A short proof of non-noetherianity of the universal enveloping algebra of the Witt algebra

Lucas Buzaglo

1 Notation

Let $\Bbbk$ be a field of characteristic 0.

Definition 1.1. The *Witt algebra* over $\Bbbk$ is a Lie algebra W with basis $\{e_n \mid n \in \mathbb{Z}\}$ and Lie bracket

$$[e_n, e_m] = (m - n)e_{n+m}.$$

Notation 1.2. Let $\sigma \in \mathrm{Aut}(\Bbbk[a, b])$ such that $\sigma(f(a, b)) = f(a + 1, b)$. Define $T = \Bbbk[a, b][t^{\pm 1}; \sigma]$, i.e. as a vector space

$$T = \bigoplus_{n \in \mathbb{Z}} \Bbbk[a, b]t^n,$$

with $t^n f(a, b) = \sigma^n(f(a, b))t^n = f(a + n, b)t^n$.

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All proofs are adapted from [SŠ17].

Proposition 2.1. *The linear map $\varphi : W \to T$, $\varphi(e_n) = -(a + bn)t^n$, defines a homomorphism $\Phi : U(W) \to T$.*

Proof. It suffices to check that $\varphi([e_n, e_m]) = \varphi(e_n)\varphi(e_m) - \varphi(e_m)\varphi(e_n)$. Indeed,

$$\begin{aligned}
\varphi(e_n)\varphi(e_m) - \varphi(e_m)\varphi(e_n) &= (a + bn)t^n(a + bm)t^m - (a + bm)t^m(a + bn)t^n \\
&= (a + bn)(a + bm + n)t^{n+m} - (a + bm)(a + bn + m)t^{n+m} \\
&= (a(n - m) + b(n^2 - m^2))t^{n+m} \\
&= (m - n)(-a - b(n + m))t^{n+m} \\
&= \varphi((m - n)e_{n+m}) \\
&= \varphi([e_n, e_m])
\end{aligned}$$

□

We will now show that $B = \Phi(U(W))$ is not left or right noetherian. In particular, this implies that $U(W)$ is not left or right noetherian, a fact originally proved in [SW14].

For $n \in \mathbb{Z} \setminus \{0\}$, let $p_n = e_n e_{3n} - e_{2n}^2 - n e_{4n}$.

Lemma 2.2. *We have $\Phi(p_n) = n^2 b(1-b)t^{4n}$.*

Proof. We compute

$$\begin{aligned}\Phi(p_n) &= \Phi(e_n e_{3n} - e_{2n}^2 - n e_{4n}) \\ &= (a+bn)t^n(a+3bn)t^{3n} - (a+2bn)t^{2n}(a+2bn)t^{2n} + n(a+4bn)t^{4n} \\ &= ((a+bn)(a+3bn+n) - (a+2bn)(a+2bn+2n) + n(a+4bn))t^{4n} \\ &= n^2 b(1-b)t^{4n}\end{aligned}$$

□

Fix $m \in \mathbb{Z} \setminus \{0\}$ and let $I = B\Phi(p_m)B$.

Lemma 2.3. *For all $n \in \mathbb{Z}$ we have $b(1-b)t^n \in I$. In particular, I does not depend on the choice of m. Consequently, $I = b(1-b)T$.*

Proof. By Lemma 2.2, $b(1-b)t^{4m} \in I$. We have

$$\begin{aligned}&\Phi(e_{n-4m})b(1-b)t^{4m} - b(1-b)t^{4m}\Phi(e_{n-4m}) \\ &\qquad = -(a+b(n-4m))t^{n-4m}b(1-b)t^{4m} + b(1-b)t^{4m}(a+b(n-4m))t^{n-4m} \\ &\qquad = (-(a+b(n-4m)) + a + b(n-4m) + 4m)b(1-b)t^n \\ &\qquad = 4mb(1-b)t^n\end{aligned}$$

and therefore $b(1-b)t^n \in I$ for all $n \in \mathbb{Z}$.

Let $S = \Bbbk[a][t^{\pm 1}; \sigma\big|_{\Bbbk[a]}]$, i.e. S is the subring of T generated by a, t and t^{-1}. Note that $I \subseteq b(1-b)T$, and since $b(1-b) \in I$ and $a = \Phi(e_0) \in B$, we have $b(1-b)S \subseteq I$. Since also $(a+bn)t^n \in B$, we easily obtain by induction on k that $b(1-b)b^k S \subseteq I$ for all $k \geq 0$, and thus $b(1-b)T \subseteq I$. □

Theorem 2.4. *B is not left or right noetherian.*

Proof. We will show that I is not finitely generated as a left or right ideal of B. For $n \in \mathbb{Z}$, let $T_n = \Bbbk[a,b]t^n \subseteq T$. By Lemma 2.3, $I = \bigoplus_{n \in \mathbb{Z}} I_n$ is a graded ideal of B, where $I_n = b(1-b)T_n$.

Assume, for a contradiction, that I is finitely generated as a left ideal of B. Since I is graded, we may assume that all generators are homogeneous. Then there exist $m_1, \ldots, m_k \in \mathbb{Z}$ such that $I = B(I_{m_1} + \ldots + I_{m_k})$. Take $m \neq m_i$, $1 \leq i \leq k$. We claim that $I_m = (B(I_{m_1} + \ldots + I_{m_k})) \cap T_m$ is contained in $(a,b)b(1-b)t^m$, where (a,b) is the ideal generated by a and b in $\Bbbk[a,b]$. By the choice of m, an element of I_{m_i} needs to be multiplied by at least one element of the form $\Phi(e_n) \in B$ in order to get an element in I_m. Therefore, it suffices to show that $\Phi(e_{m-m_i})b(1-b)t^{m_i} \in (a,b)b(1-b)t^m$. Indeed,

$$(a + b(m-m_i))t^{m-m_i}b(1-b)t^{m_i} = (a + b(m-m_i))b(1-b)t^m \in (a,b)b(1-b)t^m.$$

Hence, $I_m \subseteq (a,b)b(1-b)t^m$. In particular, this implies that $b(1-b)t^m \notin I_m$, a contradiction to Lemma 2.3.

Now assume that I is finitely generated as a right ideal of B. Similarly to before, there exist $m_1, \ldots, m_k \in \mathbb{Z}$ such that $I = (I_{m_1} + \ldots + I_{m_k})B$. Take $m \neq m_i$, $1 \leq i \leq k$. We claim that $I_m = ((I_{m_1} + \ldots + I_{m_k})B) \cap T_m$ is contained in $t^m b(1-b)(a, b-1)$. As before, it suffices to show that $b(1-b)t^{m_i}\Phi(e_{m-m_i}) \in t^m b(1-b)(a, b-1)$. Indeed,

$$b(1-b)t^{m_i}(a + b(m - m_i))t^{m-m_i} = t^m b(1-b)(a + (b-1)(m - m_i)) \in t^m b(1-b)(a, b-1).$$

Hence, $I_m \subseteq t^m b(1-b)(a, b-1)$. Again, this implies that $b(1-b)t^m \notin I_m$, a contradiction to Lemma 2.3. □

As an immediate corollary we get

Corollary 2.5. *$U(W)$ is not left or right noetherian.*

References

[SŠ17] Susan Sierra and Špela Špenko. Generalised Witt algebras and idealizers. *Journal of Algebra*, 483(C):415–428, 2017.

[SW14] Susan Sierra and Chelsea Walton. The universal enveloping algebra of the Witt algebra is not noetherian. *Advances in Mathematics*, 262:239 – 260, 2014.