A categorical proof of the Carathéodory extension theorem

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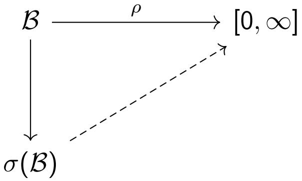
Introduction: Carathéodory extension theorem

Let $\mathcal{B}$ be a (Boolean) algebra of subsets of X and let ρ be a premeasure. We would like to extend ρ to a measure on $\sigma(\mathcal{B})$.



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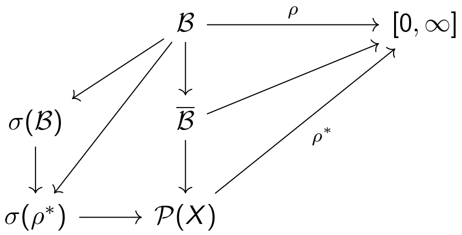

Introduction: suprema of measures

Let $(\mu_i)_{i\in I}$ be a directed collection of measures on measurable space X. Then

$$\left(\bigvee_{i\in I}\mu_i\right)(A) = \sup_{i\in I}\mu_i(A).$$

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$$\left(\bigvee_{i\in I}\mu_i\right)(A) = \sup_{i\in I}\mu_i(A).$$

For measures μ_1 and μ_2 on X,

$$(\mu_1 \vee \mu_2)(A) = \sup\left\{\sum_{n=1}^{\infty}\mu_1(A_n)\vee\mu_2(A_n) \mid \bigcup_{n=1}^{\infty}A_n = A\right\}.$$

Overview

1. Categories of lax transformations	posets of (outer) (pre)measures
2. Colimits of lax transformations	suprema of (outer) (pre)measures
3. Extensions of lax transformations	Carathéodory extension theorem

Categories of lax natural transformations

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Let $\mathcal{C}$ be a category. Let $H, G : \mathcal{C} \to \mathbf{Cat}$ be functors and let Σ be a collection of morphisms in $\mathcal{C}$.
A Σ-lax transformation $H \xrightarrow{\sigma} G$ is a lax transformation $H \xrightarrow{\sigma} G$ such that σ_f is an isormophism for every $f \in \Sigma$.

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There are inclusion functors

$$[H, G] \to \mathrm{Lax}_\Sigma[H, G] \to \mathrm{Lax}[H, G].$$

Examples of categories of lax natural transformations

1. Measures

Let $\mathbf{Set}_c$ be the category of countable sets and functions.

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$$(p_a)_{a\in A} \mapsto \left(\sum_{a\in f^{-1}(b)} p_a \right)_{b\in B} .$$

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This defines a functor $G : \mathbf{Set}_c \to \mathbf{Cat}$.

Let $i : \mathbf{Set}_c \to \mathbf{Mble}$ be the functor defined by the assignment

$$A \mapsto (A, \mathcal{P}(A)).$$

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The set (discrete category) $\mathbf{Mble}(X, i(A))$ can be identified with the set of A-indexed measurable partitions of X.

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The set (discrete category) $\mathbf{Mble}(X, i(A))$ can be identified with the set of A-indexed measurable partitions of X.
Under this identification, the function $\mathbf{Mble}(X, i(f))$ corresponds to the assignment

$$(E_a)_{a \in A} \mapsto \left(\bigcup_{a \in f^{-1}(b)} E_a \right)_{b \in B}$$

for a function $f : A \to B$ of countable sets.

Let $M(X, \mathcal{S})$ be all measures with order

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Theorem

$$[F_{\mathcal{S}}, G] \simeq M(X, \mathcal{S})$$

2. Outer measures

Let $\mathbf{Part}_c$ be the category of countable sets and partial functions and let **PartMble** be the category of measurable spaces and partial measurable maps.

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Let $\mathbf{Part}_c$ be the category of countable sets and partial functions and let **PartMble** be the category of measurable spaces and partial measurable maps. We have functors

$$G : \mathbf{Part}_c \to \mathbf{Cat} : A \mapsto [0, \infty]^A$$

and

$$\mathbf{PartMble}(X, i-) : \mathbf{Part}_c \to \mathbf{Cat}.$$

Let $M_{\text{out}}(X, \mathcal{S})$ be the poset of outer measures.

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Theorem

$[\mathbf{PartMble}(X, i-), G] \simeq M(X, \mathcal{S})$ and $\text{Lax}_\Sigma[\mathbf{PartMble}(X, i-), G] \simeq M_{\text{out}}(X, \mathcal{S})$

3. Premeasures

Let $M(X, \mathcal{B})$ be the poset of premeasures on X.

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Let $M(X, \mathcal{B})$ be the poset of premeasures on X. In a similar way as before, we can define functors $F_X, G : \mathbf{Set}_c \to \mathbf{Cat}$ such that:

Theorem

$$[F_X, G] \simeq M(X, \mathcal{B})$$

Summary

1. Measures and premeasures are natural transformations;
2. Outer measures are Σ-lax transformations.

Colimits of lax natural transformations

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Proposition

If GA has colimits of shape I for all A and Gf preserves colimits of shape I for all $f \in \Sigma$, then $\mathrm{Lax}_{\Sigma}[H, G]$ has colimits of shape I.

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In this case the colimit is calculated 'pointwise':

$$(\mathrm{colim}_i \sigma^i)_A(x) = \mathrm{colim}_i(\sigma^i_A(x)),$$

for $A \in \mathcal{C}$ and $x \in HA$.

To obtain more information about colimits in $[H, G]$, we will look for conditions such that

$$i : [H, G] \to \mathrm{Lax}_\Sigma[H, G]$$

is a reflective subcategory.

[1] Blackwell, Kelly, Power; *Two-dimensional monad theory*

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There is a natural transformations $\iota : H^\# \to H$. Let $\sigma : H \to G$ be a lax natural transformation. This corresponds to a natural transformation $\tau^\sigma : H^\# \to G$.

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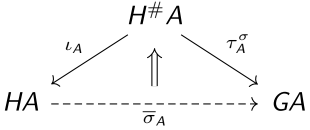
For an object A in $\mathcal{C}$, define $\overline{\sigma}_A$ as the left Kan extension of τ_A^σ along ι_A.



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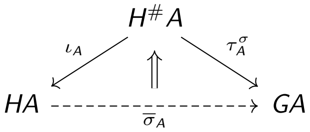

This means

$$\overline{\sigma}_A(x) = \operatorname{colim}(\iota_A \downarrow x \to H^\# A \xrightarrow{\tau^\sigma_A} GA).$$

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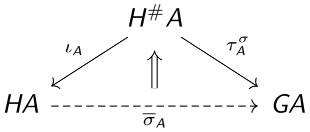

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We will now look at conditions such that the assignment

$$\sigma \mapsto \overline{\sigma}$$

defines a reflector $\mathrm{Lax}[H, G] \to [H, G]$.

Let Φ_1 be a class of categories and let

$$\Phi_2 := \{J \mid D : I \to J \text{ has a cocone for all } I \in \Phi_1\}.$$

Theorem

Let $H : \mathcal{C} \to \mathbf{Cat}$ be a functor such that:

1. HA is a discrete category for all A,
2. $\mathcal{C}$ has and H preserves limits of shape $(I^{\mathrm{op}})^+$ for $I \in \Phi_1$,
3. $\mathcal{C}$ has and H preserves pullbacks.

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Let $G : \mathcal{C} \to \mathbf{Cat}$ be a functor such that:

1. GA has colimits of shape $J \in \Phi_2$,
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Then $i : [H, G] \to \mathrm{Lax}[H, G]$ is reflective.

For some class of categories Φ.

Theorem

Let $H : \mathcal{C} \to \mathbf{Cat}$ be a functor such that:

1. ~~HA is a discrete category for all A~~,
2. $\iota_A/x \in \Phi$,
3. $\iota_A/x \to \iota_B/Hg(x)$ is final for all $g : A \to B$ in Σ.

Let $G : \mathcal{C} \to \mathbf{Cat}$ be a functor such that:

1. GA has colimits of shape $J \in \Phi$,
2. Gf preserves colimits of shape $J \in \Phi$ for all $f \in \Sigma$.

Then $i : [H, G] \to \mathrm{Lax}_\Sigma[H, G]$ is reflective.

Suppose the conditions of the theorem are satisfied.

Corollary

If GA has colimits of shape I, then $[H, G]$ has colimits of shape I and

$$\mathrm{colim}_{[H,G]}\sigma_i \simeq \overline{\mathrm{colim}_{\mathrm{Lax}[H,G]}\sigma_i}.$$

If $I \in \Phi_2$, then $\mathrm{colim}_{[H,G]}\sigma_i \simeq \mathrm{colim}_{\mathrm{Lax}[H,G]}\sigma_i$.

Suprema of (outer) (pre)measures

Let $(X, \mathcal{S})$ be a measurable space and let $F_{\mathcal{S}}$ and G as in the example of measures.

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We find that $\mathbf{Set}_c$ has and $F_{\mathcal{S}}$ preserves pullbacks and that GA is cocomplete and Gf preserves sifted colimits.

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Let $\Phi_1 := \{\{\bullet \quad \bullet\}\}$.
We find that $\mathbf{Set}_c$ has and $F_{\mathcal{S}}$ preserves pullbacks and that GA is cocomplete and Gf preserves sifted colimits.

Proposition

The poset $M(X, \mathcal{S})$ is complete and $M(X, \mathcal{S}) \to \mathrm{Lax}[F_{\mathcal{S}}, G]$ is reflective. For a directed collection $(\mu_i)_{i \in I}$ of measures

$$\left(\bigvee_{i \in I} \mu_i \right) (E) = \sup_{i \in I} \mu_i(E),$$

and for measures μ_1 and μ_2,

$$(\mu_1 \vee \mu_2)(E) = \sup \left\{ \sum_{n=1}^{\infty} \mu_1(E_n) \vee \mu_2(E_n) \mid \bigcup_{n=1}^{\infty} E_n = E \right\}.$$

Remark: We find a similar result for premeasures.

Proposition

The poset $M_{\text{out}}(X, \mathcal{S})$ is complete and suprema are computed pointwise. We also have that $M(X, \mathcal{S}) \to M_{\text{out}}(X, \mathcal{S})$ is reflective.

Summary

- Measures and premeasures are natural transformations,
- Outer measures are lax natural transformations
- The assignment $\sigma \mapsto \overline{\sigma}$ defines, under certain conditions, a reflector $\mathrm{Lax}_{\Sigma}[H, G] \to [H, G]$.
  - $M(X, \mathcal{S}) \subseteq M_{\text{out}}(X, \mathcal{S})$ is reflective,
  - $(\mu_1 \vee \mu_2)(E) = \sup \left\{ \sum_{n=1}^{\infty} \mu_1(E_n) \vee \mu_2(E_n) \mid \bigcup_{n=1}^{\infty} E_n = E \right\}$.

Extensions of lax natural transformations

Extensions of lax natural transformations

For functors $F, G, H : \mathcal{C} \to \mathbf{Cat}$ and (lax) natural transformations

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & G \\ {\scriptstyle \kappa}\big\downarrow & & \\ H & & \end{array}$$

we are looking for the right Kan extension

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & G \\ {\scriptstyle \kappa}\big\downarrow & \Downarrow & \nearrow \\ H & & \end{array} .$$

Proposition

Suppose all categories are small. Suppose GA is cocomplete for all A and Gf is cocontinuous for all $f \in \Sigma$, then

$$\mathrm{Lax}_\Sigma[H, G] \xrightarrow{-\circ\kappa} \mathrm{Lax}_\Sigma[F, G]$$

has a right adjoint.

Let $(X, \mathcal{B})$ be a premeasurable space and let $F_{\mathcal{B}}$ and $F_{\sigma(\mathcal{B})}$ as in the example of outer measures. There is a natural transformation $\kappa : F_{\mathcal{B}} \to F_{\sigma(\mathcal{B})}$.

Corollary

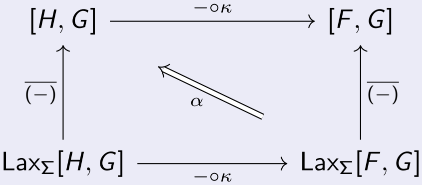
The restriction $M_{\text{out}}(X, \sigma(\mathcal{B})) \to M_{\text{out}}(X, \mathcal{B})$. has a right adjoint $(-)^*$. Furthermore,

$$\rho^* \mid_{\mathcal{B}} = \rho.$$

Let κ be a natural transformation and with conditions as before. By an adjoint lifting theorem[2]:

Proposition

There is a natural transformation



[2] Johnstone; *Adjoint lifting theorems for categories of algebras*

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$$
\begin{array}{ccc}
[H, G] & \xrightarrow{\;-\circ\kappa\;} & [F, G] \\
\uparrow{\scriptstyle \overline{(-)}} & \overset{\alpha}{\Leftarrow} & \uparrow{\scriptstyle \overline{(-)}} \\
\mathrm{Lax}_\Sigma[H, G] & \xrightarrow[\;-\circ\kappa\;]{} & \mathrm{Lax}_\Sigma[F, G]
\end{array}
$$

If α is an isomorphism, then the right adjoint $(-)^*$ can be lifted to a right adjoint for $[H, G] \xrightarrow{-\circ\kappa} [F, G]$.

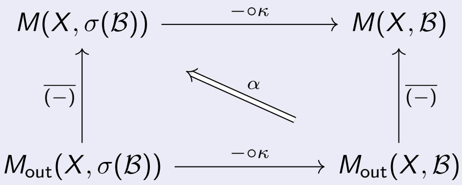
[2] Johnstone; *Adjoint lifting theorems for categories of algebras*

Corollary

The restriction $M(X, \sigma(\mathcal{B})) \to M(X, \mathcal{B})$ has a right adjoint $(-)^*$.

Proof.

We need to show that α is an isomorphism.



This requires some measure theory. □

Theorem (Carathéodory extension theorem)

Every premeasure ρ can be extended to a measure.

Summary

- Measures and premeasures are natural transformations,
- Outer measures are lax natural transformations
- The assignment $\sigma \mapsto \overline{\sigma}$ defines, under certain conditions, a reflector $\mathrm{Lax}_{\Sigma}[H, G] \to [H, G]$.
  - $M(X, \mathcal{S}) \subseteq M_{\mathrm{out}}(X)$ is reflective,
  - $(\mu_1 \vee \mu_2)(E) = \sup \left\{ \sum_{n=1}^{\infty} \mu_1(E_n) \vee \mu_2(E_n) \mid \bigcup_{n=1}^{\infty} E_n = E \right\}.$
- Under certain conditions, we can Kan extend Σ-lax transformations.
  - We can extend (outer) premeasures on $\mathcal{B}$ to (outer) measures on $\sigma(\mathcal{B})$ (Carathéodory).