Internal skein algebras and modules

Patrick Kinnear

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Abstract

Talk 6 at the Matemale spring school on Witten's finiteness conjecture for skein modules. Themed around understanding the paper [GJS19], the week was divided into talks as in [Det+].

Contents

1 Overview and motivation

The main theorem of the paper is about skein modules, but it uses a slight variation on these called **internal skein modules**. These are a slight enhancement of skein modules, to be objects internal to some category (it will be a free cocompletion of, in this case, the ribbon category $\mathcal{A} = \mathcal{U}_q(\mathrm{SL}_2) - mod^{f.d.}$). Their description can be a little delicate and involves some category theory, but they can be understood fairly topologically. But why are these enhancements needed at all?

We saw in Talk 5 [Rom] that skein categories and relative skein module functors form a TFT, so they have good gluing properties for bordisms. On the other hand, ordinary skein modules do not have these properties in general: for instance consider the schematic in Fig. 1 which suggests a skein in M need not decompose neatly into skeins for the bordisms.

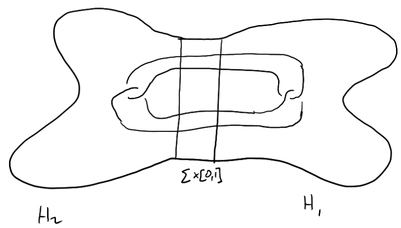


Figure 1: A skein in $H_1 \cup_\Sigma H_2$ might intersect the surface in a nontrivial way.

The internal skein modules we define will turn out to have good gluing properties (Talk 10 [Haïa]). Then, to prove our main theorem, we will take a Heegaard splitting of a closed 3-fold $M = H_1 \cup_\Sigma H_2$,and write

$$\mathrm{Sk}(M) = \mathrm{Sk}^{\mathrm{int}}(H_1) \otimes_{\mathrm{SkAlg}^{\mathrm{int}}(\Sigma)} \mathrm{Sk}^{\mathrm{int}}(H_2). \tag{1}$$

Then we will use an analytical theorem to show that the right-hand-side above is finite-dimensional.

Remark 1. In fact, the gluing property does hold for ordinary skein modules when we have a Heegaard splitting: this is because the gluing property works for the TFT, and skein modules can be obtained from the TFT by taking invariants, and it can be shown that handlebody modules are cyclic hence generated by invariants. However, general gluing does not hold for non-internal skein modules.

For finite-dimensionality, we use the following analytical theorem, which will be described in Talk 12 [Kor].

Theorem 2 ([GJS19, Thm. 3.6])**.** *Let X be a smooth Poisson scheme, and $L_1, L_2 \subset X$ be smooth Lagrangian subschemes. Then suppose that we are in the situation of the diagram:*

$$\begin{array}{ccccc} M_2 & \curvearrowleft & A & \curvearrowright & M_1 \\ \wr & & \wr & & \wr \\ DQ & & DQ & & DQ \\ \wr & & \wr & & \wr \\ L_2 & \subset & X & \supset & L_1 \end{array}$$

where there are some further conditions on A, M_1, M_2 above. Where $\hbar$ is the deformation parameter, then under these further conditions,

$$(M_2 \otimes_A M_1)[\hbar^{-1}]$$

is finite-dimensional over $\mathbb{C}((\bar{h}))$.

It is well-known (and will be reviewed in Talk 8 [Gra]) that $\mathrm{Sk}(\Sigma)$ is a deformation-quantization of $X(\Sigma)$, the character variety of Σ, which has a Poisson structure. But unfortunately the character variety of a handlebody does **not** in general define a smooth Lagrangian. On the other hand, where H is a genus g handlebody, we will see that $\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)$ is a DQ of G^{2g}, and $\mathrm{Sk}^{\mathrm{int}}(H)$ is a DQ of G^g (Talk 9 [Kar]).

Then, in view of the finiteness proof, the two main advantages of internal skein modules will be that they have good gluing properties for bordisms, and quantize smooth geometric objects. Moreover, we will be able to pass from the internal picture back to ordinary skein modules, allowing us to make our finiteness statements in the context of th eoriginal conjecture.

2 Defining internal skein algebras

2.1 The defining property

To have functoriality for bordisms, we would like to be able to deal with skeins that begin and end on the boundary of the surface. We will always be able to collect skeins that meet the surface in several places into those meeting it in a single strand, so we restrict to this case. To have a well-defined action, we will need to puncture our surface and use disk-insertion.

Let $\mathcal{A}$ be a k-linear ribbon category, assume for simplicity it is semisimple and has a fiber functor to Vect. Let Σ a surface, and $\Sigma^* = \Sigma - \bar{\mathbb{D}}$. Then there is an embedding $\mathbb{A}\mathrm{nn} \hookrightarrow \Sigma^*$ around the puncture. Choose an embedding $i : \mathbb{D} \hookrightarrow \mathbb{A}\mathrm{nn}$.

Definition 3. (Notation due to Jan Pulmann). Let $V \in \mathcal{A}$. The space $\mathrm{Sk}\Sigma_V^{\varnothing}$ is the k-module spanned by isotopy classes of $\mathcal{A}$-coloured ribbon graphs $\Gamma \subseteq \Sigma^* \times [0,1]$, such that $\partial\Gamma \subseteq \mathrm{im}(i) \times \{0\}$ and $\partial\Gamma$ is coloured by V, modulo the $\mathcal{A}$-skein relations.

See Figure 2a.

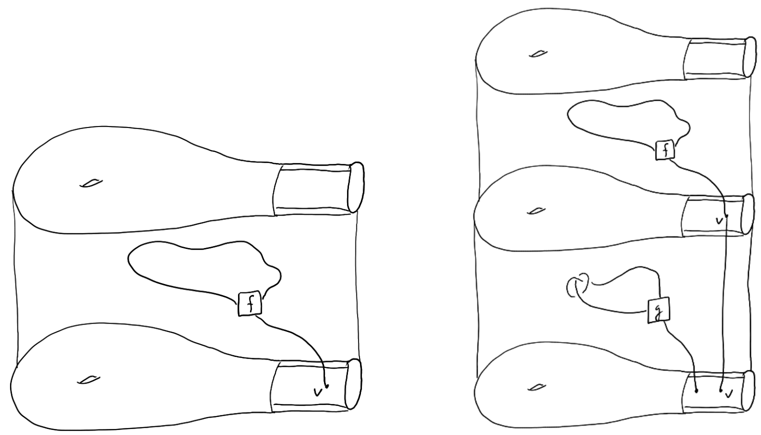
Now, we'd like to be able to consider the above spaces all together, for all $V \in \mathcal{A}$, and make them into an algebra A. What should we mean by this?

Recall that if $\mathcal{A}$ is semisimple, we will have $A = \bigoplus V_i$, a direct sum over isomorphism classes of simple objects V_i possibly with multiplicity. Where $\mathrm{Hom}_{\mathcal{A}}(V_i, A)$ is the multiplicity space and counts the multiplicity of the simple object V_i in A, we have the "isotypic" decomposition

$$A = \bigoplus_{\text{simple}} V_i \otimes \mathrm{Hom}_{\mathcal{A}}(V_i, A) \tag{2}$$

$$= \bigoplus V \otimes \mathrm{Hom}_{\mathcal{A}}(V, A)/(v \otimes \Phi \circ f \sim f(v) \otimes \Phi, f : V \to W) \tag{3}$$

where the first identification is tautological and in the second, we must account for that fact that our sum is not only over simples. Now, in general this object will live not in $\mathcal{A}$ but will be a colimit: so we will work in $\widehat{\mathcal{A}}$, the



(a) A graph ending on the boundary. (b) The stacking operation.

Figure 2: Internal skein algebras.

free cocompletion. Then to get the property we'd like, we will aim to construct $A \in \widehat{\mathcal{A}}$ such that, for all $V \in \mathcal{A}$,

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(V, A) \cong \mathrm{Sk}\Sigma_V^{\varnothing}. \tag{4}$$

Having made this identification, we will then have a description of internal skeins as

$$A = \bigoplus V \otimes \mathrm{Sk}\Sigma_V^{\varnothing} / \sim .$$

The relation $\sim$ can be interpreted topologically: using that $\mathcal{A}$ has a fiber functor to Vect, we can write elements of A as $v \otimes \Gamma$, for $v \in V$ and $\Gamma \in \mathrm{Sk}\Sigma_V^{\varnothing}$ a ribbon graph. Then if Γ has a coupon g on a single strand near the boundary, the relations say that $v \otimes \Gamma$ should be identified with $g(v) \otimes \Gamma'$, for Γ' the ribbon graph without g. Intuitively speaking, the relations allow us to absorb coupons which don't do anything interesting topologically into Σ, see Fig. 3. Later, when we glue bordisms along Σ, they will equivalently allow us to slide such coupons through the gluing surface.

Remark 4. The formula (3) is none other than a coend as introduced in Talk 5 [Rom]. This is the coend of the functor $\mathrm{Id}(-) \otimes \mathrm{Hom}(-, A) : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \widehat{\mathcal{A}}$. One way to see this is that it is the object

$$A = \int^{V \in \mathcal{A}} V \otimes \mathrm{Hom}_{\mathcal{A}}(V, A)$$

with the universal property that it equalizes the two "actions" of $\mathcal{A}$ on itself by Hom spaces: see [Lor20]. Once this notion is understood, the relations given

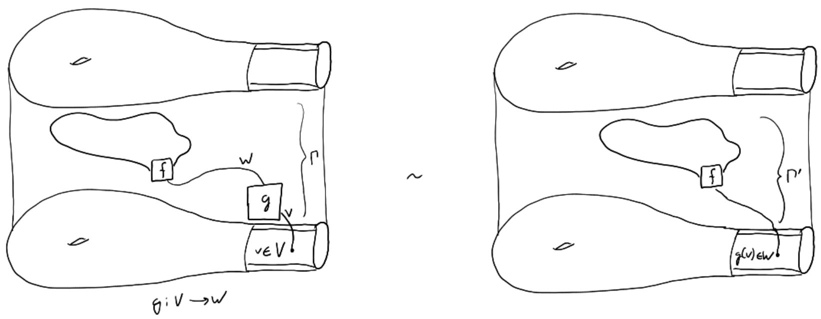


Figure 3: The coend relation for $v \otimes \Gamma \sim g(v) \otimes \Gamma'$, depicted graphically.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

Figure 4: Algebras diagrammatically.

are clear, and indeed we would replace the direct sum formula with the coend formula in the non-semisimple case.

Remark 5. This object may seem a little abstract. In the case $\mathcal{A} = \mathcal{U}_q(G) - mod^{f.d.}$, we have $\widehat{\mathcal{A}} = \mathcal{U}_q(G) - mod^{l.f.}$, and intuitively it makes sense that A will be a locally finite module from the argument given here. Moreover, in Talk 7 [Haïb] we will see a more concrete definition of A. For the purposes of proving facts about A, we will in this talk use the definition of $\widehat{\mathcal{A}}$ as $\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Vect})$.

2.2 The internal definition

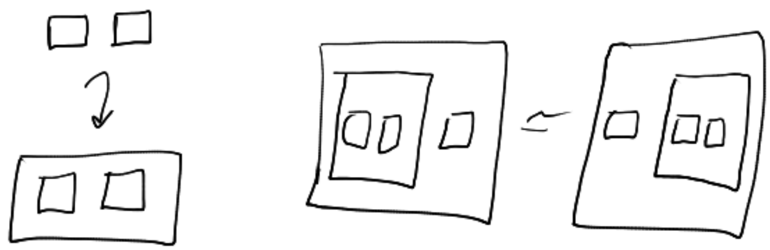
Any kind of object which is usually described as a set equipped with some further structures in terms of morphisms between its n-fold products can be made **internal** to any monoidal category (recall monoidal categories from [Mar]).

For example, an algebra can be described as a vector space A together with maps $1 : k \to A, m : A \otimes A \to A$ satisfying the diagrams in Fig. 4.

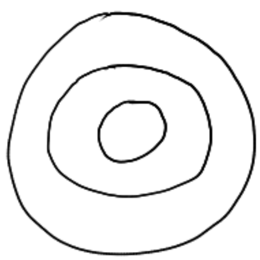
We can make this internal to categories which aren't Vect: working with an object in $\mathrm{Rep}(G)$ we have a G-equivariant algebra, for instance. Here are some more topological examples.

Definition 6. The bicategory Mfld^2 has

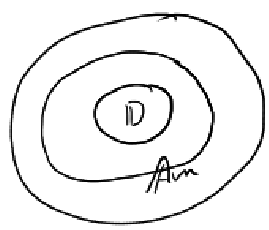
- objects: oriented smooth surfaces



(a) The algebra structure on the disk.



(b) Algebra structure on the annulus.



(c) The annulus acts on the disk.

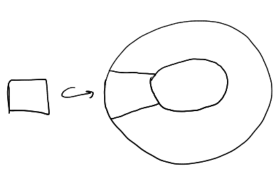
Figure 5: Some objects internal to $h_1(\mathrm{Mfld}^2)$.

- 1-morphisms: oriented smooth embeddings
- 2-morphisms: isotopies of such, mod higher isotopy.

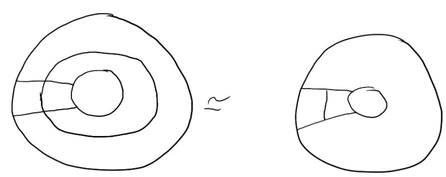
The category $h_1(\mathrm{Mfld}^2)$ has objects as above, and morphisms are isotopy classes of embeddings. Both are symmetric monoidal under disjoint union.

Example 7.
- The object $\mathbb{D}$ is an algebra object internal to $h_1(\mathrm{Mfld}^2)$. (Fig. 5a.)
- The object $\mathbb{A}\mathrm{nn}$ is an algebra, by stacking of annuli. (Fig. 5b.)
- It is clear that $\mathbb{D}$ is a module for $\mathbb{A}\mathrm{nn}$. (Fig. 5c.)
- The inclusion on the negative x-axis $\mathbb{D} \hookrightarrow \mathbb{A}\mathrm{nn}$ is a map of algebras. (Fig. 6a, 6b.)
- Moreover, $\mathbb{A}\mathrm{nn}$ acts on any punctured surface. Therefore, so does $\mathbb{D}$, via disk insertion. (Fig. 6c.)

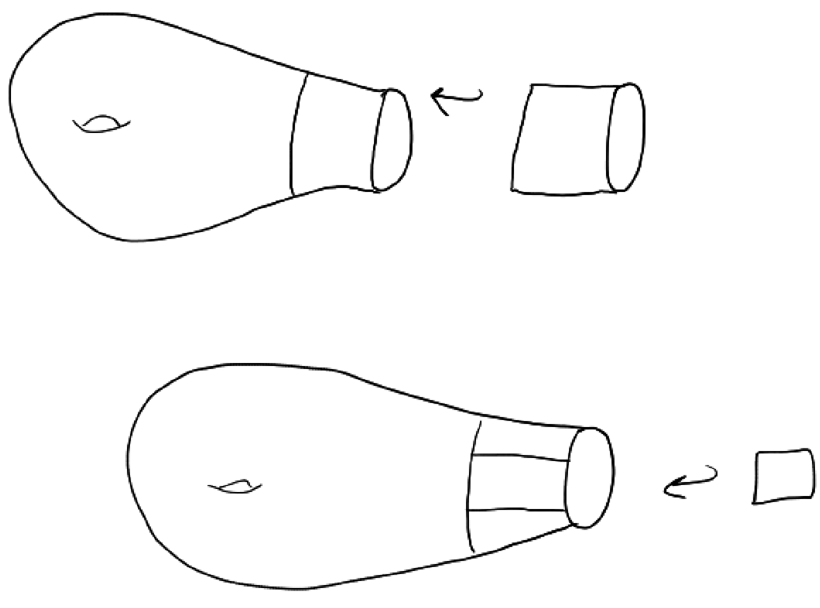
Here is a more algebraic example.



(a) $i : \mathbb{D} \hookrightarrow \mathbb{A}\mathrm{nn}$.



(b) i is an algebra map.



(c) Actions on a punctured surface.

Figure 6: Further objects internal to $h_1(\mathrm{Mfld}^2)$.

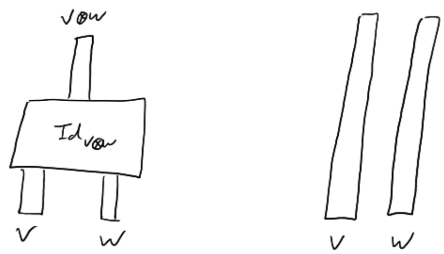


Figure 7: Two ribbon graphs that are distinct in $\mathrm{Rib}_{\mathcal{C}}$ but are identified in the skein category.

Example 8. A monoidal category $\mathcal{C}$ is like an algebra object in Cat, up to some higher data that we will gloss over in this talk. A $\mathcal{C}$-module category is like a $\mathcal{C}$-module object (not to be confused with left $\mathcal{C}$-modules as functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Vect}$). (Technically speaking, they are E_1-algebras and modules.) We say a functor $F : \mathcal{C} \to \mathcal{D}$ is (strong) monoidal if there are (natural) isomorphisms $\eta : \mathbb{1} \to F(\mathbb{1}), \mu : F(-) \otimes F(-) \to F(-)$ (perhaps satisfying some coherences).

Now we have the following.

Lemma 9. *Fixing the data of a ribbon category $\mathcal{A}$, the functor*

$$\mathrm{SkCat}_{\mathcal{A}}(-) : \mathrm{Mfld}^2 \to \mathrm{Pr}$$

is monoidal.

It is easy to see that $\mathrm{SkCat}_{\mathcal{A}}(\mathcal{D}) \simeq \mathcal{A}$, since we recall from Talk 4 [Mar] that the skein category is the quotient of $\mathrm{Rib}_{\mathcal{A}}(\Sigma)$ by relations in $\mathcal{A}$ that hold in a ball (i.e. $\mathcal{D} \times [0, 1]$). That is, referring to Fig. 7, in $\mathrm{Rib}_{\mathcal{A}}$ we considered some ribbon graphs that locally are the same morphism in $\mathcal{A}$ to be different, and in the skein category they are identified (these are the $\mathcal{A}$-skein relations, and one can check that for $\mathcal{U}_q(\mathfrak{sl}_2)$ the whole category is generated by the defining representation and the KBSRs).

The skein category functor of Lemma 9 will be compatible with the higher data involved in defining monoidal categories, module categories, etc: so it will specify the structure of an $\mathcal{A}$-module category on $\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$. Here, the surface Σ is fixed and is punctured; this gives an action of $\mathbb{A}\mathrm{nn}$ in Mfld^2, and an action of $\mathcal{D}$ via the inclusion $\mathcal{D} \hookrightarrow \mathbb{A}\mathrm{nn}$.

Now, let $\mathcal{P} : \mathcal{A} \to \mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$ be given by acting on $\varnothing \in \mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$.

Definition 10 ([GJS19, Def. 2.18])**.** The **internal skein algebra** of Σ^* is the functor $\mathcal{A}^{op} \to \mathrm{Vect}$ given by

$$V \mapsto \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V), \varnothing)$$

so it is a left $\mathcal{A}$-module.

$$\begin{tikzcd}
F(X)\otimes F(Y)\otimes F(Z) \arrow[r, "1\otimes\mu_{Y,Z}"] \arrow[d, "\mu_{X,Y}\otimes 1"'] & F(X)\otimes F(Y\otimes Z) \arrow[d, "\mu_{X,Y\otimes Z}"] \\
F(X\otimes Y)\otimes F(Z) \arrow[r, "\mu_{X\otimes Y,Z}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}_{\mathcal{D}}\otimes F(X) \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda_X"'] & F(\mathbb{1})\otimes F(X) \arrow[d, "\mu_{1,X}"] \\
& A
\end{tikzcd}$$

Figure 8: Some of the diagrams for lax monoidal functors.

Now it is easy to see that the object of $\widehat{\mathcal{A}}$ we have defined has the universal property (4), since using the Yoneda lemma we have

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(V, \mathrm{SkAlg}_{\mathcal{A}}^{\mathrm{int}}(\Sigma^*)) \cong \mathrm{SkAlg}_{\mathcal{A}}^{\mathrm{int}}(\Sigma^*)(V) \cong \mathrm{Sk}\Sigma_V^{\varnothing}$$

writing V for its image $\mathrm{Hom}(-, V) \in \widehat{\mathcal{A}}$ under the Yoneda embedding.

It remains to see that this object has an algebra structure. What are the algebras in $\widehat{A}$?

Definition 11. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. The a **lax monoidal functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with data of a morphism $\eta : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ and a natural transformation $\mu : F(-) \otimes F(-) \to F(- \otimes -)$, both satisfying the obvious coherences, such as those in Fig. 8 (cf. Fig. 4).

If η, μ are isomorphisms, we say F is strong monoidal; if they are identities, F is called strict monoidal.

Proposition 12. *The lax monoidal functors $\mathcal{A}^{op} \to \mathrm{Vect}$ are precisely the algebra objects in $\hat{\mathcal{A}}$.*

The internal skein algebra has the stacking operation

$$\mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V), \varnothing) \otimes \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(W), \varnothing) \to \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V \otimes W), \varnothing).$$

It is associative up to isotopy since disk insertion and concatenation are, and similarly for unitality: see Fig. 2b. Then the object $\mathrm{SkAlg}_{\mathcal{A}}^{\mathrm{int}}(\Sigma^*)$ is a lax monoidal functor, with the required natural transformations being the images of these isotopies under the skein category construction.

Example 13. Here is an almost-trivial example: we can work out what is the internal skein algebra of the disk. This should be given by

$$\begin{aligned}
\mathrm{SkAlg}_{\mathcal{A}}^{\mathrm{int}}(\mathbb{D}) &\cong \bigoplus_{V \in \mathcal{A}} V \otimes \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\mathcal{D})}(V, \mathbb{1}) \\
&\cong \bigoplus_{V \in \mathcal{A}} V \otimes \mathrm{Hom}_{\mathcal{A}}(V, \mathbb{1}) \\
&\cong \mathbb{1}
\end{aligned}$$

where we used the fact that $\mathrm{SkCat}_{\mathcal{A}}(\mathbb{D}) \simeq \mathcal{A}$, and the universal property of the coend formula (3).

In Talk 9 [Kar], we will see a more general way of computing the internal skein algebra for any punctured surface.

2.3 Recovery of ordinary skein algebra

Notice that, evaluating at $\mathbb{1}$, we do recover a module over $\mathrm{End}_{\mathcal{A}}(\mathbb{1}) = k$, which is clearly the ordinary skein algebra. So the ordinary skein module is obtained by taking the value on $\mathbb{1}$.

In [GJS19], this is called "taking invariants". The idea is that, for $M \in \mathrm{Rep}(G)$, then under the (contravariant) Yoneda embedding, we can regard M as an object of $\widehat{\mathrm{Rep}(G)}$ (the object $\mathrm{Hom}(-, M)$, by abuse of notation also denoted M). Then for any object of a free cocompletion we say "taking invariants" means taking its value on $\mathbb{1}$.

2.4 Monadicity and adjoints

Suppose have an adjunction, $F : \mathcal{C} \rightleftharpoons \mathcal{D} : F^R$ with adjunction data (η, ϵ). When the category $\mathcal{C}$ is monoidal and so is $F^R F$, then the object $F^R F(\mathbb{1})$ can be given an algebra structure, and one can consider the category $\mathrm{LMod}_{F^R F(\mathbb{1})}(\mathcal{C})$. The objects $F^R(X)$ can be given a module structure

$$F^R F(\mathbb{1}) \otimes F^R(V) \xrightarrow{1 \otimes \eta_F^R(V)} F^R F(\mathbb{1}) \otimes F^R F F^R(V) \cong F^R F(\mathbb{1} \otimes F^R(V)) \cong F^R F F^R(V) \xrightarrow{F^R \epsilon} F^R(V)$$

so that F^R induces a functor $\mathcal{D} \to \mathrm{LMod}_{F^R F(\mathbb{1})}(\mathcal{C})$. If this functor is an equivalence then the functors of the adjunction are said to be **monadic**.

Now, we can use our coend formula (3) to produce right adjoints. Suppose F has a right adjoint F^R. Then we'd have

$$\begin{aligned} F^R(X) &= \bigoplus V \otimes \mathrm{Hom}_{\mathcal{A}}(V, F^R(X))/\sim \\ &= \bigoplus V \otimes \mathrm{Hom}(F(V), X)/\sim . \end{aligned}$$

In fact, where the last object exists, we can make this a definition

$$F^R(X) := \bigoplus V \otimes \mathrm{Hom}(F(V), X)/\sim$$

and then using the universal property one can see that it defines a right adjoint. So when we work, for example, with free cocompletions, we have right adjoints as defined this way.

2.5 Monadic description of $\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)$.

Now, we have a functor $\mathcal{P} : \mathcal{A} \to \mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$, and this induces a functor $\widehat{\mathcal{A}} \to \widehat{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}$ which we denote, by abuse of notation, by $\mathcal{P}$ also. By the above discussion, we then have that there is a right adjoint given by

$$\mathcal{P}^R(X) = \bigoplus V \otimes \mathrm{Hom}_{\widehat{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}}(\mathcal{P}(V), X)/\sim .$$

We then have that

$$\mathcal{P}^R(\varnothing) \cong \bigoplus V \otimes \mathrm{Hom}(\mathcal{P}(V), \varnothing)/\sim \cong \mathrm{SkAlg}_{\mathcal{A}}^{\mathrm{int}}(\Sigma^*).$$

In fact, since $\mathcal{P}(\mathbb{1}) = \varnothing$, we can write

$$\mathrm{SkAlg}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^*) \cong \mathcal{P}^R\mathcal{P}(\mathbb{1}).$$

It then makes sense to consider the category $\mathrm{LMod}_{\mathrm{SkAlg}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^*)}(\widehat{\mathcal{A}})$. It is shown in [BBJ18] that the adjunction given here is monadic.

Theorem 14 ([BBJ18])**.** *There is an equivalence*

$$\mathrm{LMod}_{\mathrm{SkAlg}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^*)}(\widehat{\mathcal{A}}) \simeq \mathrm{Sk}\widehat{\mathrm{Cat}_{\mathcal{A}}}(\Sigma^*).$$

Suppose that $X \in \mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$, so that $\mathcal{P}^R(X) \in \widehat{\mathcal{A}}$. Then, using the Yoneda lemma and the defining property, we have that

$$\mathcal{P}^R(X)(V) = \mathrm{Hom}_{\widehat{\mathcal{A}}}(V, \mathcal{P}^R(X)) = \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V), X)$$

so we have that $\mathcal{P}^R$ sends X to the functor $\mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(-), X)$.

We can see the module strucutre topologically. As before, by stacking, we have a morphism

$$\mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V), \varnothing) \otimes \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(W), X) \to \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)}(\mathcal{P}(V \otimes W), X).$$

giving a left $\mathrm{SkAlg}^{\mathrm{int}}$-module structure on our functor (Fig. 9).

More specifically, we see that under $\mathcal{P}^R$, the object $\mathcal{P}(V) \in \mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)$ becomes a free left module:

$$\mathcal{P}^R\mathcal{P}(V) \cong \mathcal{P}^R\mathcal{P}(\mathbb{1} \otimes V) \cong \mathcal{P}^R\mathcal{P}(\mathbb{1}) \otimes \mathcal{P}^R\mathcal{P}(V)$$

using that $\mathcal{P}^R\mathcal{P}$ is monoidal.

3 Internal skein modules

Recall the following from Talk 5 [Rom].

Definition 15. Let N be a manifold with boundary Σ. The **relative skein module** is the functor $\mathrm{SkMod}(N) : \mathrm{SkCat}_{\mathcal{A}}(\Sigma)^{op} \to \mathrm{Vect}$ which takes an object of the skein category (i.e. a collection of objects labelling points on Σ) to the k-module of ribbon graphs in N labelled by morphsisms of $\mathcal{A}$, mod the skein relations.

It is now straightforward, based on some of the ideas we have already seen in this talk, to define internal skein modules.

Definition 16 ([GJS19, Def. 2.25])**.** The **internal skein module** of N is the functor $\mathcal{A}^{op} \to \mathrm{Vect}$ given by $\mathrm{Sk}^{\mathrm{int}}_{\mathcal{A}}(N) = \mathrm{SkMod}_{\mathcal{A}}(N, -) \circ \mathcal{P}$. It is a left module object for $\mathrm{SkAlg}^{\mathrm{int}}$ in $\widehat{\mathcal{A}}$ by stacking of skeins (Fig. 10b, 10c).

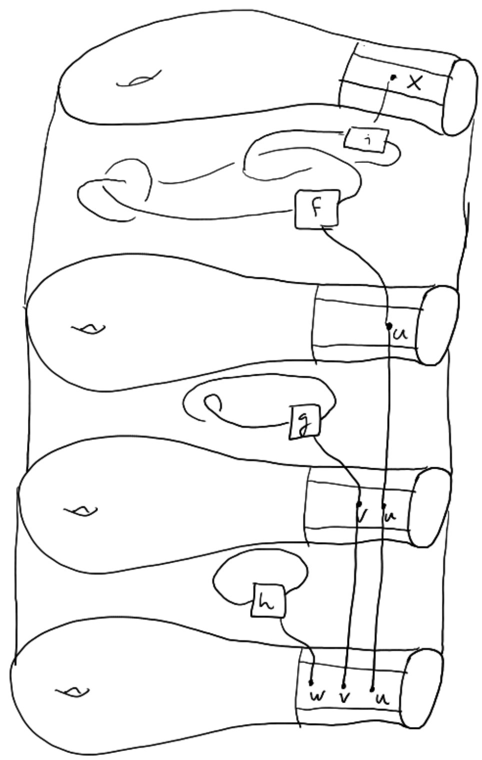
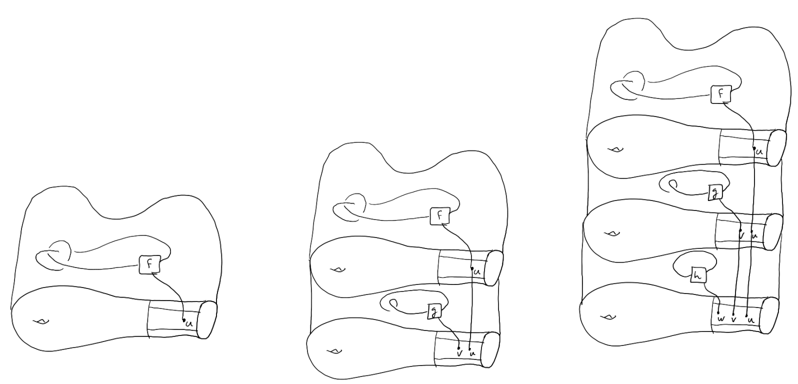


Figure 9: The definition of a functor to $\mathrm{LMod}_{\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)}(\widehat{\mathcal{A}})$.



(a) Handlebody multiplicity spaces.

(b) Stacking.

(c) Action of $\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)$.

Figure 10: Internal skein modules.

The idea is that, to give a well-defined algebra structure over which to define our skein modules, we must insist that the points on the boundary where skeins end are under control.

We also have the following more categorical argument. We've seen the functor $\mathcal{P}^R : \mathrm{SkCat}(\Sigma^*) \to \mathrm{LMod}_{\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)}(\widehat{\mathcal{A}})$. Then as we have seen, this functor induces an equivalence

$$\widehat{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^*)} \simeq \mathrm{LMod}_{\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)}(\widehat{\mathcal{A}}).$$

Now, the functor $\mathrm{SkMod}_{\mathcal{A}}(N,-) : \mathrm{SkCat}(\Sigma)^{op} \to \mathrm{Vect}$ can be restricted to a functor $\mathrm{SkCat}(\Sigma^*)^{op} \to \mathrm{Vect}$, and then uder this equivalence it should yield a module over $\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^*)$, which is precisely the internal skein module.

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