Radon-Nikodym derivatives and martingales

Ruben Van Belle

21-22 December 2022, Pisa

1 Radon-Nikodym derivatives

Radon-Nikodym theorem

Theorem (Radon-Nikodym)

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$$\nu(A) = \int_A f \mathrm{d}\mu,$$

for all A in Σ.

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The map f is called **the Radon-Nikodym derivative of ν with respect to μ** and is denoted as $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

Radon-Nikodym theorem

Consider de map $L^1(X, \Sigma, \mu) \to \{\nu \mid \nu \ll \mu\}$ that sends $f \in L^1(X, \Sigma, \mu)$ to the measure defined by

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for all $A \in \Sigma$.
The Radon-Nikodym theorem says that this is a bijection.

Radon-Nikodym theorem: examples

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra.

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. An integrable $\mathcal{F}$-measurable map $X : \Omega \to \mathbb{R}$ defines a measure ν on $(\Omega, \mathcal{G})$ by:

$$\nu(A) := \int_A X \mathrm{d}\mathbb{P} \quad (= \mathbb{E}[X 1_A]),$$

for all $A \in \mathcal{G}$.

Radon-Nikodym theorem: examples

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$$\nu(A) = \int_A X \mathrm{d}\mathbb{P} = 0.$$

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Therefore, $\nu \ll \mathbb{P}\mid_{\mathcal{G}}$ and there exists $\mathbb{P}$-almost surely unique $\mathcal{G}$-measurable integrable map $f : \Omega \to \mathbb{R}$ such that

$$\int_A X \mathrm{d}\mathbb{P} = \int_A f \mathrm{d}\mathbb{P}\mid_{\mathcal{G}},$$

or

$$(\mathbb{E}[X 1_A] = \mathbb{E}[f 1_A]),$$

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for all $A \in \mathcal{G}$.The map f is called the **conditional expectation of X with respect to $\mathcal{G}$** and is denoted as $\mathbb{E}[X \mid \mathcal{G}]$.

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$$a \mapsto \begin{cases} \frac{q_a}{p_a} \text{ if } p_a \neq 0 \\ 0 \text{ otherwise.} \end{cases}$$

It can be checked that f is the Radon-Nikodym derivative of q with respect to p.

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- Define $M_n(\Omega, \mathcal{F}, \mathbb{P})$ as the set

$$\{\mu \mid \mu \leq n\mathbb{P}\},$$

  together with the *total variation metric*.
- Define $RV_n(\Omega, \mathcal{F}, \mathbb{P})$ as the set

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These are *complete* metric spaces (Riesz-Fischer).

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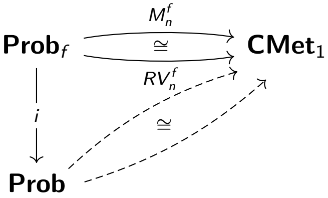
By the finite Radon-Nikodym theorem, we see that

$$\mathbf{Prob}^f \underset{RV_n^f}{\overset{M_n^f}{\rightrightarrows}} \mathbf{CMet}_1. \quad (\cong)$$

Categorically extending the finite version

It follows that also the right Kan extensions along $i : \mathbf{Prob}_f \to \mathbf{Prob}$ are isomorphic.



What do these Kan extensions look like?

Proposition

For a probability space $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathbb{P})$, we have for all $n \geq 1$ that

$$M_n(\mathbf{\Omega}) \to (\mathrm{Ran}_i M_n^f)(\mathbf{\Omega}),$$

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Proof (sketch): Let $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

$$\mathrm{Ran}_i M_n^f(\mathbf{\Omega}) \cong \int_{\mathbf{A} \in \mathbf{Prob}_f} [\mathbf{Prob}(\mathbf{\Omega}, i\mathbf{A}), M_n^f(\mathbf{A})]$$

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defined by the assignment

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This induces a morphism

$$M_n(\mathbf{\Omega}) \to \int_{\mathbf{A}} [\mathbf{Prob}(\mathbf{\Omega}, \mathbf{A}), M_n^f(\mathbf{A})] \cong (\mathrm{Ran}_i M_n^f)(\mathbf{\Omega}).$$

What do these Kan extensions look like?

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For $E \in \mathcal{F}$, consider the finite probability space

$$\mathbf{2}_E := (\{0,1\}, \mathbb{P}(E^C)\delta_0 + \mathbb{P}(E)\delta_1),$$

and note that the indicator function 1_E becomes a measure-preserving map

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It can be shown that $\mu_y \in M_n(\boldsymbol{\Omega})$. This gives a morphism $Y \to M_n(\boldsymbol{\Omega})$, making $M_n(\boldsymbol{\Omega})$ a universal wedge.

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Proposition

For a probability space $\mathbf{\Omega}$, we have for all $n \geq 1$ that

$$(\mathrm{Ran}_i RV_n^f)(\mathbf{\Omega}) \cong RV_n(\mathbf{\Omega}).$$

The proof for this results requires some measure theory.

Radon-Nikodym theorem

Combining everything gives a *bounded* Radon-Nikodym theorem, namely

$$\begin{aligned}\{\mu \mid \mu \leq n\mathbb{P}\} = M_n(\mathbf{\Omega}) &\cong \mathrm{Ran}_i M_n^f((\mathbf{\Omega}) \\ &\cong \mathrm{Ran}_i RV_n^f(\mathbf{\Omega}) \\ &\cong RV_n(\mathbf{\Omega}) = \mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}}\end{aligned}$$

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We can look at the colimit over all $n \geq 1$,

$$\begin{array}{ccccccccc} M_1\mathbf{\Omega} & \longleftarrow & M_2\mathbf{\Omega} & \longleftarrow & \dots & \longleftarrow & M_n\mathbf{\Omega} & \longleftarrow & \dots \\ \wr\Vert & & \wr\Vert & & & & \wr\Vert & & \\ RV_1\mathbf{\Omega} & \longleftarrow & RV_2\mathbf{\Omega} & \longleftarrow & \dots & \longleftarrow & RV_n\mathbf{\Omega} & \longleftarrow & \dots \end{array}$$

This gives us

$$\{\mu \mid \mu \ll \mathbb{P}\} \cong \{f : \Omega \to [0, \infty) \mid f \text{ is integrable}\} / =_{\mathbb{P}} .$$

Remark on conditional expectation

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What can we say about $M_n(g) := (\mathrm{Ran}_i M_n^f)(g)$ and $RV_n(g) := (\mathrm{Ran}_i RV_n^f)(g)$ for $g : \mathbf{\Omega}_1 \to \mathbf{\Omega}_2$?

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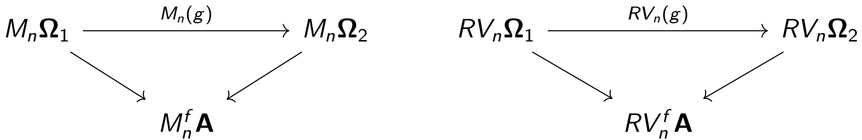
What can we say about $M_n(g) := (\mathrm{Ran}_i M_n^f)(g)$ and $RV_n(g) := (\mathrm{Ran}_i RV_n^f)(g)$ for $g : \mathbf{\Omega}_1 \to \mathbf{\Omega}_2$?
They are the unique morphisms such that



commute for morphisms $\mathbf{\Omega}_2 \to \mathbf{A}$.

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$$M_n(g)(\mu) \circ 1_E^{-1} = \mu \circ 1_{g^{-1}(E)}^{-1},$$

and

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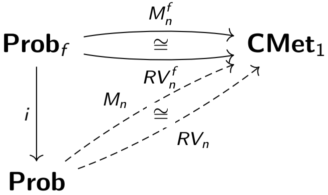
This means that

$$M_n(g)(\mu) = \mu \circ g^{-1} \quad \text{and} \quad RV_n(g)(f) = \mathbb{E}[f \mid g].$$

Summary



- <u>(Bounded) Radon-Nikodym theorem</u>:

$$M_n(\mathbf{\Omega}) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\mathbf{\Omega}) = \mathbf{Mble}(\Omega, [0, n]) / =_{\mathbb{P}} .$$

- <u>Conditional expectation</u>:

$$RV_n(g)(X) = \mathbb{E}[X \mid f].$$

2 Martingales

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A **martingale** is a collection of integrable random variables $X_i : (\Omega, \mathcal{F}_i) \to \mathbb{R}$ such that

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for all $i \leq j$.

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An L^1-*bounded martingale* $(X_n)_n$, converges $\mathbb{P}$-almost surely to a random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}$.

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Theorem

Let $p > 1$. An L^p-*bounded martingale* $(X_n)_n$ converges to a random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ in L^p and for all $n \geq 1$,

$$\mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

How does this translate categorically?

The space $\mathbf{\Omega}$ is the limit of

$$\mathbf{\Omega}_1 \xleftarrow[s_{21}]{} \mathbf{\Omega}_2 \xleftarrow[s_{32}]{} \mathbf{\Omega}_3 \longleftarrow \ldots \longleftarrow \mathbf{\Omega}_m \longleftarrow \ldots$$

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Suppose that $M_n : \mathbf{Prob} \to \mathbf{CMet}_1$ preserves this limit, then

$$\begin{aligned} RV_n(\mathbf{\Omega}) &\cong \lim_m RV_n(\mathbf{\Omega}_m) \\ &\cong \{(X_m)_m \mid RV_n(s_{m_1 m_2})(X_{m_1}) = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \mathbb{E}[X_{m_1} \mid \mathcal{F}_{n_2}] = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \text{ martingale } \} \end{aligned}$$

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It follows that for every martingale $(X_m)_m$ such that $X_m \leq n$ for all m, there exists a random variable $X : (\Omega, \mathcal{F}) \to [0, n]$ such that for all m,

$$\mathbf{E}[X \mid \mathcal{F}_m] = X_m.$$

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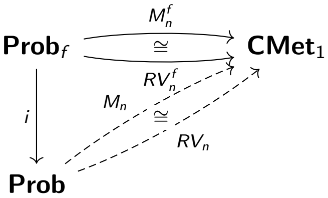

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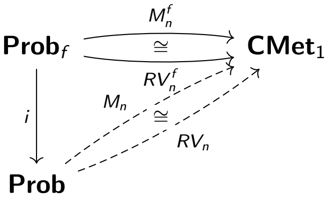

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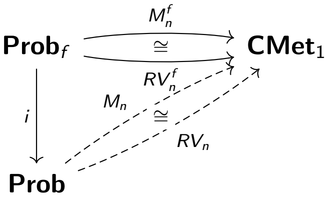

How is **Prob** enriched over $\mathbf{CMet}_1$?
Answer: $\mathbf{Prob}(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is the *completion* of

$$\{f : \mathbf{\Omega}_1 \to \mathbf{\Omega}_2 \mid \text{ measure preserving}\}$$

with the pseudometric

$$d(f_1, f_2) := \sup \left\{ \mathbb{P}_1(f_1^{-1}(A) \Delta f_2^{-1}(A)) \mid A \in \mathcal{F}_2 \right\}.$$

RV_n preserves cofiltered limits

For any finite probability space $\mathbf{A}$, we always have a map

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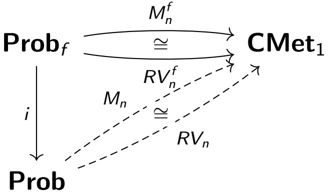
Remark: We did not use anything about RV_n^f.

Summary

Enriched version of



- (Bounded) Radon-Nikodym theorem:

$$M_n(\mathbf{\Omega}) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\mathbf{\Omega}) = \mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}} .$$

- Conditional expectation:

$$RV_n(g)(X) = \mathbb{E}[X \mid f].$$

- Martingale convergence: RV_n preserves cofilitered limits.
- Weaker Kolmogorov extension theorem : M_n preserves cofilitered limits.

What about left Kan extensions?

Let $H : \mathbf{Prob}_f \to \mathbf{CMet}_1$ be a functor. Suppose that $\mathbf{\Omega}$ is a probability space that is **not** *essentially finite*.
Then $\mathbf{Prob}(\mathbf{A}, \mathbf{\Omega}) = \emptyset$ for all finite probability spaces $\mathbf{A}$ and

$$\mathrm{Lan}_i H(\mathbf{\Omega}) = \int^{\mathbf{A}} \mathbf{Prob}(\mathbf{A}, \mathbf{\Omega}) \times H\mathbf{A} = \emptyset.$$