Structure Detection for Quadratic Programming

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Outline

1. Structure exploiting optimization
2. Benders decomposition in SCIP
3. Structure detection
4. Examples
5. Outlook

Structure-exploiting optimization

- Dantzig-Wolfe decomposition
- (Generalized) Benders decomposition
- Lagrangean decomposition
- (Multi-block) ADMM

Automatic decomposition software

- **GCG** - Dantzig-Wolfe & Benders decomposition,
- **DIP** (formerly **DECOMP**) - Dantzig-Wolfe decomposition
- **COLUNA** - Dantzig-Wolfe decomposition
- **SAS/OR** - Dantzig-Wolfe decomposition
- **CPLEX** - Benders decomposition

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SCIP: includes generalized Benders decomposition framework

Quadratic programming

$$\begin{aligned} \min \quad & \tfrac{1}{2}x^T Q x + c^T x \quad (Q \succeq 0) \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

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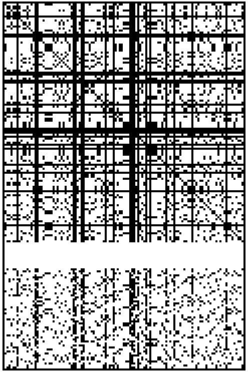
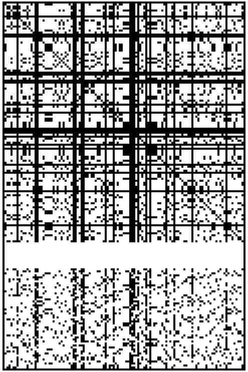


QP instances



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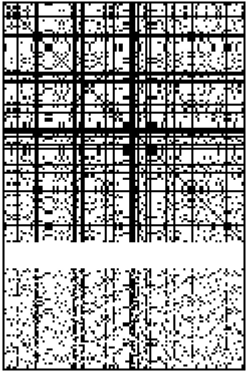
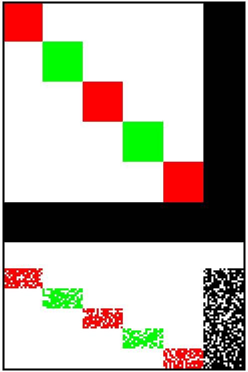
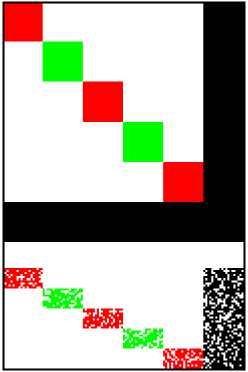




Notation



- x, y_i: linking and subproblem variables
- Q_i: Submatrix of Hessian corresponding to variables y_i
- R_i: Submatrix of Hessian connecting subproblem i with linking vars x
- A_i: Submatrix of constraints corresponding to variables y_i
- B_i: Submatrix of constraints connecting subproblem i with linking vars x

Benders decomposition

Master problem

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}x^T Q_0 x + c_0^T x + \sum \varphi_i \\
\text{s.t.} \quad & A_0 x \leq b_0 \\
& \varphi_i \geq \gamma_{\omega_i}^T (b_i - B_i x) && \forall \omega_i \in \mathcal{O} \\
& 0 \geq \gamma_{\omega_i}^T (b_i - B_i x) && \forall \omega_i \in \mathcal{F} \\
& x \in \mathbb{R}^{n_0} \\
& \varphi_i \geq 0
\end{aligned}
$$

Benders decomposition

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Subproblems

$$\begin{aligned} z_i(\hat{x}) = \min \quad & \tfrac{1}{2}y_i^T Q_i y_i + \tilde{x}^T R_i y_i + c_i^T y_i \\ s.t. \quad & A y_i \ge b_i - B_i \tilde{x} \\ & \tilde{x} = \hat{x} \\ & y_i \in \mathbb{R}^{n_i} \\ & \tilde{x} \in \mathbb{R}^{n_0} \end{aligned}$$

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& \tilde{x} = \hat{x} \\
& y_i \in \mathbb{R}^{n_i} \\
& \tilde{x} \in \mathbb{R}^{n_0}
\end{aligned}
$$

How do we find that structure?

MIP-based structure detection

$$x_{ib} = \begin{cases} 1 & \text{if var } i \text{ is in block } b \\ 0 & \text{otherwise} \end{cases}$$

$$z_b = \text{number of vars in block } b$$

MIP-based structure detection

$$\min_{x,z} \quad \sum_{b=0}^{B} z_b^2 + 2z_0(n - z_0)$$

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& \sum_{b=0}^{B} x_{ib} = 1 & \forall i \\
& & \forall c \in M : \forall i, j \in NZ(c), i \neq j \\
& x_{ia} + x_{jb} \leq 1 & \forall a, b \in \{1, \cdots, B\}, a \neq b
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& x_{ia} + x_{jb} \leq 1 & \forall a, b \in \{1, \cdots, B\}, a \neq b \\
& x_{ib} \in \{0, 1\} & \forall i, \forall b \\
& z_b \in \mathbb{Z} & \forall b
\end{aligned}
$$

MIP-based structure detection

- Related: Martin 1999 (PhD dissertation)
- "optimal" decomposition
- Not practical
- Useful as benchmark (MIP-score)

Hypergraph-based detection

- Used in linear decomposition software (GCG, DIP)
- Mature graph partitioning software available (hmetis, gpmetis)

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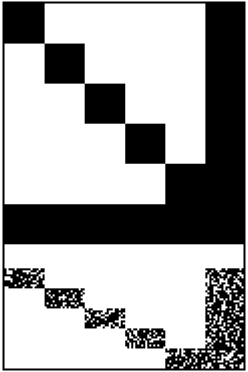
Various possibilities:

- Column-based hypergraph
- Row-based hypergraph
- **Row-column-based graph**

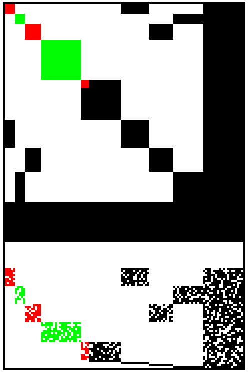
Row-Column-based Graph

- Vertex for every row and column, + dummy vertices
- Edge for every nonzero in constraint/Hessian matrix
- Use gpmetis for partitioning
- "repair" partitioning result to get valid decomposition
- Use MIP-score to compare decompositions

Example: column-graph and row-graph

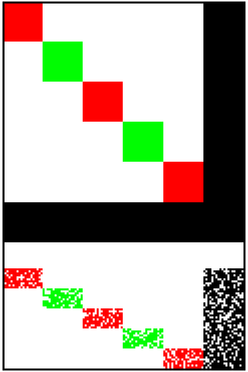


CG: score 14400

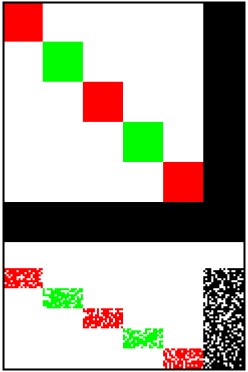


RG: score 13092

Example: MIP and row-column-graph

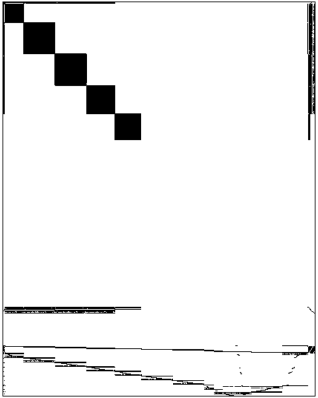


MIP: score 6000



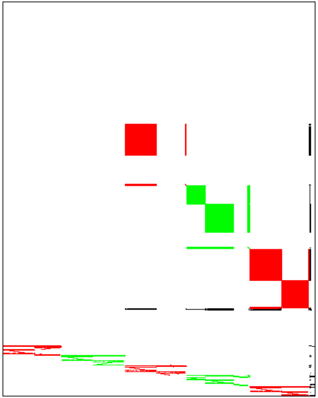
RCG: score 6000

Example: QPLIB 8906

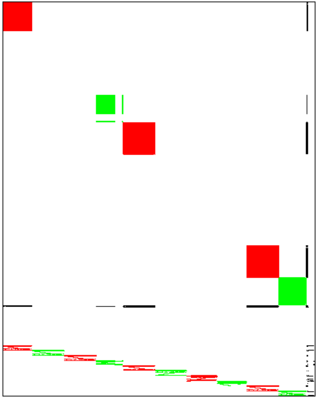


nblocks	2	3	4	5	6	7	8	9	10	11	12	13	14	15
score	8.9	6.0	6.3	**4.1**	4.7	5.5	5.4	4.2	**2.7**	3.7	6.4	7.8	9.2	10.1

Example: QPLIB 8906



5 blocks: score 4,082,907



10 blocks: score 2,668,247

Outlook

- Full numerical study (QPLIB & Maros-Meszaros testset)
- Make available through GCG
- Generic convex programming?