# Lax algebras in probability theory and topology

Ruben Van Belle

10th Workshop on Application of Categorical Probability Theory to Finance,
February 2023

# Overview

1. Lax algebras
2. Lax algebras in topology [1]
3. Lax algebras in probability theory

[1] *Monoidal topology*, D. Hofmann, G.J. Seal, W. Tholen

# 1. Lax algebras

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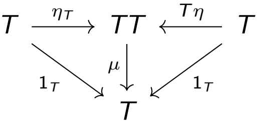


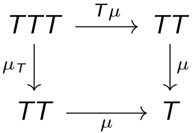

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that induces the monad $\mathbb{T}$. This is the initial such adjunction.

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2. The algebras of the free group monad are *groups*.

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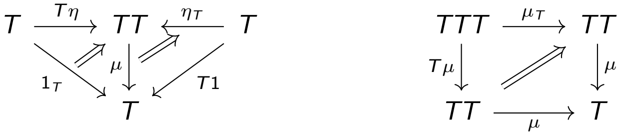


together with the appropriate coherence axioms.

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The category $\mathbf{2} - \mathbf{Rel}$ is the usual 2-category of sets and relations.

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This fits in an entire theory of extentions of monads on **Set** to lax monads on $V$-**Rel**.

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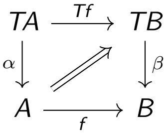

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# Lax algebras vs. (strict) algebras

For *a large class*[2] of lax extensions of monads on **Set**, the inclusion functor

$$\mathbf{Set}^{\mathbb{T}} \to V\text{-}\mathbf{Rel}^{\mathbb{T}}_{\mathrm{lax}}$$

has a left adjoint.

[2] This includes all the lax extensions from section 1 and section 2.

# 2. Topology

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# Ultrafilters

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- $\mathcal{F} \neq \emptyset$,
- If $A$, $B \in \mathcal{F}$, then there is a $C \in \mathcal{F}$ such that $C \subseteq A$ and $C \subseteq B$,
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

The collection of filters on $X$ is partially ordered by inclusion. A maximal filter on $X$ is called an *ultrafilter*.

Example: For $x \in X$, consider $\mathcal{U}_x$ as the collection of all subsets of $A$ that contain $x$. Then $\mathcal{U}_x$ is a (trivial) ultrafilter.

By Zorn's lemma (AC), non-trivial ultrafilters exist.

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By Zorn's lemma (AC), non-trivial ultrafilters exist.

## Proposition

If $\mathcal{U}$ is an ultrafilter on $X$ and $A_1 \cup \ldots \cup A_n \in \mathcal{U}$, then there is an $n_0$ such that $A_{n_0} \in \mathcal{U}$.

# Ultrafilters in topology

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Example: Let $(x_n)_n$ be a sequence in $X$. Define $\mathcal{F}$ as the filter generated by the set $(\{x_k \mid k \geq n\})_n$. Then $(x_n)_n$ converges to $x$ if and only if $\mathcal{F}$ converges to $x$.
We write $\mathcal{F} \to x$

**Proposition**

If $(X, \mathcal{T})$ is Hausdorff, then every filter converges to at most one point.

# Ultrafilters in topology

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If $X$ is compact, then every ultrafilter on $X$ converges to at least one point.

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If $X$ is compact, then every ultrafilter on $X$ converges to at least one point.

Proof: Let $\mathcal{U}$ be an ultrafilter on $X$. Suppose that $\mathcal{U}$ does not converge, then for every $x \in X$ there is an open neighbourhood $G_x$ such that $G_x \notin \mathcal{U}$. Since $X$ is compact, there is a finite collection $x_1, \ldots, x_n$ such that $X = \bigcup_{k=1}^{n} G_{x_k}$.Since $\mathcal{U}$ is an ultrafilter, there is a $k$ such that $G_{x_k} \in \mathcal{U}$. This is a contradiction.

# The ultrafilter monad

For every set $X$, let $\beta(X)$ be the set of all ultrafilters on $X$.

[3] *Equational completion, model induced triples and pro- objects*, J.F.Kennison and D. Gildenhuys

# The ultrafilter monad

For every set $X$, let $\beta(X)$ be the set of all ultrafilters on $X$.
For a map $f : X \to Y$ let $\beta f$ be the map that sends an ultrafilter $\mathcal{U}$ on $X$ to the ultrafilter generated by $f(\mathcal{U})$.

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This defines an endofunctor $\beta$ : **Set** $\to$ **Set**.

- For a set $X$, define a map $\eta_X : X \to \beta X$ by the assignment

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- Let $\mathbf{U}$ be an ultrafilter on $\beta X$. Then

$$\mu(\mathbf{U}) := \{A \mid \forall \mathcal{U} \in \mathbf{U} : A \in \mathcal{U}\}$$

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These form a monad $\beta$ on **Set**, called the **ultrafilter monad**.
This monad is the codensity monad of the inclusion $\mathbf{Set}_f \to \mathbf{Set}$ of finite sets into all sets.[3]

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# Algebras of the ultrafilter monad

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**Theorem (Manes)**

The category of algebras of the ultrafilter monad is equivalent to the category of compact Hausdorff spaces and continuous maps.

# The ultrafilter monad on **Rel**

Let $R : X \rightsquigarrow Y$ be a relation of sets.

# The ultrafilter monad on **Rel**

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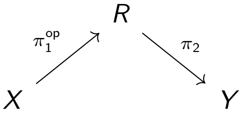

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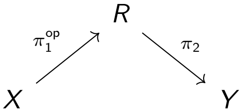


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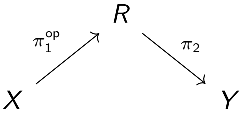


Define the relation $\beta R$ as $(\beta\pi_2) \circ (\beta\pi_1)^{\text{op}}$.
This leads to the so called *Barr extensions* of the ultrafilter monad on **Set** to the ultrafilter monad on **Rel**.

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$$\mathcal{F} R x :\Leftrightarrow \mathcal{F} \to x$$

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# Algebras of the ultrafilter monad on **Rel**

Let $X$ be a topological space. Define a relation $R : \beta X \rightsquigarrow X$ as follows:

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This forms a lax algebras over $\beta$.

**Theorem (Barr)**

The category $\mathbf{Rel}^{\beta}_{\mathrm{lax}}$ is equivalent to **Top**.

# The Čech-Stone compactification

The inclusion $\mathbf{Set}^{\beta} \to \mathbf{Rel}^{\beta}_{\mathsf{lax}}$ has a left adjoint.

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# The Čech-Stone compactification

The inclusion $\mathbf{Set}^{\beta} \to \mathbf{Rel}^{\beta}_{\mathsf{lax}}$ has a left adjoint. That means the functor

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# 3. Probability Theory

# The distribution monads

For a set $A$, we can look at the set

$$DA := \left\{ (p_a)_{a \in A} \in [0,1]^A \mid \text{finitely supported, } \sum_{a \in A} p_a = 1 \right\}.$$

# The distribution monads

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For a function $f : A \to B$, there is a function $Df : DA \to DB$ defined by the assignment:

$$(p_a)_{a\in A} \mapsto \left( \sum_{f(a)=b} p_a \right)_{b\in B}.$$

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This induces a functor $D : \mathbf{Set} \to \mathbf{Set}$.

# Distribution monads

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There is also a map $\mu_A : DDA \to DA$ that sends $(p_q)_{q \in DA}$ to

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This gives $D$ a monad structure. This monad is called the **distribution monad**.

# Distribution monads

- Using the Barr extension construction, we can lift this monad to a monad on **Rel**, which we will also denote by $D$.
- If $A$ is a metric space, topological space or measurable space, we can give $DA$ extra structure, so that $D$ becomes a monad on $\mathbf{Met}_1$,**Top** or **Mble**.

# Algebras of the distribution monads

A **convex set** is a a set $X$ together with a collection of operations $(c_\lambda : X \times X \to X)_{\lambda \in [0,1]}$ satisfying certain axioms.[4]

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## Proposition

The category $\mathbf{Set}^D$ is isomorphic to the category of convex sets and convex functions.

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**Proposition**

The category $\mathbf{Set}^D$ is isomorphic to the category of convex sets and convex functions.

Remark: For the distribution monads on $\mathbf{Met}_1$, **Top** or **Mble**, it is *not* enough to have that the structure maps $(c_\lambda)_\lambda$ are in the respective category, because we also need continuity/measurability in the $\lambda$-variable.

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A convex set has the cancellation property if and only if it can be embedded in a vector space.

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## Theorem (Stone)

A convex set has the cancellation property if and only if it can be embedded in a vector space.

## Proposition

The algebras of the distribution monads on $\mathbf{Met}_1$ and **Top** are convex sets that *have* the cancellation property.

# Lax algebras of the distribution monads

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**Proposition**

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**Proposition**

The category $\mathbf{Rel}_{\mathrm{lax}}^{D}$ is isomorphic to the category of partial convex sets and partial convex functions.

Question: Does the functor $\mathbf{Set}^{D} \to \mathbf{Rel}_{\mathrm{lax}}^{D}$ have a left adjoint?

# Probability monads

Let $X$ be a measurable space. Define $GX$ as the set

$$\{\text{probability measures on } X\}$$

together with the $\sigma$-algebra generated by the evaluation maps

$$\mathrm{ev}_A : GX \to [0,1] : \mathbb{P} \mapsto \mathbb{P}(A).$$

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This leads to a functor $G : \mathbf{Mble} \to \mathbf{Mble}$.

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[5] *A categorical approach to probability theory*, M. Giry

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For $\mathbf{P} \in GGX$, define a probability measure $\mu_X \mathbf{P}$ on $X$ by

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This defines a map $\mu_X : GGX \to GX$, giving $G$ a monad structure. This monad is called the **Giry monad**.[5]

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# Probability monads

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For the *Kantorovich monad*, *bounded Lipschitz monad* on *complete* metric spaces and the *Radon monad*, the category of algebras is equivalent to the category of algebras of the corresponding distribution monad.

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For the *Kantorovich monad*, *bounded Lipschitz monad* on *complete* metric spaces and the *Radon monad*, the category of algebras is equivalent to the category of algebras of the corresponding distribution monad.

Proof(sketch):

$$\begin{array}{ccc} DX & \xrightarrow{\alpha} & X \\ \cap\!\mid & \nearrow_{\tilde{\alpha}} & \\ GX & & \end{array}$$

Every uniformly continuous function on a dense subset can be uniquely extended to the whole space.

# Algebras of probability monads

**Corollary**

If $X$ is an algebra for the distribution monad on $\mathbf{Met}_1$, then $\overline{X}$ is an algebra for the corresponding probability monad on $\mathbf{Met}_1$.

# Algebras of probability monads

## Corollary

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We would like to know when $X$ itself is an algebra.

## Theorem

Let $X$ be an algebra for the distribution monad on $\mathbf{Met}_1$. Suppose that

$$\forall x \in X \forall y \in \overline{X} \forall \lambda \in [0,1] : \lambda x + (1-\lambda) y \in \overline{X} \setminus X \Rightarrow \lambda = 0,$$

then $X$ is an algebra for the corresponding probability monad.

# Questions

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2. Can we construct lax extensions of monads on **Mble**, **Top**, $\mathbf{Met}_1, \ldots$?
3. What would the lax algebras of these extended probability monads look like?
4. Is there an adjunction between algebras and lax algebras?

# Questions

1. What do the algebras of Giry monad look like? (These might not even satisfy the cancellation property.)
2. Can we construct lax extensions of monads on **Mble**, **Top**, $\mathbf{Met}_1, \ldots$?
3. What would the lax algebras of these extended probability monads look like?
4. Is there an adjunction between algebras and lax algebras? (Note that $\mathbb{R}$ is not an algebra for the Giry monad)

# Related work

1. **Partial evaluations**
2. **Martingales in algebras**

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