Reading group:

Invariant Gibbs measure for the three dimensional cubic nonlinear wave equation

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Contents:

Notes for the reading group on the hyperbolic Φ^4_3-model

Reading session 1 : Background materials

We consider the following wave equation on $\mathbb{T}^3$:

$$\partial_t^2 u + (1-\Delta)u + u^3 = 0 \qquad \text{(NLW)}$$

where $u : \mathbb{T} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$

- Scaling invariance for NLW (on $\mathbb{R}^3$)

  $u_\lambda = \lambda u(\lambda x, \lambda t)$ also solves (NLW)

  $\Rightarrow$ Sobolev critical exponent is $s_{crit} = \frac{1}{2}$

- Well-posedness for NLW (deterministic) in $\mathcal{H}^s = H^s \times H^{s-1}(\mathbb{T}^3)$

  (See Oh-Pocovnicu-Tzvetkov for a survey)

  - $s \geq 1$: GWP via Sobolev embedding $L^6 \hookrightarrow H^1$
  - $s \geq \frac{1}{2}$: LWP via Strichartz estimates : Lindblad-Sogge
  - $s < \frac{1}{2}$: ill-posedness : Christ-Colliander-Tao, Xia

Q : How can we go below $s < \frac{1}{2}$?

A : randomness (LWP fails but only for very specific initial data)

Let $$u^\omega_{0,\alpha} = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \quad \text{for} \quad \underset{\substack{\nearrow \\ \text{support of Gibbs measure}}}{1} \leq \alpha \leq \underset{\substack{\nwarrow \\ \text{below } H^{\frac{1}{2}}}}{2}$$

g_n = independent standard complex Gaussian s.t. $\overline{g_n} = g_{-n}$ (g_0 real valued)

$\Rightarrow u^\omega_{0,\alpha} \in \mathbb{R}$

Similarly $u^{\omega}_{1,\alpha} = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}$

h_n = independent standard complex Gaussian s.t. $\overline{h_n} = h_{-n}$ (h_0 real valued)

and $\{h_n\}_{n\in\mathbb{Z}^3}$ independent with $\{g_n\}_{n\in\mathbb{Z}^3}$

Then, we have

$$(u^{\omega}_{0,\alpha}, u^{\omega}_{1,\alpha}) \in \mathcal{H}^{\alpha-\frac{3}{2}-} \setminus \mathcal{H}^{\alpha-\frac{3}{2}} \quad \text{a.s.}$$

- Probabilistic well-posedness for NLW
  - GWP for $\alpha > \frac{3}{2}$: Burq - Tzvetkov JEMS '14

    ↑ L^2-level
  - LWP for $\frac{5}{4} < \alpha \leq \frac{3}{2}$: Oh - Pocovnicu - Tzvetkov Ann. Inst. Fourier (Grenoble)

    "renormalization" needed
  - LWP for $1 < \alpha \leq \frac{5}{4}$: Bringmann JEMS '22, Oh - Wang - Zine
  - LWP for $\alpha = 1$: Bringmann - Deng - Nahmod - Yue '22

  GOAL for the reading group : $\alpha = 1$ case

Basic analytic setup

- Linear wave equation :

$$\partial_t^2 u = (\Delta - 1) u \quad \text{on} \quad \mathbb{T}^3$$

Taking the space-time Fourier transform :

$$(i\tau)^2 \hat{u}(n,\tau) = -\langle n\rangle^2 \hat{u}(n,\tau) \qquad \langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$$

Hence, $\hat{u}$ supported on $\{|\tau| = \langle n\rangle\}$

Fourier restriction norm method (Klainerman - Machedon '93, Bourgain '93, Tao's book)

$$\|u\|_{X^{s,b}} = \| \langle n\rangle^s \langle |\tau| - \langle n\rangle \rangle^b \hat{u}(n,\tau) \|_{\ell^2_n L^2_\tau}$$

- The linear nonhomogeneous problem :

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u = F \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1) \end{cases}$$

Duhamel formulation :

$$u(t) = \partial_t S(t) \phi_0 + S(t) \phi_1 + \int_0^t S(t-t') F(t') \, dt'$$

with $S(t) = \dfrac{\sin(t \langle \nabla \rangle)}{\langle \nabla \rangle}$

- Linear estimates :

Let smooth $\eta = \begin{cases} 1 & \text{on } [-1, 1] \\ 0 & \text{on } \{|t| \geq 2\} \end{cases}$

Lemma 1 (linear homogeneous estimate)

For $s \in \mathbb{R}$, $b \in \mathbb{R}$, we have

$$\| \eta(t) \, \partial_t S(t) \phi \|_{X^{s,b}} \lesssim \| \phi \|_{H^s}$$

$$\| \eta(t) \, S(t) \phi \|_{X^{s,b}} \lesssim \| \phi \|_{H^{s-1}}$$

Lemma 2 (linear nonhomogeneous estimate)

For $s \in \mathbb{R}$, $b \in (\frac{1}{2}, 1)$, we have

$$\| I(F) \|_{X^{s,b}} \lesssim \| F \|_{X^{s-1, b-1}}$$

where $I(F) = \eta(t) \int_0^t S(t-t') F(t') \, dt'$

Lemma 3 (time localization estimate) (Deng - Nahmod - Yue '19)

For $s \in \mathbb{R}$, $b_2 > b_1 > \frac{1}{2}$, $T > 0$, we have

$$\| \eta(\tfrac{t}{T}) u \|_{X^{s, b_2}} \lesssim T^{b_2 - b_1} \| u \|_{X^{s, b_1}}$$

given that $u(x, 0) = 0$

Back to NLW with random initial data

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + u^3 = 0 \\ (u, \partial_t u)|_{t=0} = (u_{0,\alpha}^\omega, u_{1,\alpha}^\omega) \end{cases}$$

with $(u_0^\omega, u_1^\omega) = \left(\sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \right)$

- $\alpha > \frac{3}{2}$: First order expansion $u = \overset{\circ}{|} + v$

  $\overset{\circ}{|}$: solution of the linear wave equation with random initial data

  $$\begin{cases} \partial_t^2 \overset{\circ}{|} + (1-\Delta)\overset{\circ}{|} = 0 \\ (\overset{\circ}{|}, \partial_t \overset{\circ}{|})|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases} \iff \overset{\circ}{|} = \partial_t S(t) u_0^\omega + S(t) u_1^\omega$$

  v : remainder term satisfying

  $$\begin{cases} \partial_t^2 v + (1-\Delta)v + (\overset{\circ}{|} + v)^3 = 0 \\ (v, \partial_t v)|_{t=0} = (0,0) \end{cases}$$

- $\frac{5}{4} < \alpha \le \frac{3}{2}$: $\overset{\circ}{|}$ spatial regularity $\alpha - \frac{3}{2} - < 0$

  $\Rightarrow$ renormalization needed to make sense of $\overset{\circ}{|}^2, \overset{\circ}{|}^3$

  $P_{\le N} u$ or $u_{\le N}$: Fourier truncation on $\{|n| \le N\}$

  - Renormalized NLW :

    $$\begin{cases} \partial_t^2 u_{\le N} + (1-\Delta) u_{\le N} + P_{\le N}(:\! u_{\le N}^3 \!:) = 0 \\ (u_{\le N}, \partial_t u_{\le N})|_{t=0} = (P_{\le N} u_0^\omega, P_{\le N} u_1^\omega) \end{cases}$$

    where $:\! u_{\le N}^3 \!: = u_{\le N}^3 - \sigma_N u_{\le N}$ for some constant σ_N s.t.

    $:\! u_{\le N}^3 \!:$ converges to a limit $:\! u^3 \!:$

  - Second order expansion : $u_{\le N} = \overset{\circ}{|}_{\le N} + \overset{\circ\,\circ\,\circ}{\Psi}_{\le N} + v_{\le N}$

$\Psi_{\le N}$ solution of

$$\begin{cases} (\partial_t^2 + 1 - \Delta)\Psi_{\le N} + P_{\le N}(:\Phi_{\le N}^3:) = 0 \\ (\Psi_{\le N}, \partial_t \Psi_{\le N})|_{t=0} = (0,0) \end{cases}$$

$\Psi_{\le N}$ inherits the regularity from $:\Phi_{\le N}^2: = \Phi_{\le N}^2 - \sigma_N$: $2(\alpha - \frac{3}{2} -) + \underset{\uparrow}{1}$

by the Duhamel operator

- $1 < \alpha \le \frac{5}{4}$: second order expansion $u_{\le N} = \Phi_{\le N} + \Psi_{\le N} + v_{\le N}$

high × low × low - interaction : $I\left[\Phi_{\le N} P_{\lesssim 1} v_{\le N} P_{\lesssim 1} v_{\le N}\right]$ I: Duhamel operator

$\Rightarrow$ no gain through multilinear dispersive effects

Solution : Para-controlled approach

Idea : Write $v_{\le N} = X + Y$

X carries the rough regularity of $v_{\le N}$

Y smoother

Reading session 2 : The 3-d cubic NLW with Gaussian initial data

The frequency-localized NLW :

$$\begin{cases} (\partial_t^2 + 1 - \Delta)\, u_{\le N} = -P_{\le N}\big(:(P_{\le N} u_{\le N})^3: + \gamma_{\le N} u_{\le N} \big) \\ \big(u_{\le N}(0),\ \langle \nabla \rangle^{-1} \partial_t u_{\le N}(0) \big) = (\phi^{\cos}, \phi^{\sin}) \end{cases}$$

- $P_{\le N}$ sharp frequency truncation on $\{|n|_\infty \le N\}$
- $:(P_{\le N} u_{\le N})^3:$ Wick ordered cubic power

  $$:(P_{\le N} f)^3: = (P_{\le N} f)^3 - 3\sigma_{\le N}^2 P_{\le N} f$$

  $$\sigma_{\le N}^2 = \sum_{|n|_\infty \le N} \frac{1}{\langle n \rangle^2} \sim N$$

- $\gamma_{\le N}$ additional renormalization (Definition 6.2)

  $$\gamma_{\le N} = \Gamma_{\le N} + (\gamma_{\le N} - \Gamma_{\le N})$$

  $$\Gamma_{\le N}(n) = 6 \cdot \mathbb{1}_{\le N}(n) \sum_{\substack{m_1, m_2, m_3 \in \mathbb{Z}^3 \\ m_{123} = n}} \Big(\prod_{j=1}^{3} \mathbb{1}_{\le N}(m_j) \langle m_j \rangle^{-2} \Big)$$

  $\mathbb{1}_{\le N}(n) = \mathbb{1}_{\{|n|_\infty \le N\}}$
  $m_{123} = m_1 + m_2 + m_3$

  $$\gamma_{\le N} = \Gamma_{\le N}(0)$$

- $$(\phi^{\cos}, \phi^{\sin}) = \Big(\sum_{n \in \mathbb{Z}^3} \frac{g_n}{\langle n \rangle} e^{i n \cdot x},\ \sum_{n \in \mathbb{Z}^3} \frac{h_n}{\langle n \rangle} e^{i n \cdot x} \Big)$$

  $\{g_n, h_n\}_{n \in \mathbb{Z}^3}$ independent standard complex Gaussians

  with $\overline{g_n} = g_{-n}$, $\overline{h_n} = h_{-n}$

  g_0, h_0 real-valued

- Random objects

  - The linear random object $\lollipop$:

    $$\begin{cases} (\partial_t^2 + 1 - \Delta) \lollipop = 0 \\ \big(\lollipop(0),\ \langle \nabla \rangle^{-1} \partial_t \lollipop(0) \big) = (\phi^{\cos}, \phi^{\sin}) \end{cases}$$

    Define $\lollipop_{\le N} = P_{\le N} \lollipop$

    Spatial regularity for $\lollipop_{\le N}$: $-\frac{1}{2}-$

    We also define

    $$\cherry_{\le N} = :(\lollipop_{\le N})^2: = \lollipop_{\le N}^2 - \sigma_{\le N}^2 \qquad \trident_{\le N} = :(\lollipop_{\le N})^3: = \lollipop_{\le N}^3 - \sigma_{\le N}^2 \lollipop_{\le N}$$

- The cubic random object $\overset{\circ\circ\circ}{\Psi}$:

$$\begin{cases} (\partial_t^2+1-\Delta)\overset{\circ\circ\circ}{\Psi}_{\le N} = P_{\le N}\overset{\circ\circ\circ}{\vee}_{\le N} \\ \left(\overset{\circ\circ\circ}{\Psi}_{\le N}(0),\ \langle\nabla\rangle^{-1}\partial_t\overset{\circ\circ\circ}{\Psi}_{\le N}(0)\right) = (0,0) \end{cases}$$

Spatial regularity for $\overset{\circ\circ\circ}{\Psi}_{\le N}$: $0-$ (shown later)

- The quintic random object $\overset{\circ\circ\circ\circ\circ}{\Psi}$:

$$\begin{cases} 3(\partial_t^2+1-\Delta)\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} = P_{\le N}\left(3\overset{\circ\circ}{\vee}_{\le N}\overset{\circ\circ\circ}{\Psi}_{\le N} - \Gamma_{\le N}\overset{\circ}{|}\right) \\ \left(\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N}(0),\ \langle\nabla\rangle^{-1}\partial_t\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N}(0)\right) = (0,0) \end{cases}$$

Spatial regularity for $\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N}$: $\frac{1}{2}-$ (shown later)

We write $u_{\le N} = \overset{\circ}{|} - \overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + v_{\le N}$, where $v_{\le N}$ solves :

$$\begin{aligned}
(\partial_t^2+1-\Delta)v_{\le N} &= -P_{\le N}\left(:(P_{\le N}u_{\le N})^3: + \Gamma_{\le N}u_{\le N}\right) - (\gamma_{\le N}-\Gamma_{\le N})P_{\le N}u_{\le N} \\
&\quad + P_{\le N}\overset{\circ\circ\circ}{\vee}_{\le N} - P_{\le N}\left(3\overset{\circ\circ}{\vee}_{\le N}\overset{\circ\circ\circ}{\Psi}_{\le N} - \Gamma_{\le N}\overset{\circ}{|}\right) \\
&= -\cancel{P_{\le N}\overset{\circ\circ\circ}{\vee}_{\le N}} - P_{\le N}\left[3\overset{\circ\circ}{\vee}_{\le N}\left(-\cancel{\overset{\circ\circ\circ}{\Psi}_{\le N}} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)\right] \\
&\quad - P_{\le N}\left[3\overset{\circ}{|}_{\le N}\left(-\overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)^2\right] \\
&\quad - P_{\le N}\left[\left(-\overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)^3\right] \\
&\quad - P_{\le N}\left[\Gamma_{\le N}\left(\cancel{\overset{\circ}{|}} - \overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + v_{\le N}\right)\right] \\
&\quad - (\gamma_{\le N}-\Gamma_{\le N})\left(\overset{\circ}{|}_{\le N} - \overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right) \\
&\quad + \cancel{P_{\le N}\overset{\circ\circ\circ}{\vee}_{\le N}} - \cancel{P_{\le N}\left(3\overset{\circ\circ}{\vee}_{\le N}\overset{\circ\circ\circ}{\Psi}_{\le N}\right)} + \cancel{P_{\le N}\Gamma_{\le N}}\overset{\circ}{|} \\
&= -P_{\le N}\Big[3\overset{\circ\circ}{\vee}_{\le N}\left(3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right) && \text{(i)} \\
&\quad + 3\overset{\circ}{|}_{\le N}\left(-\overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)^2 && \text{(ii)} \\
&\quad + \left(-\overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)^3 && (3.12) \quad \text{(iii)} \\
&\quad + \Gamma_{\le N}\left(-\overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right) && \text{(iv)} \\
&\quad + (\gamma_{\le N}-\Gamma_{\le N})\left(\overset{\circ}{|}_{\le N} - \overset{\circ\circ\circ}{\Psi}_{\le N} + 3\overset{\circ\circ\circ\circ\circ}{\Psi}_{\le N} + P_{\le N}v_{\le N}\right)\Big] && \text{(v)}
\end{aligned}$$

$$\left(v_{\le N}(0), \langle\nabla\rangle^{-1}\partial_t v_{\le N}(0)\right) = (0,0)$$

Spatial regularity for $v_{\le N}$: $\frac{1}{2}-$ (shown later)

- The 1533-cancellation

Problem: both $\mathbb{E}[\mathfrak{l}_{\le N} \Psi\!\!\Psi_{\le N}]$ and $\mathbb{E}[\Psi_{\le N} \Psi_{\le N}]$ diverges logarithmically (Lemma 6.24)

Solution: $\mathbb{E}[6\, \mathfrak{l}_{\le N} \Psi\!\!\Psi_{\le N} + \Psi_{\le N} \Psi_{\le N}]$ has a well-defined limit

We define

$$\mathcal{C}^{(1,5)}_{\le N}(t) = \mathbb{E}\big[\mathfrak{l}_{\le N}(t,x)\, \Psi\!\!\Psi_{\le N}(t,x)\big], \quad \mathcal{C}^{(3,3)}_{\le N}(t) = \mathbb{E}\big[\Psi_{\le N}(t,x)\, \Psi_{\le N}(t,x)\big] \tag{3.21}$$

$$\mathcal{C}_{\le N}(t) = 6\, \mathcal{C}^{(1,5)}_{\le N}(t) + \mathcal{C}^{(3,3)}_{\le N}(t)$$

Note: $\mathcal{C}^{(1,5)}_{\le N}(t)$ and $\mathcal{C}^{(3,3)}_{\le N}(t)$ are independent of x

$\Leftarrow$ translation-invariance of the random initial data

(Lemma 6.19 and 6.20: explicit computation of $\mathcal{C}^{(1,5)}_{\le N}(t)$ and $\mathcal{C}^{(3,3)}_{\le N}(t)$)

We write

$$\mathfrak{l}_{\le N} \Psi\!\!\Psi_{\le N} = \overbrace{\Big(\mathfrak{l}_{\le N} \Psi\!\!\Psi_{\le N} - \mathcal{C}^{(1,5)}_{\le N}\Big)}^{\text{limit exists as } N\to\infty} + \mathcal{C}^{(1,5)}_{\le N}$$

$$\Psi_{\le N} \Psi_{\le N} = \underbrace{\Big(\Psi_{\le N} \Psi_{\le N} - \mathcal{C}^{(3,3)}_{\le N}\Big)}_{\text{limit exists as } N\to\infty} + \mathcal{C}^{(3,3)}_{\le N}$$

* Some (non-)1533-cancellation patterns in (3.12)

E.g. 1: (3.12 ii) contains $18\, \mathfrak{l}_{\le N} \Psi\!\!\Psi_{\le N} P_{\le N} v_{\le N}$, (3.12 iii) contains $3 \Psi_{\le N} \Psi_{\le N} P_{\le N} v_{\le N}$

↳ coefficients match 6 : 1 ↵

E.g. 2: (3.12 ii) contains $-18\, \mathfrak{l}_{\le N} \Psi_{\le N} \Psi\!\!\Psi_{\le N}$, (3.12 iii) contains $-\Psi_{\le N} \Psi_{\le N} \Psi_{\le N}$

↳ coefficients do not match 6:1 ↵

See Lemma 3.12 below for all occurances of 1533-cancellations

- Symbols (Definition 3.9)

$S^b = \{\mathfrak{l}, \Psi, \Psi\!\!\Psi, v\}$

$S^b_0 = \{\Psi, \Psi\!\!\Psi, v\}$ spatial regularity $\geqslant 0-$

$S^b_{1/2} = \{\Psi\!\!\Psi, v\}$ spatial regularity $\geqslant \frac{1}{2}-$

- Modified product (Definition 3.10) (adjusted to the 1533-cancellation)

$\varsigma^{(1)} \in S^b$, $\varsigma^{(2)}, \varsigma^{(3)} \in S^b_0$

$$\Pi^*_{\le N}(\mathring{\mathsf{I}}_{\le N}, \Psi_{\le N}, \Psi_{\le N}) := \mathring{\mathsf{I}}_{\le N}\left((\Psi_{\le N})^2 - \mathscr{C}^{(3,3)}_{\le N}\right)$$

$$\Pi^*_{\le N}(\mathring{\mathsf{I}}_{\le N}, \Psi\!\!\Psi_{\le N}, \Psi\!\!\Psi_{\le N}) := \mathring{\mathsf{I}}_{\le N}\Psi\!\!\Psi_{\le N}\Psi\!\!\Psi_{\le N} - 2\,\mathscr{C}^{(1,5)}_{\le N}\Psi\!\!\Psi_{\le N}$$

$$\Pi^*_{\le N}(\mathring{\mathsf{I}}_{\le N}, \Psi\!\!\Psi_{\le N}, \varsigma^{(3)}_{\le N}) := \mathring{\mathsf{I}}_{\le N}\Psi\!\!\Psi_{\le N}\varsigma^{(3)}_{\le N} - \mathscr{C}^{(1,5)}_{\le N}\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(3)}_{\le N} \ne \Psi\!\!\Psi_{\le N}$$

$$\Pi^*_{\le N}(\mathring{\mathsf{I}}_{\le N}, \Psi_{\le N}, \varsigma^{(3)}_{\le N}) := \mathring{\mathsf{I}}_{\le N}\Psi_{\le N}\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(3)}_{\le N} \ne \Psi_{\le N}, \Psi\!\!\Psi_{\le N}$$

$$\Pi^*_{\le N}(\mathring{\mathsf{I}}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N}) := \mathring{\mathsf{I}}_{\le N}\varsigma^{(2)}_{\le N}\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \Psi_{\le N}, \Psi\!\!\Psi_{\le N}$$

$$\Pi^*_{\le N}(\Psi_{\le N}, \Psi_{\le N}, \Psi_{\le N}) := (\Psi_{\le N})^3 - 3\,\mathscr{C}^{(3,3)}_{\le N}\Psi_{\le N}$$

$$\Pi^*_{\le N}(\Psi_{\le N}, \Psi_{\le N}, \varsigma^{(3)}_{\le N}) := \left((\Psi_{\le N})^2 - \mathscr{C}^{(3,3)}_{\le N}\right)\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(3)}_{\le N} \ne \Psi_{\le N}$$

$$\Pi^*_{\le N}(\Psi_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N}) := \Psi_{\le N}\varsigma^{(2)}_{\le N}\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \Psi_{\le N}$$

$$\Pi^*_{\le N}(\varsigma^{(1)}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N}) := \varsigma^{(1)}_{\le N}\varsigma^{(2)}_{\le N}\varsigma^{(3)}_{\le N} \qquad \text{if } \varsigma^{(1)}_{\le N} \ne \mathring{\mathsf{I}}_{\le N},\ \varsigma^{(1)}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \Psi_{\le N}$$

- Grouping

Lemma 3.12 $\exists$ coefficient maps

$$A_1 : S^b \to \mathbb{Z}, \quad A_3 : S^b \times S^b_0 \times S^b_0 \to \mathbb{Z}, \quad \tilde{A}_1 : S^b \to \mathbb{Z}$$

such that

(3.12 ii) + (3.12 iii)

$$= 3\,\mathring{\mathsf{I}}_{\le N}\left(-\Psi_{\le N} + 3\Psi\!\!\Psi_{\le N} + P_{\le N}v_{\le N}\right)^2 + \left(-\Psi_{\le N} + 3\Psi\!\!\Psi_{\le N} + P_{\le N}v_{\le N}\right)^3 \tag{3.26}$$

$$= -18\,\mathscr{C}^{(1,5)}_{\le N}\mathring{\mathsf{I}}_{\le N} - \mathscr{C}_{\le N}\sum_{\varsigma \in S^b} A_1(\varsigma)\varsigma_{\le N} - \sum_{\varsigma^{(1)} \in S^b}\sum_{\varsigma^{(2)},\varsigma^{(3)} \in S^b_0} A_3(\varsigma^{(1)}, \varsigma^{(2)}, \varsigma^{(3)})\,\Pi^*_{\le N}(\varsigma^{(1)}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N})$$

$$(3.12\ \mathrm{v}) = (\gamma_{\le N} - \Gamma_{\le N})\left(\mathring{\mathsf{I}}_{\le N} - \Psi_{\le N} + 3\Psi\!\!\Psi_{\le N} + P_{\le N}v_{\le N}\right)$$

$$= -(\gamma_{\le N} - \Gamma_{\le N})\sum_{\varsigma \in S^b}\tilde{A}_1(\varsigma)\varsigma_{\le N} \tag{3.29}$$

Proof : For (3.29), put

$$\left(\tilde{A}_1(\mathring{\mathsf{I}}), \tilde{A}_1(\Psi), \tilde{A}_1(\Psi\!\!\Psi), \tilde{A}_1(v)\right) = (-1, 1, -3, -1)$$

For (3.26), let

$$\mathcal{L} := \mathrm{span}\left(\{\mathscr{C}_{\le N}\varsigma_{\le N} : \varsigma \in S^b\} \cup \{\Pi^*_{\le N}(\varsigma^{(1)}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N}) : \varsigma^{(1)} \in S^b,\ \varsigma^{(2)}, \varsigma^{(3)} \in S^b_0\}\right)$$

Calculations below up to elements in $\mathcal{L}$ (written as mod $\mathcal{L}$) :

$$
\begin{aligned}
&3\,\text{ϙ}_{\le N}\left(-\Psi_{\le N}+3\widetilde{\Psi}_{\le N}+P_{\le N}v_{\le N}\right)^2\\
&=3\,\text{ϙ}_{\le N}(\Psi_{\le N})^2+27\,\text{ϙ}_{\le N}(\widetilde{\Psi}_{\le N})^2+3\,\text{ϙ}_{\le N}\overset{\text{in }\mathcal{L}}{\cancel{(P_{\le N}v_{\le N})^2}}\\
&\quad-6\,\overset{\text{in }\mathcal{L}}{\cancel{\text{ϙ}_{\le N}\Psi_{\le N}}}P_{\le N}v_{\le N}+18\,\text{ϙ}_{\le N}\widetilde{\Psi}_{\le N}\left(-\Psi_{\le N}+P_{\le N}v_{\le N}\right)\\
&=3\,\text{ϙ}_{\le N}(\Psi_{\le N})^2+27\,\text{ϙ}_{\le N}(\widetilde{\Psi}_{\le N})^2+18\,\text{ϙ}_{\le N}\widetilde{\Psi}_{\le N}\left(-\Psi_{\le N}+P_{\le N}v_{\le N}\right)\qquad\text{mod }\mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
&\left(-\Psi_{\le N}+3\widetilde{\Psi}_{\le N}+P_{\le N}v_{\le N}\right)^3\\
&=-(\Psi_{\le N})^3+27\overset{\text{in }\mathcal{L}}{\cancel{(\widetilde{\Psi}_{\le N})^3}}+\overset{\text{in }\mathcal{L}}{\cancel{(P_{\le N}v_{\le N})^3}}+3(\Psi_{\le N})^2\left(3\widetilde{\Psi}_{\le N}+P_{\le N}v_{\le N}\right)\\
&\quad+\overset{\text{in }\mathcal{L}}{\cancel{27(\widetilde{\Psi}_{\le N})^2\left(-\Psi_{\le N}+P_{\le N}v_{\le N}\right)}}+\overset{\text{in }\mathcal{L}}{\cancel{3(P_{\le N}v_{\le N})^2\left(-\Psi_{\le N}+3\widetilde{\Psi}_{\le N}\right)}}-\overset{\text{in }\mathcal{L}}{\cancel{18\Psi_{\le N}\widetilde{\Psi}_{\le N}P_{\le N}v_{\le N}}}\\
&=-(\Psi_{\le N})^3+3(\Psi_{\le N})^2\left(3\widetilde{\Psi}_{\le N}+P_{\le N}v_{\le N}\right)\qquad\text{mod }\mathcal{L}
\end{aligned}
$$

Add above two equations, we obtain

$$
\begin{aligned}
(3.26)&=3\,\text{ϙ}_{\le N}(\Psi_{\le N})^2+\left[27\,\text{ϙ}_{\le N}(\widetilde{\Psi}_{\le N})^2+9(\Psi_{\le N})^2\widetilde{\Psi}_{\le N}\right]-\left[18\,\text{ϙ}_{\le N}\widetilde{\Psi}_{\le N}\Psi_{\le N}+(\Psi_{\le N})^3\right]\\
&\quad+\left(18\,\text{ϙ}_{\le N}\widetilde{\Psi}_{\le N}+3(\Psi_{\le N})^2\right)P_{\le N}v_{\le N}\qquad\text{mod }\mathcal{L}
\end{aligned}
\qquad(3{,}31)
$$

Note that

$$
\begin{aligned}
3\,\text{ϙ}_{\le N}(\Psi_{\le N})^2&=3\mathcal{C}^{(3,3)}_{\le N}\text{ϙ}_{\le N}+3\Pi^*_{\le N}(\text{ϙ}_{\le N},\Psi_{\le N},\Psi_{\le N})&&\text{by Def 3.10}\\
&=-18\,\mathcal{C}^{(1,5)}_{\le N}\text{ϙ}_{\le N}+3\mathcal{C}_{\le N}\text{ϙ}_{\le N}+3\Pi^*_{\le N}(\text{ϙ}_{\le N},\Psi_{\le N},\Psi_{\le N})&&\text{by (3.21)}\qquad(3.31-1)\\
&=-18\,\mathcal{C}^{(1,5)}_{\le N}\text{ϙ}_{\le N}\qquad\text{mod }\mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
&27\,\text{ϙ}_{\le N}(\widetilde{\Psi}_{\le N})^2+9(\Psi_{\le N})^2\widetilde{\Psi}_{\le N}\\
&=27\,\Pi^*_{\le N}(\text{ϙ}_{\le N},\widetilde{\Psi}_{\le N},\widetilde{\Psi}_{\le N})+54\mathcal{C}^{(1,5)}_{\le N}\widetilde{\Psi}_{\le N}\\
&\quad+9\,\Pi^*_{\le N}(\Psi_{\le N},\Psi_{\le N},\widetilde{\Psi}_{\le N})+9\mathcal{C}^{(3,3)}_{\le N}\widetilde{\Psi}_{\le N}\qquad\text{by Def 3.10}\qquad(3.31-2)\\
&=27\,\Pi^*_{\le N}(\text{ϙ}_{\le N},\widetilde{\Psi}_{\le N},\widetilde{\Psi}_{\le N})+9\,\Pi^*_{\le N}(\Psi_{\le N},\Psi_{\le N},\widetilde{\Psi}_{\le N})+9\mathcal{C}_{\le N}\widetilde{\Psi}_{\le N}\qquad\text{by (3.21)}\\
&=0\qquad\text{mod }\mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
&18\, \<1>_{\le N} \<5>_{\le N} \<3>_{\le N} + \left(\<3>_{\le N}\right)^3 \\
&= 18\, \Pi^*_{\le N}\left(\<1>_{\le N}, \<5>_{\le N}, \<3>_{\le N}\right) + 18\, \mathcal{C}^{(1,5)}_{\le N} \<3>_{\le N} \\
&\quad + \Pi^*_{\le N}\left(\<3>_{\le N}, \<3>_{\le N}, \<3>_{\le N}\right) + 3\, \mathcal{C}^{(3,3)}_{\le N} \<3>_{\le N} \qquad \text{by Def 3.10} \qquad (3.31\text{-}3) \\
&= 18\, \Pi^*_{\le N}\left(\<1>_{\le N}, \<5>_{\le N}, \<3>_{\le N}\right) + \Pi^*_{\le N}\left(\<3>_{\le N}, \<3>_{\le N}, \<3>_{\le N}\right) + 3\, \mathcal{C}_{\le N} \<3>_{\le N} \qquad \text{by (3.21)} \\
&= 0 \qquad \mod \mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
&\left(18\, \<1>_{\le N} \<5>_{\le N} + 3 \left(\<3>_{\le N}\right)^2\right) P_{\le N} v_{\le N} \\
&= 18 \left(\<1>_{\le N} \<5>_{\le N} - \mathcal{C}^{(1,5)}_{\le N}\right) P_{\le N} v_{\le N} + 3\left(\left(\<3>_{\le N}\right)^2 - \mathcal{C}^{(3,3)}_{\le N}\right) P_{\le N} v_{\le N} + 3\, \mathcal{C}_{\le N} P_{\le N} v_{\le N} \qquad \text{by (3.21)} \qquad (3.31\text{-}4) \\
&= 18\, \Pi^*_{\le N}\left(\<1>_{\le N}, \<5>_{\le N}, P_{\le N} v_{\le N}\right) + 3\, \Pi^*_{\le N}\left(\<3>_{\le N}, \<3>_{\le N}, P_{\le N} v_{\le N}\right) + 3\, \mathcal{C}_{\le N} P_{\le N} v_{\le N} \qquad \text{by Def 3.10} \\
&= 0 \qquad \mod \mathcal{L}
\end{aligned}
$$

Combining (3.31), (3.31-1), (3.31-2), (3.31-3), and (3.31-4) finishes (3.26) □

By Lemma 3.12, we have

$$
\begin{aligned}
&(\partial_t^2 + 1 - \Delta) v_{\le N} \\
&= - P_{\le N}\left[9\, v_{\le N} \<5>_{\le N} - \Gamma_{\le N} \<3>_{\le N} + 3\, \Gamma_{\le N} \<5>_{\le N} - 18\, \mathcal{C}^{(1,5)}_{\le N} \<1>_{\le N} \right] && (3.32) \\
&\quad - P_{\le N}\left[3\, v_{\le N} P_{\le N} v_{\le N} + \Gamma_{\le N} v_{\le N} \right] && (3.34) \\
&\quad + \sum_{\zeta^{(1)} \in \mathcal{S}^b} \sum_{\zeta^{(2)}, \zeta^{(3)} \in \mathcal{S}^b_0} A_3(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})\, P_{\le N} \Pi^*_{\le N}(\zeta^{(1)}_{\le N}, \zeta^{(2)}_{\le N}, \zeta^{(3)}_{\le N}) && (3.35) \\
&\quad + \mathcal{C}_{\le N} \sum_{\zeta \in \mathcal{S}^b} A_1(\zeta) P_{\le N} \zeta_{\le N} + (r_{\le N} - \Gamma_{\le N}) \sum_{\zeta \in \mathcal{S}^b} \widetilde{A}_1(\zeta) P_{\le N} \zeta_{\le N} && (3.36)
\end{aligned}
$$

Reading session 3 : Paracontrolled approach

Products (parabolic thinking) :

- Deterministic case :

  f, g , f regularity s_1 , g regularity s_2

  Then, fg well-defined if $s_1 + s_2 > 0$ ($s_1 + s_2 \geq 0$ for Sobolev regularities)

  fg regularity $\min(s_1, s_2)$

- Stochastic objects : (Gubinelli - Koch - Oh '21)

  f, g stochastic objects

  fg well-defined (up to renormalization)

  fg regularity $\min(s_1, s_2, s_1 + s_2)$

Multilinear smoothing :

E.g. Cubic Schrödinger

$$F = \int_0^t e^{i(t-t')\Delta} (e^{it'\Delta}\phi)^3 \, dt' \qquad \phi = \sum_n g_n(\omega) e^{in\cdot x}$$

$$\hat{F}(n) \sim \int_0^t \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \neq n_1, n_3}} e^{it'(|n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2)} g_{n_1} \overline{g_{n_2}} g_{n_3} \, dt' \longrightarrow \text{gain of regularity from the integral}$$

$$\sim \sum_* \frac{1}{\langle \Omega \rangle} g_{n_1} \overline{g_{n_2}} g_{n_3} \qquad \Omega = |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = 2(n - n_1)\cdot(n - n_3)$$

$$\lesssim \sqrt{\sum_* \frac{1}{\langle \Omega \rangle^2}} \qquad \text{Wiener chaos estimate}$$

$$\lesssim \sqrt{\sum_{d \in \mathbb{Z}^n} \frac{1}{\langle d \rangle^2} \sum_{* \text{ and } \Omega = d} 1} \qquad \text{divisor counting}$$

The equation for $v_{\le N}$:

$$(\partial_t^2 + 1 - \Delta) v_{\le N}$$

$$= - P_{\le N} \left[9 v_{\le N} \Psi_{\le N} - \Gamma_{\le N} \Psi_{\le N} + 3 \Gamma_{\le N} \Psi_{\le N} - 18\, \mathcal{C}_{\le N}^{(1,5)} \Psi_{\le N} \right] \qquad (3.32)$$

$$- P_{\le N} \left[3 v_{\le N} P_{\le N} v_{\le N} + \Gamma_{\le N} v_{\le N} \right] \qquad (3.34)$$

$$+ \sum_{\varsigma^{(1)} \in S^b} \sum_{\varsigma^{(2)}, \varsigma^{(3)} \in S_0^b} A_3(\varsigma^{(1)}, \varsigma^{(2)}, \varsigma^{(3)}) P_{\le N} \Pi_{\le N}^{*} (\varsigma_{\le N}^{(1)}, \varsigma_{\le N}^{(2)}, \varsigma_{\le N}^{(3)}) \qquad (3.35)$$

$$+ \mathcal{C}_{\le N} \sum_{\varsigma \in S^b} A_1(\varsigma) P_{\le N} \varsigma_{\le N} + (\gamma_{\le N} - \Gamma_{\le N}) \sum_{\varsigma \in S^b} \tilde{A}_1(\varsigma) P_{\le N} \varsigma_{\le N} \qquad (3.36)$$

- Some problematic terms

  - (high x high → low) x low - interaction

$$I\big[P_{\lesssim 1}(\rho_N \cdot P_N v_{\le N})\, P_{\lesssim 1} v_{\le N}\big]$$

P_N : frequency truncation on $\{\frac{N}{2} < |n|_\infty \le N\}$

ρ_N : truncation of ρ on $\{\frac{N}{2} < |n|_\infty \le N\}$

I = Duhamel operator

frequencies : $\downarrow n_1$ $\quad\downarrow n_2$ $\quad\downarrow n_3$

multilinear dispersive symbol :

$n_{123} = n_1 + n_2 + n_3$

$$\big|\langle n_{123}\rangle - \langle n_{12}\rangle - \langle n_3\rangle\big| \lesssim \langle n_{12}\rangle + \langle n_3\rangle \lesssim 1 \implies \text{no derivative gain}$$

ρ_N regularity $-\frac{1}{2}-$ $\implies$ need $v_{\le N}$ regularity $\frac{1}{2}+$

$\Downarrow$

unlikely due to following problematic interactions

  - high x low x low - interaction

$$I\big[\rho_N\, P_{\lesssim 1} v_{\le N}\, P_{\lesssim 1} v_{\le N}\big]$$

frequencies : $\downarrow n_1$ $\quad\downarrow n_2$ $\quad\downarrow n_3$

multilinear dispersive symbol :

$$\big|\langle n_{123}\rangle - \langle n_1\rangle - \langle n_2\rangle - \langle n_3\rangle\big| \lesssim \langle n_2\rangle + \langle n_3\rangle \lesssim 1 \implies \text{no derivative gain}$$

$$\text{regularity} : \underset{\substack{\uparrow \\ \text{from } \rho_N}}{-\frac{1}{2}} + \underset{\substack{\uparrow \\ \text{from } I}}{1} = \frac{1}{2}- \implies \text{not enough for contraction}$$

$\implies$ put in $X^{(1)}_{\le N}$

  - high x high x low - interaction

$$I\big[:\rho_N \rho_N:\, P_{\lesssim 1} v_{\le N}\big]$$

frequencies : $\downarrow n_1$ $\quad\downarrow n_2$ $\quad\downarrow n_3$

$$(n_1, n_2, n_3) \longmapsto \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle \quad \text{equidistributed over} \lesssim N$$

$\implies \frac{1}{2}$ derivative gain (Lemma 10.5)

$$\text{regularity} : 2\cdot\underset{\substack{\uparrow \\ \text{from } :\rho_N\rho_N:}}{(-\tfrac{1}{2}-)} + \underset{\substack{\uparrow \\ \text{multilinear smoothing}}}{\frac{1}{2}} + \underset{\substack{\uparrow \\ \text{from } I}}{1} = \frac{1}{2}- \implies \text{not enough for contraction}$$

$\implies$ put in $X^{(2)}_{\le N}$

Write $v_{\le N} = X^{(1)}_{\le N} + X^{(2)}_{\le N} + Y_{\le N}$ ↷ remainder term with regularity $s > \frac{1}{2}$

∘ resonant - interaction

$$I\left[\<1>_N P_{\lesssim 1} (\<1>_N P_N Y_{\le N}) \right]$$

regularity : $-\frac{1}{2}- \; + \; -\frac{1}{2}- \; + \; s \; + \; 1 \; = \; s- \; < \; s$ ⇒ not okay

from $\<1>_N$ from $Y_{\le N}$ from I ⇒ put in $X^{(1)}_{\le N}$

- The para-controlled approach

∘ Dyadically - localized modified product (Definition 3.13)

$N \ge 1$, $1 \le N_1 , N_2 \le N$, we define

$$\mathcal{C}^{(1,5)}_{\le N}[N_1, N_2](t) := \mathbb{E}\left[\<1>_{N_1} P_{N_2} \<5>_{\le N} \right]$$

$$\mathcal{C}^{(3,3)}_{\le N}[N_1, N_2](t) := \mathbb{E}\left[P_{N_1} \<3>_{\le N} P_{N_2} \<3>_{\le N} \right]$$

$S^b = \{ \<1>, \<3>, \<5>, v \}$

$S^b_0 = \{ \<3>, \<5>, v \}$

$\varsigma^{(1)} \in S^b$, $\varsigma^{(2)}, \varsigma^{(3)} \in S^b_0$, $1 \le N_1, N_2, N_3 \le N$, we define

$$\Pi^*_{\le N}(P_{N_1} \<1>_{\le N}, P_{N_2} \<3>_{\le N}, P_{N_3} \<3>_{\le N}) := \<1>_{N_1} \left(P_{N_2} \<3>_{\le N} P_{N_3} \<3>_{\le N} - \mathcal{C}^{(3,3)}_{\le N}[N_2, N_3] \right)$$

$$\Pi^*_{\le N}(P_{N_1} \<1>_{\le N}, P_{N_2} \<5>_{\le N}, P_{N_3} \<5>_{\le N}) := \<1>_{N_1} P_{N_2} \<5>_{\le N} P_{N_3} \<5>_{\le N} - \mathcal{C}^{(1,5)}_{\le N}[N_1, N_2] P_{N_3} \<5>_{\le N}$$
$$- \mathcal{C}^{(1,5)}_{\le N}[N_1, N_3] P_{N_2} \<5>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<1>_{\le N}, P_{N_2} \<5>_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := \<1>_{N_1} P_{N_2} \<5>_{\le N} P_{N_3} \varsigma^{(3)}_{\le N} - \mathcal{C}^{(1,5)}_{\le N}[N_1, N_2] P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(3)}_{\le N} \ne \<5>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<1>_{\le N}, P_{N_2} \<3>_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := \<1>_{N_1} P_{N_2} \<3>_{\le N} P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(3)}_{\le N} \ne \<3>_{\le N}, \<5>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<1>_{\le N}, P_{N_2} \varsigma^{(2)}_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := \<1>_{N_1} P_{N_2} \varsigma^{(2)}_{\le N} P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \<3>_{\le N}, \<5>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<3>_{\le N}, P_{N_2} \<3>_{\le N}, P_{N_3} \<3>_{\le N}) := P_{N_1} \<3>_{\le N} P_{N_2} \<3>_{\le N} P_{N_3} \<3>_{\le N} - \mathcal{C}^{(3,3)}_{\le N}[N_2, N_3] P_{N_1} \<3>_{\le N}$$
$$- \mathcal{C}^{(3,3)}_{\le N}[N_1, N_3] P_{N_2} \<3>_{\le N} - \mathcal{C}^{(3,3)}_{\le N}[N_1, N_2] P_{N_3} \<3>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<3>_{\le N}, P_{N_2} \<3>_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := \left(P_{N_1} \<3>_{\le N} P_{N_2} \<3>_{\le N} - \mathcal{C}^{(3,3)}_{\le N}[N_1, N_2] \right) P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(3)}_{\le N} \ne \<3>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \<3>_{\le N}, P_{N_2} \varsigma^{(2)}_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := P_{N_1} \<3>_{\le N} P_{N_2} \varsigma^{(2)}_{\le N} P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \<3>_{\le N}$$

$$\Pi^*_{\le N}(P_{N_1} \varsigma^{(1)}_{\le N}, P_{N_2} \varsigma^{(2)}_{\le N}, P_{N_3} \varsigma^{(3)}_{\le N}) := P_{N_1} \varsigma^{(1)}_{\le N} P_{N_2} \varsigma^{(2)}_{\le N} P_{N_3} \varsigma^{(3)}_{\le N} \quad \text{if } \varsigma^{(1)}_{\le N} \ne \<1>_{\le N} , \; \varsigma^{(1)}_{\le N}, \varsigma^{(2)}_{\le N}, \varsigma^{(3)}_{\le N} \ne \<3>_{\le N}$$

- Trilinear para-product operators (Definition 3.14)

(1) high × low × low

$\varsigma^{(2)}, \varsigma^{(3)} \in S_0^b$, we define

$$\Pi^{hi,lo,lo}_{\leq N}(\mathring{|}_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}) := \sum_{\substack{1\leq N_1,N_2,N_3\leq N \\ N_2,N_3\leq N_1^\eta}} \Pi^*_{\leq N}(\mathring{|}_{N_1}, P_{N_2}\varsigma^{(2)}_{\leq N}, P_{N_3}\varsigma^{(3)}_{\leq N})$$

$0<\eta<<1$

$$\Pi^{hi,lo,lo}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, \Psi_{\leq N}) := \sum_{\substack{1\leq N_1,N_2,N_3\leq N \\ N_2,N_3\leq N_1^\eta}} \mathring{|}_{N_1}\left(\mathring{|}_{N_2} P_{N_3}\Psi_{\leq N} - \mathcal{C}^{(1,5)}_{\leq N}[N_2,N_3]\right)$$

$$\Pi^{hi,lo,lo}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, v_{\leq N}) := \sum_{\substack{1\leq N_1,N_2,N_3\leq N \\ N_2,N_3\leq N_1^\eta}} :\mathring{|}_{N_1}\mathring{|}_{N_2}: P_{N_3}v_{\leq N}$$

(2) high × high × low

$\varsigma^{(3)} \in \{\Psi, v\}$, we define

$$\Pi^{hi,hi,lo}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, \varsigma^{(3)}_{\leq N}) := \sum_{\substack{1\leq N_1,N_2,N_3\leq N \\ \min(N_1,N_2)>\max(N_1,N_2)^\eta \\ N_3\leq \max(N_1,N_2)^\eta}} :\mathring{|}_{N_1}\mathring{|}_{N_2}: P_{N_3}\varsigma^{(3)}_{N_3}$$

(3) resonant

$\mathbb{1}_N(n) = \mathbb{1}_{\{\frac{N}{2}<|n|_\infty\leq N\}}$, $n_{12} = n_1+n_2$, $n_{123} = n_1+n_2+n_3$

$Y : \mathbb{R}\times\mathbb{T}^3 \to \mathbb{R}$ any function

$$\Pi^{res}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, P_{\leq N}Y) := \sum_{\substack{1\leq N_1,N_2,N_3\leq N \\ N_3>\max(N_1,N_2)^\eta}} \sum_{N_{13},N_{23}} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\left(\mathbb{1}_{\{N_{13}\leq N_2^\eta\}} + \mathbb{1}_{\{N_{23}\leq N_1^\eta\}}\right)$$
$$\times\left(\prod_{j=1}^{3}\mathbb{1}_{N_j}(n_j)\right)\times\mathbb{1}_{N_{13}}(n_{13})\,\mathbb{1}_{N_{23}}(n_{23})\,\mathring{|}_{N_1}(n_1)\,\mathring{|}_{N_2}(n_2)\,P_{N_3}\widehat{Y}(n_3)\,e^{i\langle n_{123},x\rangle}\Big]$$

- Para-controlled equations

We write

$$v_{\leq N} = X^{(1)}_{\leq N} + X^{(2)}_{\leq N} + Y_{\leq N},$$

where

$$(\partial_t^2 + 1 - \Delta)X^{(1)}_{\leq N}$$
$$= -6P_{\leq N}\Pi^{hi,lo,lo}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, 3\Psi_{\leq N} + v_{\leq N}) \qquad (3.46)$$
$$+\sum_{\varsigma^{(2)},\varsigma^{(3)}\in S_0^b} A_3(\mathring{|}, \varsigma^{(2)}, \varsigma^{(3)})P_{\leq N}\Pi^{hi,lo,lo}_{\leq N}(\mathring{|}_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}) \qquad (3.47)$$
$$+ A_1(\mathring{|})\mathcal{C}_{\leq N}\mathring{|}_{\leq N} + \widetilde{A}_1(\mathring{|})(\gamma_{\leq N} - \Gamma_{\leq N})\mathring{|}_{\leq N} \qquad (3.48)$$
$$- 3P_{\leq N}\Pi^{res}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, Y_{\leq N}) \qquad (3.49)$$

$$(X^{(1)}_{\leq N}(0), \partial_t X^{(1)}_{\leq N}(0)) = (0,0)$$

and

$$(\partial_t^2 + 1 - \Delta) X^{(2)}_{\leq N} = -3 P_{\leq N} \Pi^{\mathrm{hi,hi,lo}}_{\leq N}\big(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, 3\overset{\Psi}{\Psi}_{\leq N} + v_{\leq N}\big) \qquad (3.51)$$

$$\big(X^{(2)}_{\leq N}(0), \partial_t X^{(2)}_{\leq N}(0)\big) = (0, 0)$$

Para-controlled operators : (Definition 3.16)

$$\mathbb{X}^{(1)}_{\leq N} = \mathbb{X}^{(1)}_{\leq N}[v_{\leq N}, Y_{\leq N}] = \mathcal{I}\big[(3.46) + (3.47) + (3.48) + (3.49)\big]$$

$$\mathbb{X}^{(2)}_{\leq N} = \mathbb{X}^{(2)}_{\leq N}[v_{\leq N}] = \mathcal{I}\big[(3.51)\big]$$

Para-controlled symbols : (Definition 3.17)

$$S^p := \{\mathring{|}, \Psi, \overset{\Psi}{\Psi}, \mathbb{X}^{(1)}, \mathbb{X}^{(2)}, Y\}$$

$$S^p_0 := \{\Psi, \overset{\Psi}{\Psi}, \mathbb{X}^{(1)}, \mathbb{X}^{(2)}, Y\}$$

$$S^p_{1/2} := \{\overset{\Psi}{\Psi}, \mathbb{X}^{(1)}, \mathbb{X}^{(2)}, Y\}$$

Equation for $Y_{\leq N}$:

$$(\partial_t^2 + 1 - \Delta) Y_{\leq N}$$

$$v_{\leq N}\left\{\begin{aligned} &= - P_{\leq N}\Big[9 v_{\leq N} \overset{\Psi}{\Psi}_{\leq N} - \Gamma_{\leq N}\Psi_{\leq N} + 3\Gamma_{\leq N}\overset{\Psi}{\Psi}_{\leq N} - 18\,\mathcal{C}^{(1,5)}_{\leq N}\mathring{|}_{\leq N}\Big] \\ &\quad - P_{\leq N}\Big[3 v_{\leq N} P_{\leq N} v_{\leq N} + \Gamma_{\leq N} v_{\leq N}\Big] \\ &\quad + \sum_{\varsigma^{(1)}\in S^b}\ \sum_{\varsigma^{(2)},\varsigma^{(3)}\in S^b_0} A_3(\varsigma^{(1)}, \varsigma^{(2)}, \varsigma^{(3)})\, P_{\leq N}\Pi^{*}_{\leq N}(\varsigma^{(1)}_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}) \\ &\quad + \mathcal{C}_{\leq N}\sum_{\varsigma\in S^b} A_1(\varsigma) P_{\leq N}\varsigma_{\leq N} + (\gamma_{\leq N} - \Gamma_{\leq N})\sum_{\varsigma\in S^b}\widetilde{A}_1(\varsigma) P_{\leq N}\varsigma_{\leq N} \qquad \text{by (3.32) -- (3.36)} \end{aligned}\right.$$

$$X^{(1)}_{\leq N}\left\{\begin{aligned} &+ 6 P_{\leq N}\Pi^{\mathrm{hi,lo,lo}}_{\leq N}\big(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, 3\overset{\Psi}{\Psi}_{\leq N} + v_{\leq N}\big) \\ &- \sum_{\varsigma^{(2)},\varsigma^{(3)}\in S^b_0} A_3(\mathring{|}, \varsigma^{(2)}, \varsigma^{(3)})\, P_{\leq N}\Pi^{\mathrm{hi,lo,lo}}_{\leq N}(\mathring{|}_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}) \qquad \text{cancel terms involving } \mathring{|} \\ &- A_1(\mathring{|})\,\mathcal{C}_{\leq N}\mathring{|}_{\leq N} - \widetilde{A}_1(\mathring{|})\,(\gamma_{\leq N} - \Gamma_{\leq N})\,\mathring{|}_{\leq N} \\ &+ 3 P_{\leq N}\Pi^{\mathrm{res}}_{\leq N}(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, Y_{\leq N}) \end{aligned}\right.$$

$$X^{(2)}_{\leq N}\Big\{\ + 3 P_{\leq N}\Pi^{\mathrm{hi,hi,lo}}_{\leq N}\big(\mathring{|}_{\leq N}, \mathring{|}_{\leq N}, 3\overset{\Psi}{\Psi}_{\leq N} + v_{\leq N}\big)$$

$$= - P_{\le N}\Big[9\, \overset{\circ\,\circ}{\mathsf{V}}_{\le N} \overset{\circ\circ\circ}{{\scriptstyle\circ}\mathsf{Y}{\scriptstyle\circ}}_{\le N} - \Gamma_{\le N} \overset{\circ\,\circ}{\mathsf{Y}}_{\le N} - 18\, \mathcal{C}^{(1,5)}_{\le N} \overset{\circ}{\mathsf{l}}_{\le N} - 9\big(2\,\Pi^{\mathrm{hi,lo,lo}}_{\le N} + \Pi^{\mathrm{hi,hi,lo}}_{\le N}\big)\big(\overset{\circ}{\mathsf{l}}_{\le N}, \overset{\circ}{\mathsf{l}}_{\le N}, \overset{\circ\circ\circ}{{\scriptstyle\circ}\mathsf{Y}{\scriptstyle\circ}}_{\le N}\big)\Big] \tag{3.57}$$

$$- 3 P_{\le N}\Big[\overset{\circ\,\circ}{\mathsf{V}}_{\le N} \mathbb{X}^{(1)}_{\le N} - \big(2\,\Pi^{\mathrm{hi,lo,lo}}_{\le N} + \Pi^{\mathrm{hi,hi,lo}}_{\le N}\big)\big(\overset{\circ}{\mathsf{l}}_{\le N}, \overset{\circ}{\mathsf{l}}_{\le N}, \mathbb{X}^{(1)}_{\le N}\big)\Big] \tag{3.59}$$

$$- P_{\le N}\Big[3\, \overset{\circ\,\circ}{\mathsf{V}}_{\le N} \mathbb{X}^{(2)}_{\le N} - \big(6\,\Pi^{\mathrm{hi,lo,lo}}_{\le N} + 3\,\Pi^{\mathrm{hi,hi,lo}}_{\le N}\big)\big(\overset{\circ}{\mathsf{l}}_{\le N}, \overset{\circ}{\mathsf{l}}_{\le N}, \mathbb{X}^{(2)}_{\le N}\big) + \Gamma_{\le N}\big(3\, \overset{\circ\circ\circ}{{\scriptstyle\circ}\mathsf{Y}{\scriptstyle\circ}}_{\le N} + v_{\le N}\big)\Big] \tag{3.60}$$

$$- 3 P_{\le N}\Big[\overset{\circ\,\circ}{\mathsf{V}}_{\le N} Y_{\le N} - \big(2\,\Pi^{\mathrm{hi,lo,lo}}_{\le N} + \Pi^{\mathrm{hi,hi,lo}}_{\le N} + \Pi^{\mathrm{res}}_{\le N}\big)\big(\overset{\circ}{\mathsf{l}}_{\le N}, \overset{\circ}{\mathsf{l}}_{\le N}, Y_{\le N}\big)\Big] \tag{3.61}$$

$$+ \sum_{\zeta^{(2)}, \zeta^{(3)} \in S^{p}_{0}} A_3\big(\overset{\circ}{\mathsf{l}}, \zeta^{(2)}, \zeta^{(3)}\big)\, P_{\le N}\big(\Pi^{*}_{\le N} - \Pi^{\mathrm{hi,lo,lo}}_{\le N}\big)\big(\overset{\circ}{\mathsf{l}}_{\le N}, \zeta^{(2)}_{\le N}, \zeta^{(3)}_{\le N}\big) \tag{3.62}$$

$$+ \sum_{\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)} \in S^{p}_{0}} A_3\big(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}\big)\, P_{\le N} \Pi^{*}_{\le N}\big(\zeta^{(1)}_{\le N}, \zeta^{(2)}_{\le N}, \zeta^{(3)}_{\le N}\big) \tag{3.63}$$

$$+ \mathcal{C}_{\le N} \sum_{\zeta \in S^{b}_{0}} A_1(\zeta) P_{\le N} \zeta_{\le N} + \big(\gamma_{\le N} - \Gamma_{\le N}\big) \sum_{\zeta \in S^{b}_{0}} \widetilde{A}_1(\zeta) P_{\le N} \zeta_{\le N} \tag{3.64}$$

Reading session 4 : Main estimates and local well-posedness

- Local well-posedness

Recall : $u_{\le N} = \mathsf{I} - \Psi_{\le N} + 3\,\Psi\!\!\!\Psi_{\le N} + v_{\le N}$

$v_{\le N} = X^{(1)}_{\le N} + X^{(2)}_{\le N} + Y_{\le N}$

$X^{(1)}_{\le N}$ satisfies (3.46) - (3.49) $\qquad \mathbb{X}^{(1)}_{\le N}[v_{\le N}, Y_{\le N}] = I\big[(3.46) + \cdots + (3.49)\big]$

$X^{(2)}_{\le N}$ satisfies (3.51) $\qquad \mathbb{X}^{(2)}_{\le N}[v_{\le N}] = I\big[(3.51)\big]$

$Y_{\le N}$ satisfies (3.57) - (3.64)

Proposition 3.1 (Qualitative local well-posedness)

For any $0 < \tau \ll 1$, there exists $\Omega_\tau \subseteq \Omega$ s.t.

(i) (High probability) $\quad \mathbb{P}(\Omega_\tau) \geq 1 - c_1^{-1}\exp(-c_1 \tau^{-c_1})$ $\qquad$ c_1 small constant

(ii) (Convergence) $\quad$ For all $w \in \Omega_\tau$, the solutions $u_{\le N}$ converge in

$$L^\infty_t \mathcal{H}^{-\frac{1}{2}-\varepsilon}_x([-\tau,\tau]\times\mathbb{T}^3)$$

as $N \to \infty$.

Proposition 3.25 (Quantitative local well-posedness)

For all $A \geq 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geq 1 - c^{-1}\exp(-cA^c)$ s.t. :

c constant independent of A

(i) (Para-controlled solutions)

For all $0 < \tau < A^{-\Theta}$ and $N \geq 1$, the para-controlled solutions $(v_{\le N}, X^{(1)}_{\le N}, X^{(2)}_{\le N}, Y_{\le N})$ exist on $[-\tau,\tau]\times\mathbb{T}^3$. Furthermore, they satisfy

$$\|u_{\le N}\|_{X^{-\frac{1}{2}-\varepsilon,b}([-\tau,\tau])} \le CA$$

$$\|v_{\le N}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} \le CA$$

$$\max_{j=1,2} \|X^{(j)}_{\le N}\|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1,b})([-\tau,\tau])} \le CA$$

$$\|Y_{\le N}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])} \le CA$$

$\Theta \gg 1$

C constant

$0 \ll \delta_2 \ll \delta_1 \ll 1$

$0 < b - \frac{1}{2} \ll 1$

(ii) (Difference estimates)

For all $0 < \tau < A^{-\Theta}$ and $N_1, N_2 \geq 1$, the differences satisfy

$$\| u_{\le N_1} - u_{\le N_2} \|_{X^{-\frac{1}{2}-\varepsilon, b}([-\tau,\tau])} \le \min(N_1, N_2)^{-\theta}$$

$\theta \ll 1$
$\theta = \Theta^{-1}$

$$\| v_{\le N_1} - v_{\le N_2} \|_{X^{\frac{1}{2}-\delta_1, b}([-\tau,\tau])} \le \min(N_1, N_2)^{-\theta}$$

$$\max_{j=1,2} \| X^{(j)}_{\le N_1} - X^{(j)}_{\le N_2} \|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1, b})([-\tau,\tau])} \le \min(N_1, N_2)^{-\theta}$$

$$\| Y_{\le N_1} - Y_{\le N_2} \|_{X^{\frac{1}{2}+\delta_2, b}([-\tau,\tau])} \le \min(N_1, N_2)^{-\theta}$$

Proposition 3.1 follows directly from Proposition 3.25 by letting $A = \tau^{-\theta}$

- Main estimates

Proposition 3.20 For all $A \ge 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \ge 1 - c^{-1}\exp(-cA^c)$

s.t. : For $j=1,2$, $N \ge 1$, $T \geqslant 1$, $0 \in J \subseteq [-T, T]$ closed interval,

$$\| \mathbb{X}^{(j)}_{\le N}[V_{\le N}, Y_{\le N}] \|_{X^{\frac{1}{2}-\delta_1, b}(J)} + \| \mathbb{X}^{(j)}_{\le N}[V_{\le N}, Y_{\le N}] \|_{L^\infty_t C^{\frac{1}{2}-\delta_1}_x(J)}$$
$$\le A T^\alpha |J|^{b_+ - b} \left(1 + \| V_{\le N} \|^2_{X^{-1,b}(J)} + \| Y_{\le N} \|_{X^{\frac{1}{2}+\delta_2, b}(J)} \right)$$

α constant

Proposition 3.22 (Terms involving two linear random objects)

For all $A \ge 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \ge 1 - c^{-1}\exp(-cA^c)$ s.t. :

For $N \ge 1$, $T \geqslant 1$, $0 \in J \subseteq [-T, T]$ closed interval :

(i) (Explicit random objects)

$0 < b - \frac{1}{2} \ll b_+ - \frac{1}{2} \ll 1$

$$\Big\| P_{\le N} \Big[9 v_{\le N} \Psi_{\le N} - \Gamma_{\le N} \Upsilon_{\le N} - 18\, \mathcal{C}^{(1,5)}_{\le N}\, q_{\le N}$$
$$- 9 \left(2 \Pi^{hi,lo,lo}_{\le N} + \Pi^{hi,hi,lo}_{\le N} \right) (q_{\le N}, q_{\le N}, \Psi_{\le N}) \Big] \Big\|_{X^{-\frac{1}{2}+\delta_2, b_+ - 1}(J)} \le A T^\alpha$$

(ii) (Para-controlled calculus)

$$\Big\| P_{\leq N}\Big[\vee_{\leq N} \mathbb{X}^{(1)}_{\leq N} - \big(2\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,hi,lo}_{\leq N}\big)\big(⫯_{\leq N}, ⫯_{\leq N}, \mathbb{X}^{(1)}_{\leq N}\big)\Big]\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$

$$+ \Big\| P_{\leq N}\Big[3\vee_{\leq N} \mathbb{X}^{(2)}_{\leq N} - \big(6\Pi^{hi,lo,lo}_{\leq N} + 3\Pi^{hi,hi,lo}_{\leq N}\big)\big(⫯_{\leq N}, ⫯_{\leq N}, \mathbb{X}^{(2)}_{\leq N}\big) + P_{\leq N}\big(3\,⚘_{\leq N} + v_{\leq N}\big)\Big]\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$

$$\leq A T^{\alpha}\Big(1 + \|v_{\leq N}\|^2_{X^{-1,b}(J)} + \|Y_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}(J)}\Big)$$

(iii) (The $Y_{\leq N}$ term)

$$\Big\| P_{\leq N}\Big[\vee_{\leq N} Y_{\leq N} - \big(2\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,hi,lo}_{\leq N} + \Pi^{res}_{\leq N}\big)\big(⫯_{\leq N}, ⫯_{\leq N}, Y_{\leq N}\big)\Big]\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$

$$\leq A T^{\alpha} \|Y_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}(J)}$$

Proposition 3.23 (Terms involving one linear random object)

For all $A \geq 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geq 1 - c^{-1}\exp(-cA^c)$ s.t. :

For $N \geq 1$, $T \geq 1$, $0 \in J \subseteq [-T, T]$ closed interval, $\varsigma^{(2)}, \varsigma^{(3)} \in S^P_0$:

$$\Big\| \big(\Pi^{*}_{\leq N} - \Pi^{hi,lo,lo}_{\leq N}\big)\big(⫯_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}\big)\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$

$S^P_0 = \{\Psi, ⚘, X^{(1)}, X^{(2)}, Y\}$

$$\leq A T^{\alpha}\Big(1 + \|v_{\leq N}\|^4_{X^{-1,b}(J)} + \|Y_{\leq N}\|^2_{X^{\frac{1}{2}+\delta_2,b}(J)}\Big)$$

Proposition 3.24 (Terms involving zero linear random object)

For all $A \geq 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geq 1 - c^{-1}\exp(-cA^c)$ s.t. :

For $N \geq 1$, $T \geq 1$, $0 \in J \subseteq [-T, T]$ closed interval :

(i) $\varsigma^{(1)}, \varsigma^{(2)}, \varsigma^{(3)} \in S^P_0$,

$$\Big\| \Pi^{*}_{\leq N}\big(\varsigma^{(1)}_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}\big)\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$

$$\leq A T^{\alpha}\Big(1 + \|v_{\leq N}\|^6_{X^{-1,b}(J)} + \|Y_{\leq N}\|^3_{X^{\frac{1}{2}+\delta_2,b}(J)}\Big)$$

(ii) $\varsigma \in S^b_0$, $S^b_0 = \{\Psi, ⚘, v\}$

$$\|C_{\leq N}\varsigma_{\leq N}\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \leq A T^{\alpha}\Big(1 + \|v_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}(J)}\Big)$$

$$\|(\gamma_{\leq N} - \Gamma_{\leq N})\varsigma_{\leq N}\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \leq A T^{\alpha}\Big(1 + \|v_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}(J)}\Big)$$

Proof of Proposition 3.25 :

Define the ball

$$\mathbb{B}_A := \Big\{ (v_{\leq N}, X^{(1)}_{\leq N}, X^{(2)}_{\leq N}, Y_{\leq N}) : \|v_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} \leq CA ,$$
$$\|X^{(1)}_{\leq N}\|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1,b})([-\tau,\tau])} \leq CA , \ \|X^{(2)}_{\leq N}\|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1,b})([-\tau,\tau])} \leq CA ,$$
$$\|Y_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])} \leq CA \Big\}$$

Define the map

$$\Upsilon_{\leq N}[v_{\leq N}, X^{(1)}_{\leq N}, X^{(2)}_{\leq N}, Y_{\leq N}] := (\Upsilon^{v}_{\leq N}, \Upsilon^{X^{(1)}}_{\leq N}, \Upsilon^{X^{(2)}}_{\leq N}, \Upsilon^{Y}_{\leq N})[v_{\leq N}, X^{(1)}_{\leq N}, X^{(2)}_{\leq N}, Y_{\leq N}] ,$$

where

$$\Upsilon^{v}_{\leq N} := \Upsilon^{X^{(1)}}_{\leq N} + \Upsilon^{X^{(2)}}_{\leq N} + \Upsilon^{Y}_{\leq N}$$
$$\Upsilon^{X^{(1)}}_{\leq N} := I\big[(3.46) + \cdots + (3.49) \big] = \mathbb{X}^{(1)}_{\leq N}[v_{\leq N}, Y_{\leq N}]$$
$$\Upsilon^{X^{(2)}}_{\leq N} := I\big[(3.51) \big] = \mathbb{X}^{(2)}_{\leq N}[v_{\leq N}]$$
$$\Upsilon^{Y}_{\leq N} := I\big[(3.57) + \cdots + (3.64) \big]$$

WTS: $\Upsilon_{\leq N}$ maps $\mathbb{B}_A$ back to itself

Pick $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geq 1 - c^{-1}\exp(-cA^{c})$ s.t. Proposition 3.20, 3.22, 3.23, 3.24 hold

Pick an arbitrary $(v_{\leq N}, X^{(1)}_{\leq N}, X^{(2)}_{\leq N}, Y_{\leq N}) \in \mathbb{B}_A$

Proposition 3.20 $\Rightarrow$

$$\max_{j=1,2} \|\Upsilon^{X^{(j)}}_{\leq N}\|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1,b})([-\tau,\tau])}$$
$$\leq A\tau^{b_+-b}\big(1 + \|v_{\leq N}\|^2_{X^{-1,b}([-\tau,\tau])} + \|Y_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])} \big)$$
$$\leq 3CA^3\tau^{b_+-b} \leq \frac{CA}{4} \qquad 0<\tau<A^{-\Theta} , \ \Theta \gg 1$$

Proposition 3.22, 3.23, 3.24 $\Rightarrow$

$$\|I[(3.57)+\cdots+(3.64)]\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])}$$

by the non-homogeneous linear estimate and the time-localization estimate

$$\leq CA\tau^{b_+-b}\big(1 + \|v_{\leq N}\|^6_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} + \|Y_{\leq N}\|^3_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])} \big)$$
$$\leq 3C^7A^7\tau^{b_+-b} \leq \frac{CA}{4}$$
$$\Rightarrow \ \|\Upsilon^{Y}_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])} \leq \frac{CA}{4}$$

Triangle inequality $\Rightarrow$

$$\|\Upsilon^{v}_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} \leq \|\Upsilon^{X^{(1)}}_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} + \|\Upsilon^{X^{(2)}}_{\leq N}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} + \|\Upsilon^{Y}_{\leq N}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])}$$
$$\leq CA$$

$\Rightarrow$ Self-mapping property of $\Upsilon_{\leq N}$ on B_A

$\Rightarrow$ Contraction property of $\Upsilon_{\leq N}$ is similar (to check)

(need minor generalizations of Proposition 3.20, 3.22, 3.23, 3.24)

Difference estimate :

Similar with additional gain in the maximal frequency-scale (to check) :

$$\|u_{\leq N_1} - u_{\leq N_2}\|_{X^{-\frac{1}{2}-\varepsilon,b}([-\tau,\tau])} + \|v_{\leq N_1} - v_{\leq N_2}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])}$$
$$+ \sum_{j=1,2} \|X^{(j)}_{\leq N_1} - X^{(j)}_{\leq N_2}\|_{(L^\infty_t C^{\frac{1}{2}-\delta_1}_x \cap X^{\frac{1}{2}-\delta_1,b})([-\tau,\tau])} + \|Y_{\leq N_1} - Y_{\leq N_2}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])}$$
$$\leq C^7 A^7 \tau^{b_+-b}\Big(\min(N_1,N_2)^{-\theta} + \|v_{\leq N_1} - v_{\leq N_2}\|_{X^{\frac{1}{2}-\delta_1,b}([-\tau,\tau])} + \|Y_{\leq N_1} - Y_{\leq N_2}\|_{X^{\frac{1}{2}+\delta_2,b}([-\tau,\tau])}\Big)$$

Since $0 < \tau < A^{-\Theta}$ for $\Theta \gg 1$, the difference estimates follow $\square$

Reading session 5 : Structures of stochastic objects

Algebraic and graphical aspects of stochastic diagrams

- The linear random object ϙ :

$$\widehat{\text{ϙ}}(t,n) = \cos(t\langle n\rangle)\frac{g_n(\omega)}{\langle n\rangle} + \sin(t\langle n\rangle)\frac{h_n(\omega)}{\langle n\rangle}$$

We write $g_n = \int_0^1 1\, dW_s^{\cos}(n)$, $h_n = \int_0^1 1\, dW_s^{\sin}(n)$

- $\{W_s^{\cos}(n), W_s^{\sin}(n)\}_{n\in\mathbb{Z}^3}$ independent $\mathbb{C}$-valued standard Brownian motions
- For all $n\in\mathbb{Z}^3$, $\overline{W_s(n)} = W_s(-n)$
- $W_s^{\cos}(0)$, $W_s^{\sin}(0)$ are $\mathbb{R}$-valued standard Brownian motions

Lemma 6.10 (Covariance of ϙ)

For all $t, t'\in\mathbb{R}$ and $n, n'\in\mathbb{Z}^3$, we have

$$\mathbb{E}\left[\widehat{\text{ϙ}}(t,n)\,\widehat{\text{ϙ}}(t',n')\right] = \delta_{n+n'=0}\frac{\cos((t-t')\langle n\rangle)}{\langle n\rangle^2}$$

Proof : $\mathbb{E}\left[\widehat{\text{ϙ}}(t,n)\,\widehat{\text{ϙ}}(t',n')\right]$

$$= \sum_{\varphi,\varphi'=\cos,\sin}\Big(\varphi(t\langle n\rangle)\,\varphi'(t'\langle n'\rangle)\ \mathbb{E}\Big[\frac{1}{\langle n\rangle}\Big(\int_0^1 1\, dW_s^{\varphi}(n)\Big)\frac{1}{\langle n'\rangle}\Big(\int_0^1 1\, dW_s^{\varphi'}(n')\Big)\Big]$$

$$= \delta_{n+n'=0}\frac{1}{\langle n\rangle^2}\sum_{\varphi=\cos,\sin}\varphi(t\langle n\rangle)\,\varphi(t'\langle n\rangle) \qquad \text{(by independence)}$$

$$= \delta_{n+n'=0}\frac{1}{\langle n\rangle^2}\Big(\cos(t\langle n\rangle)\cos(t'\langle n\rangle) + \sin(t\langle n\rangle)\sin(t'\langle n\rangle)\Big)$$

$$= \delta_{n+n'=0}\frac{\cos((t-t')\langle n\rangle)}{\langle n\rangle^2}$$

$\square$

- Multiple stochastic integrals (Oh-Wang-Zine , Bringmann 22', Nualart 06')

Given $f\in L^2((\mathbb{R}_+\times\mathbb{Z}^3)^k)$, we define

$$I_k[f] = \sum_{n_1,\dots,n_k\in\mathbb{Z}^3}\int_{[0,\infty)^k} f(n_1,s_1,\dots,n_k,s_k)\, dW_{s_1}(n_1)\cdots dW_{s_k}(n_k)$$

- Contraction

$k, \ell \in \mathbb{N}$, $0 \le r \le \min(k,\ell)$, $f \in L^2((\mathbb{R}_+ \times \mathbb{Z}^3)^k)$, $g \in L^2((\mathbb{R}_+ \times \mathbb{Z}^3)^\ell)$

$$(f \otimes_r g)(z_1, \dots, z_{k+\ell-2r}) = \sum_{m_1, \dots, m_r \in \mathbb{Z}^3} \int_{\mathbb{R}_+^r} f(z_1, \dots, z_{k-r}, \zeta_1, \dots, \zeta_r)$$
$$\times\, g(z_{k+1-r}, \dots, z_{k+\ell-2r}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_r)\, ds_1 \cdots ds_r ,$$

where $\zeta_j = (m_j, s_j)$ and $\tilde{\zeta}_j = (-m_j, s_j)$

- Product formula (Proposition 1.1.3 in Nualart)

$k, \ell \in \mathbb{N}$, $f \in L^2((\mathbb{R}_+ \times \mathbb{Z}^3)^k)$, $g \in L^2((\mathbb{R}_+ \times \mathbb{Z}^3)^\ell)$ symmetric

$$I_k[f] \cdot I_\ell[g] = \sum_{r=0}^{\min(k,\ell)} r! \binom{k}{r} \binom{\ell}{r} I_{k+\ell-2r}[f \otimes_r g]$$

- The cubic random object Ψ :

$$\begin{cases} (\partial_t^2 + 1 - \Delta)\Psi_{\le N} = P_{\le N} :(\varphi_{\le N})^3: \\ (\Psi_{\le N}(0), \langle \nabla \rangle^{-1} \partial_t \Psi_{\le N}(0)) = (0, 0) \end{cases}$$

$$\Rightarrow \quad \Psi_{\le N}(t,x) = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle} P_{\le N} :(\varphi_{\le N})^3:(t',x)\, dt'$$

We can compute that

$$\widehat{:(\varphi_{\le N})^3:}(t,n) = \sum_{\substack{\varphi_1, \varphi_2, \varphi_3 \\ \in \{\cos, \sin\}}} \sum_{n_{123} = n} \left(\prod_{j=1}^3 \frac{\mathbb{1}_{\le N}(n_j)}{\langle n_j \rangle} \right) \left(\prod_{j=1}^3 \varphi_j(t\langle n_j \rangle) \right) \left(\prod_{j=1}^3 \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \right)$$
$$- 3 \sum_{|n'|_\infty \le N} \frac{1}{\langle n' \rangle^2} \sum_{\varphi \in \{\cos, \sin\}} \frac{\varphi(t\langle n \rangle)}{\langle n \rangle} \int_0^1 1\, dW_s^\varphi(n)$$

$n_{123} = n_1 + n_2 + n_3$

$\mathbb{1}_{\le N}(n) = \mathbb{1}_{\{|n|_\infty \le N\}}$

By the product formula,

$$\int_0^1 dW_{s_1}^{\varphi_1}(n_1) \int_0^1 dW_{s_2}^{\varphi_2}(n_2) = \int_{[0,1]^2} dW_{s_1}^{\varphi_1}(n_1)\, dW_{s_2}^{\varphi_2}(n_2) + \mathbb{1}_{\{n_1 = -n_2, \varphi_1 = \varphi_2\}}$$

$$\int_0^1 dW_{s_1}^{\varphi_1}(n_1) \int_0^1 dW_{s_2}^{\varphi_2}(n_2) \int_0^1 dW_{s_3}^{\varphi_3}(n_3) = \int_{[0,1]^3} dW_{s_1}^{\varphi_1}(n_1)\, dW_{s_2}^{\varphi_2}(n_2)\, dW_{s_3}^{\varphi_3}(n_3) + 2\, \mathbb{1}_{\{n_2 = -n_3, \varphi_2 = \varphi_3\}} \int_0^1 1\, dW_{s_1}^{\varphi_1}(n_1)$$
$$+ \mathbb{1}_{\{n_1 = -n_2, \varphi_1 = \varphi_2\}} \int_0^1 1\, dW_{s_3}^{\varphi_3}(n_3)$$

Thus,

$$\widehat{:(\text{ϙ}_{\leq N})^3:}(t,n) = \sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}} \sum_{n_{123}=n} \Big(\prod_{j=1}^{3}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\prod_{j=1}^{3}\varphi_j(t\langle n_j\rangle)\Big)\int_{[0,1]^3} 1 \bigotimes_{j=1}^{3} dW_{s_j}^{\varphi_j}(n_j)$$

$$+3\sum_{|n'|_\infty\leq N}\frac{1}{\langle n'\rangle^2}\sum_{\varphi,\varphi'\in\{\cos,\sin\}}\varphi'(t\langle n'\rangle)^2\frac{\varphi(t\langle n\rangle)}{\langle n\rangle}\int_0^1 1\, dW_s^{\varphi}(n)$$

$$-3\sum_{|n'|_\infty\leq N}\frac{1}{\langle n'\rangle^2}\sum_{\varphi\in\{\cos,\sin\}}\frac{\varphi(t\langle n\rangle)}{\langle n\rangle}\int_0^1 1\, dW_s^{\varphi}(n)$$

$$= \sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}} \sum_{n_{123}=n} \Big(\prod_{j=1}^{3}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\prod_{j=1}^{3}\varphi_j(t\langle n_j\rangle)\Big)\int_{[0,1]^3} 1 \bigotimes_{j=1}^{3} dW_{s_j}^{\varphi_j}(n_j)$$

This implies

$$\widehat{\Psi}_{\leq N}(t,n) = \frac{\mathbb{1}_{\leq N}(n)}{\langle n\rangle}\sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}} \sum_{n_{123}=n}\Big(\prod_{j=1}^{3}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\int_0^t \sin((t-t')\langle n\rangle)\prod_{j=1}^{3}\varphi_j(t'\langle n_j\rangle)\,dt'\Big)\int_{[0,1]^3} 1\bigotimes_{j=1}^{3} dW_{s_j}^{\varphi_j}(n_j) \quad (6.40)$$

- The quintic random object  :

$\Big\{$ $3(\partial_t^2+1-\Delta)$  $_{\leq N} = P_{\leq N}\big(3 V_{\leq N}\Psi_{\leq N} - \Gamma_{\leq N}\,\text{ϙ}\big)$

$\Big($  $_{\leq N}(0)$, $\langle\nabla\rangle^{-1}\partial_t$  $_{\leq N}(0)\Big) = (0,0)$

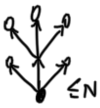
$V_{\leq N} = :(\text{ϙ}_{\leq N})^2:$

Note that

$$\widehat{V}_{\leq N}(t,n) = \sum_{\substack{\varphi_1,\varphi_2\\ \in\{\cos,\sin\}}}\sum_{n_{12}=n}\Big(\prod_{j=1}^{2}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\prod_{j=1}^{2}\varphi_j(t\langle n_j\rangle)\Big)\Big(\prod_{j=1}^{2}\int_0^1 1\,dW_s^{\varphi_j}(n_j)\Big) - \sum_{|n'|_\infty\leq N}\frac{1}{\langle n'\rangle^2}$$

$$= \sum_{\substack{\varphi_1,\varphi_2\\ \in\{\cos,\sin\}}}\sum_{n_{12}=n}\Big(\prod_{j=1}^{2}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\prod_{j=1}^{2}\varphi_j(t\langle n_j\rangle)\Big)\int_{[0,1]^2} 1\bigotimes_{j=1}^{2} dW_{s_j}^{\varphi_j}(n_j)$$

Thus, by the product formula,

$3P_{\leq N}(V_{\leq N}\Psi_{\leq N}) = 3$  $+ 18$  $+ 18$  (6.43)

one pairing

two pairings

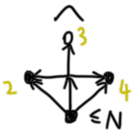
where

$$(t,n) = \mathbb{1}_{\leq N}(n)\sum_{\substack{\varphi_1,\dots,\varphi_5\\ \in\{\cos,\sin\}}}\sum_{n_{12345}=n}\Bigg[\frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle}\Big(\prod_{j=1}^{5}\frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big)\Big(\prod_{j=1,5}\varphi_j(t\langle n_j\rangle)\Big)$$

$$\times\Big(\int_0^t \sin((t-t')\langle n_{234}\rangle)\prod_{j=2,3,4}\varphi_j(t'\langle n_j\rangle)\,dt'\Big)\int_{[0,1]^5} 1\bigotimes_{j=1}^{5} dW_{s_j}^{\varphi_j}(n_j)\Bigg] \quad (6.44)$$

$n_1 = -n_2\ ,\ \varphi_1 = \varphi_2$

$$(t,n) = \mathbb{1}_{\leq N}(n) \sum_{\substack{\varphi_3,\varphi_4,\varphi_5 \\ \in\{\cos,\sin\}}} \sum_{n_{345}=n} \sum_{n_2} \Bigg[\frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle} \frac{\mathbb{1}_{\leq N}(n_2)}{\langle n_2\rangle^2} \left(\prod_{j=3}^{5} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle} \right) \varphi_5(t\langle n_5\rangle)$$
$$\times \left(\int_0^t \sin((t-t')\langle n_{234}\rangle) \prod_{j=3,4} \varphi_j(t'\langle n_j\rangle) \left(\sum_{\varphi\in\{\cos,\sin\}} \varphi(t\langle n_2\rangle)\varphi(t'\langle n_2\rangle) \right) dt' \right)$$
$$\times \int_{[0,1]^3} 1 \bigotimes_{j=3}^{5} dW_{s_j}^{\varphi_j}(n_j) \Bigg]$$
$$= \mathbb{1}_{\leq N}(n) \sum_{\substack{\varphi_3,\varphi_4,\varphi_5 \\ \in\{\cos,\sin\}}} \sum_{n_{345}=n} \sum_{n_2} \Bigg[\frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle} \frac{\mathbb{1}_{\leq N}(n_2)}{\langle n_2\rangle^2} \left(\prod_{j=3}^{5} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle} \right) \varphi_5(t\langle n_5\rangle)$$
$$\times \left(\int_0^t \sin((t-t')\langle n_{234}\rangle) \cos((t-t')\langle n_2\rangle) \prod_{j=3,4} \varphi_j(t'\langle n_j\rangle) \right) \int_{[0,1]^3} 1 \bigotimes_{j=3}^{5} dW_{s_j}^{\varphi_j}(n_j) \Bigg] \tag{6.45}$$



$n_1 = -n_2\ ,\ \varphi_1 = \varphi_2$

$n_5 = -n_4\ ,\ \varphi_5 = \varphi_4$

$$(t,n) = \mathbb{1}_{\leq N}(n) \sum_{\varphi_3\in\{\cos,\sin\}} \sum_{n_3=n} \sum_{n_2,n_4} \Bigg[\frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle} \frac{\mathbb{1}_{\leq N}(n_2)}{\langle n_2\rangle^2} \frac{\mathbb{1}_{\leq N}(n_3)}{\langle n_3\rangle} \frac{\mathbb{1}_{\leq N}(n_4)}{\langle n_4\rangle^2} \Bigg(\int_0^t \sin((t-t')\langle n_{234}\rangle)$$
$$\times \varphi_3(t'\langle n_3\rangle) \left(\sum_{\varphi_2\in\{\cos,\sin\}} \varphi_2(t\langle n_2\rangle)\varphi_2(t'\langle n_2\rangle) \right) \left(\sum_{\varphi_4\in\{\cos,\sin\}} \varphi_4(t\langle n_2\rangle)\varphi_4(t'\langle n_2\rangle) \right) dt' \Bigg)$$
$$\times \int_{[0,1]} 1\, dW_{s_3}^{\varphi_3}(n_3) \Bigg]$$
$$= \mathbb{1}_{\leq N}(n) \sum_{\varphi_3\in\{\cos,\sin\}} \sum_{n_3=n} \sum_{n_2,n_4} \Bigg[\frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle} \frac{\mathbb{1}_{\leq N}(n_2)}{\langle n_2\rangle^2} \frac{\mathbb{1}_{\leq N}(n_3)}{\langle n_3\rangle} \frac{\mathbb{1}_{\leq N}(n_4)}{\langle n_4\rangle^2} \Bigg(\int_0^t \sin((t-t')\langle n_{234}\rangle)$$
$$\times \varphi_3(t'\langle n_3\rangle) \cos((t-t')\langle n_2\rangle) \cos((t-t')\langle n_4\rangle)\, dt' \Bigg) \int_{[0,1]} 1\, dW_{s_3}^{\varphi_3}(n_3) \Bigg] \tag{6.46}$$

Deeper look at (6.46) :

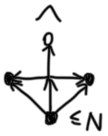
- t-dependent version of $\Gamma_{\leq N}$ (Definition 6.11) :

$$\Gamma_{\leq N}(n,t) = 6 \cdot \mathbb{1}_{\leq N}(n) \sum_{n_{123}=n} \left[\prod_{j=1}^{3} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle^2} \cos(t\langle n_j\rangle) \right] \tag{6.47}$$

Note : $\Gamma_{\leq N}(n,0) = \Gamma_{\leq N}(n)$

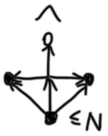
- The renormalized resonant part of the quintic object

Lemma 6.12 For all $N \geq 1$, we have



$$18 \quad (t,n) - \Gamma_{\leq N}(n) \quad (t,n) = -\Gamma_{\leq N}(n,t)\langle n\rangle^{-1} \int_{[0,1]} 1\, dW_s^{\cos}(n)$$
$$- \sum_{\varphi\in\{\cos,\sin\}} \left[\left(\int_0^t \Gamma_{\leq N}(n,t-t')(\partial_t\varphi)(t'\langle n\rangle)\, dt' \right) \int_{[0,1]} 1\, dW_s^{\varphi}(n) \right] \tag{6.48}$$

Proof: By (6.46), letting $n_6 := -n_{234} = -n_{24} - n$, and switching variables $(n_2, n_4, n_6) \to (-n_2, -n_4, -n_6)$, we obtain

18  (t,n)

$$= 18 \cdot \mathbb{1}_{\leq N}(n) \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \sum_{n_{246} = n} \Bigg[\Big(\prod_{j=2,4} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j \rangle^2} \Big) \frac{\mathbb{1}_{\leq N}(n_6)}{\langle n_6 \rangle} \Big(\int_0^t \cos((t-t')\langle n_2 \rangle)$$
$$\times \cos((t-t')\langle n_4 \rangle) \sin((t-t')\langle n_6 \rangle) \varphi(t'\langle n \rangle) \, dt' \Big) \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Bigg]$$

$$= 18 \cdot \mathbb{1}_{\leq N}(n) \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \sum_{n_{246} = n} \Bigg[\Big(\prod_{j=2,4,6} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j \rangle^2} \Big) \Big(\int_0^t \cos((t-t')\langle n_2 \rangle)$$
$$\times \cos((t-t')\langle n_4 \rangle) \partial_{t'} \big(\cos((t-t')\langle n_6 \rangle) \big) \varphi(t'\langle n \rangle) \, dt' \Big) \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Bigg]$$

symmetry of n_2, n_4, n_6
$$= 18 \cdot \frac{\mathbb{1}_{\leq N}(n)}{\langle n \rangle} \sum_{\varphi \in \{\cos, \sin\}} \sum_{n_{246} = n} \Bigg[\Big(\prod_{j=2,4,6} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j \rangle^2} \Big) \Big(\int_0^t \frac{1}{3} \partial_{t'} \Big(\prod_{j=2,4,6} \cos((t-t')\langle n_j \rangle) \Big) \varphi(t'\langle n \rangle) \, dt' \Big)$$
$$\times \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Bigg]$$

by (6.47)
$$= \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \Big[\int_0^t \partial_{t'} \big(\Gamma_{\leq N}(n, t-t') \big) \varphi(t'\langle n \rangle) \, dt' \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Big]$$

integration by parts
$$= \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \Gamma_{\leq N}(n) \varphi(t\langle n \rangle) \int_{[0,1]} 1 \, dW_s^{\varphi}(n) - \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \Gamma_{\leq N}(n,t) \varphi(0) \int_{[0,1]} 1 \, dW_s^{\varphi}(n)$$
$$- \sum_{\varphi \in \{\cos, \sin\}} \Big[\int_0^t \Gamma_{\leq N}(n, t-t') (\partial_t \varphi)(t'\langle n \rangle) \, dt' \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Big]$$

definition of ϱ
$$= \Gamma_{\leq N}(n) \, \hat{\varrho}(t,n) - \Gamma_{\leq N}(n,t) \langle n \rangle^{-1} \int_{[0,1]} 1 \, dW_s^{\cos}(n)$$
$$- \sum_{\varphi \in \{\cos, \sin\}} \Big[\int_0^t \Gamma_{\leq N}(n, t-t') (\partial_t \varphi)(t'\langle n \rangle) \, dt' \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Big] \qquad \square$$

- The resistor (Definition 6.13):

18  $_{\leq N} = 18$  $_{\leq N} := \mathcal{I} \Big[18$  $- \Gamma_{\leq N} \varrho \Big]$ (6.51)

$\mathcal{I}$ = Duhamel

Exact representation of the resistor (Corollary 6.15):

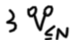
18  $_{\leq N}(t,n)$

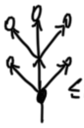
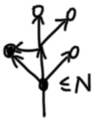
$$= -\langle n \rangle^{-2} \Big(\int_0^t \sin((t-t')\langle n \rangle) \Gamma_{\leq N}(n,t') \, dt' \Big) \int_{[0,1]} 1 \, dW_s^{\cos}(n)$$
$$- \langle n \rangle^{-1} \sum_{\varphi \in \{\cos, \sin\}} \Big[\int_0^t \int_0^{t'} \sin((t-t')\langle n \rangle) \Gamma_{\leq N}(n, t'-t'') (\partial_t \varphi)(t''\langle n \rangle) \, dt'' dt' \int_{[0,1]} 1 \, dW_s^{\varphi}(n) \Big]$$

- The linear x quintic - object $_{\le N} \cdot$ $_{\le N}$

Goal : derive a formula for $\mathcal{C}^{(1,5)}_{\le N} = \mathbb{E}\big[$ $_{\le N}$ $_{\le N}\big]$

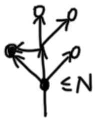
Note : (6.43) and (6.51) give

$$3$$  $$_{\le N} = I\Big[P_{\le N}\Big(3$$  $$_{\le N}$$  $$_{\le N} - \Gamma_{\le N}$$  $$\Big)\Big]$$

$$= 3$$  $$_{\le N} + 18$$  $$_{\le N} + 18$$  $$_{\le N}$$

$$\mathbb{E}\Big[$$  $$_{\le N}$$  $$_{\le N}\Big] = 0$$

$\mathbb{E}\big[I_k[f]\,\overline{I_\ell[g]}\big] = 0$ if $k \neq \ell$

$$\mathbb{E}\Big[$$  $$_{\le N}$$  $$_{\le N}\Big] = 0$$

Need to estimate $\mathbb{E}\Big[$ $_{\le N}$ $_{\le N}\Big]$

Product formula $\Rightarrow$ $I_1[f]\cdot I_1[g] = I_2[f\otimes g] + f\otimes_1 g$

$$\underset{\mathbb{E}\,=\,0}{\downarrow}$$

Lemma 6.19 For all $N \geq 1$, we have

$$3\,\mathcal{C}^{(1,5)}_{\le N}(t) = -\frac{1}{2}\sum_n \langle n\rangle^{-2}\int_0^t\int_0^t \sin\big((t-t')\langle n\rangle\big)\,\sin\big((t-t'')\langle n\rangle\big)\,\Gamma_{\le N}(n,t'-t'')\,dt''\,dt'$$
$$-\sum_n\Big[\langle n\rangle^{-3}\cos(t\langle n\rangle)\int_0^t \sin\big((t-t')\langle n\rangle\big)\,\Gamma_{\le N}(n,t')\,dt'\Big] \qquad (6.59)$$

Proof : Above analysis $\Rightarrow$

$$3\,\mathcal{C}^{(1,5)}_{\le N}(t) = 18\,\mathbb{E}\Big[$$  $$_{\le N}$$  $$_{\le N}\Big]$$

$$= -\sum_n \langle n\rangle^{-3}\cos(t\langle n\rangle)\int_0^t \sin\big((t-t')\langle n\rangle\big)\,\Gamma_{\le N}(n,t')\,dt' \qquad (6.60)$$

by Corollary 6.15

$$-\sum_{\varphi\in\{\cos,\sin\}}\sum_n \langle n\rangle^{-2}\int_0^t\int_0^{t'} \sin\big((t-t')\langle n\rangle\big)\,\varphi(t\langle n\rangle)\,(\partial_t\varphi)(t''\langle n\rangle)\,\Gamma_{\le N}(n,t'-t'')\,dt''\,dt' \qquad (6.61)$$

(6.60) = the second term of (6.59)

For (6.61) ,

$$\sum_{\varphi\in\{\cos,\sin\}} \varphi(t\langle n\rangle)\,(\partial_t\varphi)(t''\langle n\rangle) = -\cos(t\langle n\rangle)\sin(t''\langle n\rangle) + \sin(t\langle n\rangle)\cos(t''\langle n\rangle)$$

$$= \sin\big((t-t'')\langle n\rangle\big)$$

Thus ,

$$(6.61) = -\sum_n \langle n\rangle^{-2} \int_0^t \int_0^{t'} \sin\big((t-t')\langle n\rangle\big)\,\sin\big((t-t'')\langle n\rangle\big)\,\Gamma_{\le N}(n, t'-t'')\,dt''\,dt'$$

$$= -\sum_n \langle n\rangle^{-2} \int_0^t \int_{t''}^{t} \sin\big((t-t')\langle n\rangle\big)\,\sin\big((t-t'')\langle n\rangle\big)\,\Gamma_{\le N}(n, t'-t'')\,dt'\,dt''$$

$$= -\sum_n \langle n\rangle^{-2} \int_0^t \int_{t'}^{t} \sin\big((t-t')\langle n\rangle\big)\,\sin\big((t-t'')\langle n\rangle\big)\,\Gamma_{\le N}(n, t'-t'')\,dt''\,dt'$$

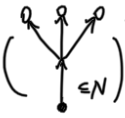
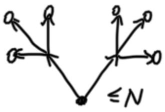
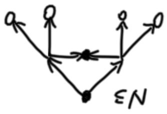
$$\Rightarrow \quad 2\times(6.61) = -\sum_n \langle n\rangle^{-2} \int_0^t \int_0^{t} \sin\big((t-t')\langle n\rangle\big)\,\sin\big((t-t'')\langle n\rangle\big)\,\Gamma_{\le N}(n, t'-t'')\,dt''\,dt' \qquad \square$$

Reading session 6 : The 1533 - cancellation and basic counting estimates

- The cubic × cubic - object $\Psi_{\le N} \cdot \Psi_{\le N}$

  Goal : derive a formula for $\mathcal{C}^{(3,3)}_{\le N} = \mathbb{E}\left[(\Psi_{\le N})^2\right]$

  By the product formula,

$\Big($  $\Big)^2 =$  $+ 9$  $+ 18$  $+ 6$ 

0 pairing
I_6
$\downarrow$
$\mathbb{E} = 0$

1 pairing
I_4
$\downarrow$
$\mathbb{E} = 0$

2 pairings
I_2
$\downarrow$
$\mathbb{E} = 0$

3 pairings
I_0

$\Rightarrow \mathcal{C}^{(3,3)}_{\le N} = 6$ 

Recall from (6.40)

$$\widehat{\Psi}_{\le N}(t,n) = \frac{\mathbb{1}_{\le N}(n)}{\langle n\rangle} \sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}} \sum_{n_{123}=n} \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{\le N}(n_j)}{\langle n_j\rangle}\right)\left(\int_0^t \sin((t-t')\langle n\rangle) \prod_{j=1}^{3} \varphi_j(t'\langle n_j\rangle)\, dt'\right) \int_{[0,1]^3} 1 \bigotimes_{j=1}^{3} dW^{\varphi_j}_{s_j}(n_j)$$

Thus, by letting $n = n_{123}$, we have

$$6 \; \cdots = 6 \sum_{\substack{n,n_1,n_2,n_3\in\mathbb{Z}^3\\ n=n_{123}}} \Bigg[\frac{\mathbb{1}_{\le N}(n)}{\langle n\rangle^2} \prod_{j=1}^{3} \frac{\mathbb{1}_{\le N}(n_j)}{\langle n_j\rangle^2} \Bigg(\int_0^t \int_0^t \sin((t-t')\langle n\rangle)\ \sin((t-t'')\langle n\rangle)$$

$$\Bigg(\sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}} \prod_{j=1}^{3} \varphi_j(t'\langle n_j\rangle)\, \varphi_j(t''\langle n_j\rangle) \Bigg)\, dt''\, dt' \Bigg) \Bigg]$$

$$= 6 \sum_{\substack{n,n_1,n_2,n_3\in\mathbb{Z}^3\\ n=n_{123}}} \Bigg[\frac{\mathbb{1}_{\le N}(n)}{\langle n\rangle^2} \prod_{j=1}^{3} \frac{\mathbb{1}_{\le N}(n_j)}{\langle n_j\rangle^2} \Bigg(\int_0^t \int_0^t \sin((t-t')\langle n\rangle)\ \sin((t-t'')\langle n\rangle)$$

$$\prod_{j=1}^{3} \cos((t'-t'')\langle n_j\rangle)\, dt''\, dt' \Bigg) \Bigg] \qquad (6.63)$$

By the definition of $\Gamma_{\le N}(n,t)$ in (6.47), we obtain :

Lemma 6.20 For all $N \geq 1$, we have

$$\mathcal{C}^{(3,3)}_{\le N}(t) = \sum_{n\in\mathbb{Z}^3} \left[\frac{1}{\langle n\rangle^2} \int_0^t \int_0^t \sin((t-t')\langle n\rangle)\ \sin((t-t'')\langle n\rangle)\ \Gamma_{\le N}(n, t'-t'')\, dt''\, dt' \right]$$

- The 1533 - cancellation

Recall : $\mathcal{C}_{\le N} = 6\,\mathcal{C}^{(1,5)}_{\le N} + \mathcal{C}^{(3,3)}_{\le N}$

Proposition 6.21 For all $N \ge 1$, we have

$$\mathcal{C}_{\le N}(t) = -2 \sum_{n \in \mathbb{Z}^3} \left[\langle n \rangle^{-3} \cos(t\langle n \rangle) \int_0^t \sin((t-t')\langle n \rangle)\, \Gamma_{\le N}(n,t')\, dt' \right]$$

Proof :

$$\begin{aligned} \mathcal{C}_{\le N}(t) &= - \sum_n \langle n \rangle^{-2} \int_0^t \int_0^t \sin((t-t')\langle n \rangle)\, \sin((t-t'')\langle n \rangle)\, \Gamma_{\le N}(n, t'-t'')\, dt''\, dt' \\ &\quad -2 \sum_n \left[\langle n \rangle^{-3} \cos(t\langle n \rangle) \int_0^t \sin((t-t')\langle n \rangle)\, \Gamma_{\le N}(n,t')\, dt' \right] \Bigg\} \begin{matrix} 6\,\mathcal{C}^{(1,5)}_{\le N} \\ \text{Lemma 6.19} \end{matrix} \\ &\quad + \sum_n \left[\langle n \rangle^{-2} \int_0^t \int_0^t \sin((t-t')\langle n \rangle)\, \sin((t-t'')\langle n \rangle)\, \Gamma_{\le N}(n, t'-t'')\, dt''\, dt' \right] \Bigg\} \begin{matrix} \mathcal{C}^{(3,3)}_{\le N} \\ \text{Lemma 6.20} \end{matrix} \\ &= -2 \sum_n \left[\langle n \rangle^{-3} \cos(t\langle n \rangle) \int_0^t \sin((t-t')\langle n \rangle)\, \Gamma_{\le N}(n,t')\, dt' \right] \end{aligned}$$

□

Control of $\mathcal{C}_{\le N}$:

Lemma 6.23 For all $\chi \in C_c^\infty(\mathbb{R})$, we have

$$\| \chi(t)\, \mathcal{C}_{\le N}(t) \|_{H_t^{1-\varepsilon}} \lesssim_\chi 1 .$$

uniform in N

Proof : Relabeling n as n_0 and dyadic decomposition :

$$\mathbb{1}_N(n) = \begin{cases} \mathbb{1}_{\{|n|_\infty \le 1\}} & N=1 \\ \mathbb{1}_{\{\frac{N}{2} < |n|_\infty \le N\}} & \text{else} \end{cases}$$

$$\begin{aligned} \mathcal{C}_{\le N}(t) &= -12 \sum_{\substack{N_0,N_1,N_2,N_3 : \\ N_{max} \le N}} \sum_{\substack{n_0,n_1,n_2,n_3 \in \mathbb{Z}^3 : \\ n_0 = n_{123}}} \Bigg[\left(\prod_{j=0}^3 \mathbb{1}_{N_j}(n_j) \right) \langle n_0 \rangle^{-3} \langle n_1 \rangle^{-2} \langle n_2 \rangle^{-2} \langle n_3 \rangle^{-2} \\ &\qquad \times \cos(t\langle n_0 \rangle) \int_0^t \sin((t-t')\langle n_0 \rangle) \prod_{j=1}^3 \cos(t'\langle n_j \rangle)\, dt' \Bigg] \\ &=: \sum_{\substack{N_0,N_1,N_2,N_3 : \\ N_{max} \le N}} \mathcal{C}[N_0,N_1,N_2,N_3](t) \end{aligned}$$

Definition 6.11 for $\Gamma_{\le N}(n,t)$

Proposition 6.21

WTS

$$| \mathcal{C}[N_0,N_1,N_2,N_3](t) | \lesssim \langle t \rangle\, N_{max}^{-1+\frac{\varepsilon}{2}} \qquad (6.67)$$

$$| \partial_t \mathcal{C}[N_0,N_1,N_2,N_3](t) | \lesssim \langle t \rangle\, N_{max}^{\frac{\varepsilon}{2}} \qquad (6.67')$$

Perform the t'-integral :

$$
\begin{aligned}
&\left|\mathcal{C}[N_0,N_1,N_2,N_3](t)\right| \\
&\lesssim \langle t\rangle N_0^{-3}N_1^{-2}N_2^{-2}N_3^{-2} \sum_{\substack{\pm_0,\pm_1,\\ \pm_2,\pm_3}} \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big)\Big(1+\Big|\sum_{j=0}^{3}(\pm_j)\langle n_j\rangle\Big|\Big)^{-1} \right] \quad \text{level-set decomposition} \\
&\lesssim \langle t\rangle N_0^{-3}N_1^{-2}N_2^{-2}N_3^{-2} \sum_{\substack{\pm_0,\pm_1,\\ \pm_2,\pm_3}} \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \sum_{m\in\mathbb{Z}} \frac{1}{\langle m\rangle} \mathbb{1}\Big\{\Big|\sum_{j=0}^{3}(\pm_j)\langle n_j\rangle - m\Big| \le 1\Big\} \right] \\
&\lesssim \langle t\rangle N_{max}^{\frac{\varepsilon}{2}} N_0^{-3}N_1^{-2}N_2^{-2}N_3^{-2} \sup_{m\in\mathbb{Z}} \sum_{\substack{\pm_0,\pm_1,\\ \pm_2,\pm_3}} \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \mathbb{1}\Big\{\Big|\sum_{j=0}^{3}(\pm_j)\langle n_j\rangle - m\Big| \le 1\Big\} \right]
\end{aligned}
$$

By (5.16) in Lemma 5.4 (shown later) ,

$$
\sup_{m\in\mathbb{Z}} \sum_{\substack{\pm_0,\pm_1,\\ \pm_2,\pm_3}} \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \mathbb{1}\Big\{\Big|\sum_{j=0}^{3}(\pm_j)\langle n_j\rangle - m\Big| \le 1\Big\} \right] \lesssim (N_0N_1N_2N_3)^2 N_{min}N_{max}^{-1}
$$

Thus,

$$
\left|\mathcal{C}[N_0,N_1,N_2,N_3](t)\right| \lesssim \langle t\rangle N_{max}^{\frac{\varepsilon}{2}} N_0^{-1} N_{min} N_{max}^{-1} \lesssim \langle t\rangle N_{max}^{-1+\frac{\varepsilon}{2}} \quad \Longrightarrow \quad (6.67)
$$

Also, we have

$$
\begin{aligned}
&\partial_t \mathcal{C}[N_0,N_1,N_2,N_3](t) \\
&= -\sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \langle n_0\rangle^{-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2} \sin(t\langle n_0\rangle) \int_0^t \sin((t-t')\langle n_0\rangle) \prod_{j=1}^{3} \cos(t'\langle n_j\rangle)\, dt' \right] \\
&\quad + \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_0=n_{123}}} \left[\Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \langle n_0\rangle^{-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2} \cos(t\langle n_0\rangle) \int_0^t \cos((t-t')\langle n_0\rangle) \prod_{j=1}^{3} \cos(t'\langle n_j\rangle)\, dt' \right]
\end{aligned}
$$

Similar steps as above =>

$$
\left|\partial_t \mathcal{C}[N_0,N_1,N_2,N_3](t)\right| \lesssim \langle t\rangle N_{max}^{\frac{\varepsilon}{2}} N_{min} N_{max}^{-1} \lesssim \langle t\rangle N_{max}^{\frac{\varepsilon}{2}} \quad \Longrightarrow \quad (6.67')
$$

Thus,

$$
\begin{aligned}
\left\| \chi(t)\, \mathcal{C}_{\le N}(t) \right\|_{H_t^{1-\varepsilon}} &\le \sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N}} \left\| \chi(t)\,\mathcal{C}[N_0,N_1,N_2,N_3](t) \right\|_{H_t^{1-\varepsilon}} \\
\text{interpolation} \quad &\lesssim \sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N}} \left\| \chi(t)\,\mathcal{C}[N_0,N_1,N_2,N_3](t) \right\|_{L_t^2}^{\varepsilon} \left\| \chi(t)\,\mathcal{C}[N_0,N_1,N_2,N_3](t) \right\|_{H_t^1}^{1-\varepsilon} \\
(6.67)+(6.67') \quad &\lesssim_{\chi} \sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N}} \Big(N_{max}^{-1+\frac{\varepsilon}{2}}\Big)^{\varepsilon} \Big(N_{max}^{\frac{\varepsilon}{2}}\Big)^{1-\varepsilon} \\
&\lesssim \sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N}} N_{max}^{-\frac{\varepsilon}{2}} \lesssim 1
\end{aligned}
$$

□

Some integer lattice counting estimate

- A basic counting lemma

Lemma 5.1 Given dyadic numbers A, N, and $a\in\mathbb{Z}^3$ with $|a|_\infty \sim A$, we have

$$\sup_{m\in\mathbb{Z}} \#\{n\in\mathbb{Z}^3 : |n|_\infty \sim N, |\langle a+n\rangle \pm \langle n\rangle - m| \leq 1\} \lesssim \min(A,N)^{-1}N^3 \tag{5.1}$$

$$\sup_{m\in\mathbb{Z}} \#\{n\in\mathbb{Z}^3 : |n|_\infty \sim N, |\langle a+n\rangle + \langle n\rangle - m| \leq 1\} \lesssim N^2 \tag{5.2}$$

Proof: ℓ^2 and ℓ^∞-norms on $\mathbb{Z}^3$ are comparable $\Rightarrow$ $|a|\sim A$, $|n|\sim N$

$|\langle\xi\rangle - |\xi|| \leq 1$ for all $\xi\in\mathbb{R}^3$ $\Rightarrow$ $\langle\cdot\rangle$ can be replaced by $|\cdot|$

$\xi \mapsto \langle a+\xi\rangle \pm \langle\xi\rangle$ globally Lipschitz $\Rightarrow$

$$\#\{n\in\mathbb{Z}^3 : |n|_\infty \sim N, |\langle a+n\rangle \pm \langle n\rangle - m| \leq 1\}$$

$$\lesssim \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, |\langle a+\xi\rangle \pm \langle\xi\rangle - m| \lesssim 1\}\right)$$

$$\lesssim \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, ||a+\xi| \pm |\xi| - m| \lesssim 1\}\right)$$

Decompose :

$$\mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, ||a+\xi| \pm |\xi| - m| \lesssim 1\}\right)$$

$$\lesssim \sum_{\substack{m_1,m_2\in\mathbb{Z} \\ |m_1\pm m_2 - m|\lesssim 1}} \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, |a+\xi| = m_1 + O(1), |\xi| = m_2 + O(1)\}\right) \tag{$*$}$$

$$\lesssim N \sup_{m_1,m_2\in\mathbb{Z}} \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, |a+\xi| = m_1 + O(1), |\xi| = m_2 + O(1)\}\right)$$

↳ at most $\sim N$ non-trivial choices of m_2

m_2 fixed and $|m_1 \pm m_2 - m| \lesssim 1$ $\Rightarrow$ at most ~ 1 non-trivial choices of m_1

WTS

$$\mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, |a+\xi| = m_1 + O(1), |\xi| = m_2 + O(1)\}\right) \lesssim \min(A,N)^{-1}N^2$$

By rotational invariance of the Lebesgue measure, we can assume $a = |a|e_3$

By polar coordinates $\xi = (r\sin\alpha\cos\beta, r\sin\alpha\sin\beta, r\cos\alpha)$,

$$\mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N, |a+\xi| = m_1 + O(1), |\xi| = m_2 + O(1)\}\right)$$

$$\sim \int_{\{r\sim N\}}\int_0^\pi\int_0^{2\pi} \mathbb{1}\{r = m_2 + O(1)\}\,\mathbb{1}\{\sqrt{|a|^2 + 2r|a|\cos\alpha + r^2} = m_1 + O(1)\}\, r^2\sin\alpha \, d\beta\, d\alpha\, dr$$

$$\lesssim N^2 \int_0^\infty\int_0^\pi \mathbb{1}\{r = m_2 + O(1)\}\,\mathbb{1}\{\sqrt{|a|^2 + 2r|a|\cos\alpha + r^2} = m_1 + O(1)\}\, \sin\alpha\, d\alpha\, dr$$

Since $\sqrt{|a|^2+2r|a|\cos\alpha+r^2} = m_1 + O(1)$ and $|m_1| \lesssim |a| + |\xi| \lesssim \max(A,N)$,

we have $$\cos\alpha = 1 - \frac{(a+r)^2}{2|a|r} + \frac{m_1^2}{2|a|r} + O\left(\max(A,N)\, A^{-1}N^{-1}\right)$$

Thus, for a fixed r, $\cos\alpha$ is contained in an interval of size $\sim \min(A,N)^{-1}$

By a change of variable $\theta \to \cos\theta$, we have

$$N^2 \int_0^\infty \int_0^\pi \mathbb{1}\{r = m_2 + O(1)\}\, \mathbb{1}\{\sqrt{|a|^2+2r|a|\cos\alpha+r^2} = m_1 + O(1)\} \sin\alpha \, d\alpha\, dr$$
$$\lesssim \min(A,N)^{-1} N^2 \int_0^\infty \mathbb{1}\{r = m_2 + O(1)\}\, dr$$
$$\lesssim \min(A,N)^{-1} N^2 \qquad \Rightarrow \quad (5.1)$$

For (5.2), case $A \gtrsim N$ follows from (5.1)

Assume $A \ll N$

Main difference is at $(*)$:

$$\mathrm{Leb}\left(\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ \big||a+\xi| + |\xi| - m\big| \lesssim 1\}\right)$$
$$\lesssim \sum_{\substack{m_1,m_2\in\mathbb{Z} \\ |m_1+m_2-m|\lesssim 1 \\ |m_1-m_2|\lesssim |a|\sim A}} \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N,\ |a+\xi| = m_1 + O(1),\ |\xi| = m_2 + O(1)\}\right)$$
$$\lesssim A \sup_{m_1,m_2\in\mathbb{Z}} \mathrm{Leb}\left(\{\xi\in\mathbb{R}^3 : |\xi|\sim N,\ |a+\xi| = m_1 + O(1),\ |\xi| = m_2 + O(1)\}\right)$$

↳ at most $\sim A$ non-trivial choices of $m_1 - m_2$

at most ~ 1 non-trivial choices of $m_1 + m_2$

All other steps are the same as those for (5.1) □

- Lattice point counting I

Lemma 5.4 Given $q = 2, 3,$ or 4, $\pm_j \in \{\pm\}$ and dyadic numbers $N_j \geq 1$ $(1 \leq j \leq q)$, and $(n_{ex}, m) \in \mathbb{Z}^3 \times \mathbb{Z}$, consider the set

$$\mathcal{M}_q = \left\{(n_1,\dots,n_q)\in(\mathbb{Z}^3)^q : \langle n_j\rangle \sim N_j,\ \sum_{j=1}^q (\pm_j) n_j = n_{ex},\ \left|\sum_{j=1}^q (\pm_j)\langle n_j\rangle - m\right| \leq 1\right\}.$$

Assume $\langle n_{ex}\rangle \sim M$, and $n_{ex} = 0$ $(M = 1)$ when $q = 4$. Let $N^{(1)} \geq \dots \geq N^{(q)}$ be a decreasing rearrangement of N_j, and define $\pm^{(j)}$ correspondingly.

(1) If $q=2$, we have

$$\#\mathcal{M}_2 \lesssim \begin{cases} \max\left((N^{(2)})^2, (N^{(2)})^3 M^{-1}\right) & \text{if } \pm^{(1)} = \mp^{(2)}; \\ (N^{(2)})^2 & \text{if } \pm^{(1)} = \pm^{(2)}. \end{cases} \tag{5.10}$$

If moreover $\pm_1 = \mp_2$ and n_1, n_2 satisfy the Γ-condition:

$$\text{either } |n_1|_\infty \leq \Gamma \leq |n_2|_\infty \text{ or } |n_2|_\infty \leq \Gamma \leq |n_1|_\infty, \tag{5.5}$$

then we have

$$\#\mathcal{M}_2 \lesssim (N^{(2)})^2 M. \tag{5.11}$$

(2) If $q=3$, we have

$$\#\mathcal{M}_3 \lesssim (N^{(2)})^3 (N^{(3)})^3 \left(\mathrm{med}(N^{(2)}, N^{(3)}, M)\right)^{-1} \lesssim (N^{(2)})^3 (N^{(3)})^2. \tag{5.12}$$

(3) If $q=4$, we have

$$\#\mathcal{M}_4 \lesssim (N^{(2)})^3 (N^{(3)})^2 (N^{(4)})^3. \tag{5.13}$$

If moreover $|(\pm_1) n_1 + (\pm_2) n_2| \lesssim L$, then we have

$$\#\mathcal{M}_4 \lesssim L (N_1 N_2 N_3 N_4)^2 \left(\max(N_1, N_2)\right)^{-1} \left(\max(N_3, N_4)\right)^{-1} \tag{5.14}$$

(4) Summarizing (5.10), (5.12), (5.13), we have

$$\#\mathcal{M}_q \lesssim (N_1 \cdots N_q)^2 \cdot (N^{(1)})^{-1} \quad \text{if } q \leq 3; \tag{5.15}$$

$$\#\mathcal{M}_q \lesssim (N_1 \cdots N_q)^2 \cdot N^{(4)} (N^{(1)})^{-1} \quad \text{if } q = 4. \tag{5.16}$$

Proof: Except for (5.14), we assume $N_1 \gtrsim \cdots \gtrsim N_q$, so $N^{(j)} = N_j$

(1) (5.10) follows from Lemma 5.1 by letting $a = n_{ex} \mp_1 n_1$, N replaced by N_2, and A replaced by N.

For (5.11), WLOG $|(n_1)'| \geq \Gamma \gtrsim |(n_2)'|$ or $|(n_2)'| \geq \Gamma \gtrsim |(n_1)'|$ $(\cdot)'$ first coordinate

$$\overset{n_1 = n_2 \pm n_{ex}}{\Longrightarrow} |(n_2)'| \in [\Gamma - O(M), \Gamma + O(M)]$$

$$\Longrightarrow (n_2)' \text{ has } \lesssim M \text{ choices}$$

$$\Longrightarrow n_2 \text{ has } \lesssim N_2^2 M \text{ choices}$$

(2) Let $|n_{ex} - (\pm_3) n_3| \sim R$.

For fixed n_3, $\#(n_1, n_2) \lesssim N_2^3 \min(N_2, R)^{-1}$ by (5.10)

$\# n_3 \lesssim \min(N_3, R)^3$

If $M \lesssim N_3$, then $R \lesssim N_3 \leqslant N_2$ and $\mathrm{med}(N_2, N_3, M) \sim N_3$, so

$$\# M_3 \lesssim \sum_{R \lesssim N_3} R^3 \cdot N_2^3 R^{-1} \leqslant N_2^3 N_3^2 .$$

If $M \gg N_3$, then $R \sim M$ and $\mathrm{med}(N_2, N_3, M) \sim \min(N_2, R)$, so

$$\# M_3 \lesssim N_3^3 \cdot N_2^3 \min(N_2, R)^{-1} .$$

Either case $\Rightarrow$ (5.12)

(3) (5.13) follows from (5.12) by fixing n_4 ($\sim N_4^3$ choices)

For (5.14), WLOG $N_1 \geqslant N_2$ and $N_3 \geqslant N_4$

Fix the value of $(\pm_1) n_1 + (\pm_2) n_2$, which has $\lesssim L^3$ choices

Use (5.10) separately $\Rightarrow$

$$\begin{aligned} \# M_4 &\lesssim L^3 \cdot (N_2^2 + N_2^3 L^{-1})(N_4^2 + N_4^3 L^{-1}) \\ &\lesssim L^3 \cdot (N_1 N_2^2 L^{-1})(N_3 N_4^2 L^{-1}) \\ &= L (N_1 N_2 N_3 N_4)^2 (N_1 N_3)^{-1} \quad \Rightarrow \quad (5.14) \end{aligned}$$

$N_1 \gtrsim \max(N_2, L)$

$N_3 \gtrsim \max(N_4, L)$

(4) (5.15) and (5.16) follow directly from (5.10), (5.12), (5.13) $\square$

Reading session 7 : A reduction argument of tensor estimates

- Tensor and p-moment estimates reductions (Section 5.7)

We focus on a bilinear estimate

$0 < b - \frac{1}{2} < b_+ - \frac{1}{2} \ll 1$

$$X^{\frac{1}{2},b} \times X^{\frac{1}{2},b} \longrightarrow X^{-\frac{1}{2},b_+-1}, \quad (w_2, w_3) \mapsto P_{N_0}[\rho_{N_1} P_{N_2} w_2 P_{N_3} w_3] \tag{5.109}$$

By letting

$$\widetilde{w}_j^{+}(\lambda_j, n_j) = \mathbb{1}_{[-\langle n_j\rangle, +\infty)}(\lambda_j)\, \widehat{w}_j(\lambda_j + \langle n_j\rangle, n_j)$$

$$\widetilde{w}_j^{-}(\lambda_j, n_j) = \mathbb{1}_{(-\infty, \langle n_j\rangle)}(\lambda_j)\, \widehat{w}_j(\lambda_j - \langle n_j\rangle, n_j),$$

we can write

$$w_j(t,x) = \sum_{\pm_j} \sum_{n_j \in \mathbb{Z}^3} \int_{\mathbb{R}} e^{i(\pm_j\langle n_j\rangle + \lambda_j)t} e^{i\langle n_j, x\rangle}\, \widetilde{w}_j^{\pm_j}(\lambda_j, n_j)\, d\lambda_j \tag{5.110}$$

and we have

$$\max_{\pm_j} \| \langle \lambda_j\rangle^{b} \langle n_j\rangle^{\frac{1}{2}} \widetilde{w}_j^{\pm_j}(\lambda_j, n_j) \|_{L^2_{\lambda_j} \ell^2_{n_j}} \sim \|w_j\|_{X^{\frac{1}{2},b}}$$

To match (5.110), we write

$$\rho = \sum_{\pm_1} \sum_{n_1 \in \mathbb{Z}^3} \frac{g_{n_1}^{\pm}}{\langle n_1\rangle} e^{\pm_1 it\langle n_1\rangle} e^{i\langle n_1, x\rangle}, \tag{5.111}$$

where $\{g_{n_1}^{\pm}\}$ are i.i.d standard Gaussians

The cubic tensor ;

$$h_{n_0 n_1 n_2 n_3}(t, \lambda_1, \lambda_2, \lambda_3) := \mathbb{1}_{\{n_0 = n_{123}\}} \cdot \mathbb{1}_{N_0}(n_0) \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right) \cdot \chi(t)\, e^{i(\pm_1\langle n_1\rangle \pm_2\langle n_2\rangle \pm_3\langle n_3\rangle + \lambda_1 + \lambda_2 + \lambda_3)t}$$

$$\widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) = \mathbb{1}_{\{n_0 = n_{123}\}} \cdot \mathbb{1}_{N_0}(n_0) \cdot \widehat{\chi}(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \Omega) \cdot \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right),$$

where $\Omega = \pm_0\langle n_0\rangle \pm_1\langle n_1\rangle \pm_2\langle n_2\rangle \pm_3\langle n_3\rangle$.

Equipped with (5.110) and (5.111), we can write

$$\begin{aligned}
\langle\nabla\rangle^{-\frac{1}{2}} P_{N_0}[\rho_{N_1} P_{N_2} w_2 P_{N_3} w_3] &= \sum_{\pm_1,\pm_2,\pm_3} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{n_0, n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}_{\{n_0 = n_{123}\}} \cdot \mathbb{1}_{N_0}(n_0) \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right) \\
&\qquad \times \langle n_0\rangle^{-\frac{1}{2}} \langle n_2\rangle \langle n_3\rangle \cdot e^{i(\pm_1\langle n_1\rangle \pm_2\langle n_2\rangle \pm_3\langle n_3\rangle + \lambda_2 + \lambda_3)t} \cdot e^{i\langle n_0, x\rangle} \\
&\qquad \times g_{n_1}^{\pm_1} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \Big]\, d\lambda_3\, d\lambda_2 \\
&= \sum_{\pm_1,\pm_2,\pm_3} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{n_0, n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\langle n_0\rangle^{-\frac{1}{2}} \langle n_2\rangle \langle n_3\rangle\, h_{n_0 n_1 n_2 n_3}(t, 0, \lambda_2, \lambda_3) \cdot e^{i\langle n_0, x\rangle} \\
&\qquad \times g_{n_1}^{\pm_1} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \Big]\, d\lambda_3\, d\lambda_2,
\end{aligned}$$

so that

$$\mathcal{F}\Big(\langle\nabla\rangle^{-\frac{1}{2}} P_{N_0}\big[\varrho_{N_1} P_{N_2} w_2 P_{N_3} w_3\big]\Big)(\lambda \pm_0 \langle n_0\rangle, n_0)$$
$$= \sum_{\pm_1,\pm_2,\pm_3} \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\langle n_0\rangle^{-\frac{1}{2}} \langle n_2\rangle \langle n_3\rangle\, \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)$$
$$\times\, g_{n_1}^{\pm_1} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3)\Big]\, d\lambda_3\, d\lambda_2 .$$

We have

$$\Big\| \langle\lambda\rangle^{b_+-1} \mathcal{F}\Big(\langle\nabla\rangle^{-\frac{1}{2}} P_{N_0}\big[\varrho_{N_1} P_{N_2} w_2 P_{N_3} w_3\big]\Big)(\lambda \pm_0 \langle n_0\rangle, n_0)\Big\|_{L^2_\lambda \ell^2_{n_0}}$$
$$\lesssim \int_{\mathbb{R}^2} \Big\| \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\langle\lambda\rangle^{b_+-1} \langle n_0\rangle^{-\frac{1}{2}} \langle n_2\rangle\langle n_3\rangle\, \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3) \cdot g_{n_1}^{\pm_1} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3)\Big]\Big\|_{L^2_\lambda \ell^2_{n_0}} d\lambda_2 d\lambda_3$$
$$\lesssim N_0^{-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} \cdot \sup_{\lambda_2,\lambda_3\in\mathbb{R}} \Big[\big(\langle\lambda_2\rangle\langle\lambda_3\rangle\big)^{-(b_- - \frac{1}{2})} \Big\| \sum_{n_1\in\mathbb{Z}^3} \langle\lambda\rangle^{b_+-1} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|_{L^2_\lambda(\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0})}\Big]$$
$$\times \int_{\mathbb{R}^2} \prod_{j=2}^{3} \langle\lambda_j\rangle^{b_- - \frac{1}{2}} \big\| \langle n_j\rangle^{\frac{1}{2}} |\widetilde{w}_j^{\pm_j}(\lambda_j, n_j)| \big\|_{\ell^2_{n_j}} d\lambda_2 d\lambda_3 \qquad 0 < b_- - \tfrac{1}{2} \ll b - \tfrac{1}{2} \ll b_+ - \tfrac{1}{2} \ll 1$$
$$\lesssim \sup_{\lambda_2,\lambda_3\in\mathbb{R}} \Big[\big(\langle\lambda_2\rangle\langle\lambda_3\rangle\big)^{-(b_- - \frac{1}{2})} \Big\| \sum_{n_1\in\mathbb{Z}^3} \langle\lambda\rangle^{b_+-1} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|_{L^2_\lambda(\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0})}\Big]$$
$$\times\, N_0^{-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} \cdot \prod_{j=2}^{3} \|w_j\|_{X^{\frac{1}{2},b}} \qquad \text{Cauchy-Schwarz in } \lambda_2 \text{ and } \lambda_3$$

It suffices to control

$$\Big\| \big(\langle\lambda_2\rangle\langle\lambda_3\rangle\big)^{-(b_- - \frac{1}{2})} \Big\| \sum_{n_1\in\mathbb{Z}^3} \langle\lambda\rangle^{b_+-1} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|_{L^2_\lambda(\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0})} \Big\|_{L^\infty_{\lambda_2,\lambda_3}} \qquad (5.114)$$

Let $q = (b_- - \frac{1}{2})^{-5}$. By Sobolev embedding,

$$\|F(\lambda, \lambda_2, \lambda_3)\|_{L^\infty_{\lambda_2,\lambda_3}} \lesssim \|F(\lambda, \lambda_2, \lambda_3)\|_{L^q_{\lambda_2,\lambda_3}} + \|\nabla_{\lambda_2,\lambda_3} F(\lambda, \lambda_2, \lambda_3)\|_{L^q_{\lambda_2,\lambda_3}}$$

Note : any $(\lambda, \lambda_2, \lambda_3)$ derivative of $\widetilde{h}$ satisfies the same estimates as h itself

$\Rightarrow$ suffices to control the $\|F(\lambda, \lambda_2, \lambda_3)\|_{L^q_{\lambda_2,\lambda_3}}$ term

Take $p > q$, we apply Minkowski's inequality to get

$$\mathbb{E}\Big[\,|(5.114)|^p\Big]^{1/p} \lesssim \Big\| \big(\langle\lambda_2\rangle\langle\lambda_3\rangle\big)^{-(b_- - \frac{1}{2})} \Big\| \sum_{n_1\in\mathbb{Z}^3} \langle\lambda\rangle^{b_+-1} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|_{\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0}} \Big\|_{L^p_\omega L^q_{\lambda_2,\lambda_3} L^2_\lambda}$$
$$\lesssim \Big\| \big(\langle\lambda_2\rangle\langle\lambda_3\rangle\big)^{-(b_- - \frac{1}{2})} \Big\| \sum_{n_1\in\mathbb{Z}^3} \langle\lambda\rangle^{b_+-1} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|_{\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0}} \Big\|_{L^q_{\lambda_2,\lambda_3} L^2_\lambda L^p_\omega}$$
$$\overset{\text{Hölder}}{\lesssim} \sup_{\lambda_2,\lambda_3} \Big\| \langle\lambda\rangle^{b_+-1} \Big(\mathbb{E} \Big\| \sum_{n_1\in\mathbb{Z}^3} \widetilde{h}_{n_0 n_1 n_2 n_3}(\lambda, 0, \lambda_2, \lambda_3)\, g_{n_1}^{\pm_1} \Big\|^p_{\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0}} \Big)^{1/p} \Big\|_{L^2_\lambda}$$
$$\overset{\text{level-set decomposition}}{\lesssim} N_1^{-1} N_2^{-1} N_3^{-1} \sup_{\lambda_2,\lambda_3} \Big\| \langle\lambda\rangle^{b_+-1} A_1(\lambda, 0, \lambda_2, \lambda_3) \sup_{m\in\mathbb{Z}} \Big(\mathbb{E} \Big\| \sum_{n_1\in\mathbb{Z}^3} h^{b,m}_{n_0 n_1 n_2 n_3}\, g_{n_1}^{\pm_1} \Big\|^p_{\ell^2_{n_2}\times \ell^2_{n_3} \to \ell^2_{n_0}} \Big)^{1/p} \Big\|_{L^2_\lambda} ,$$

where

$$h^{b,m}_{nn_1n_2n_3} := \mathbb{1}_N(n) \cdot \prod_{j=1}^{3} \mathbb{1}_{N_j}(n_j) \cdot \mathbb{1}\{n = n_{123}\} \cdot \mathbb{1}\{|\Omega - m| \leq 1\} \quad \text{base tensor} \quad (5.23)$$

$$A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) := \sum_{\substack{m' \in \mathbb{Z} \\ |m'| \leq N_{max}}} |\hat{\chi}(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - m')| \quad (5.40)$$

In the proof of Lemma 5.9, by letting $\Lambda := \lambda_1 + \lambda_2 + \lambda_3$,

$$|A_1| \lesssim \sum_{m' \in \mathbb{Z}} \frac{1}{\langle \lambda - \Lambda - m' \rangle^{1+}} \lesssim 1$$

$$|A_1| \lesssim \sum_{\substack{m' \in \mathbb{Z} \\ |m'| \leq N_{max}}} \frac{1}{\langle \lambda - \Lambda - m' \rangle} \lesssim \sum_{\substack{m' \in \mathbb{Z} \\ |m'| \leq N_{max}}} \frac{\langle m' \rangle}{\langle \lambda - \Lambda \rangle} \lesssim N_{max}^2 \cdot \langle \lambda - \Lambda \rangle^{-1}$$

$$\Rightarrow |A_1| \lesssim \min(1, N_{max}^2 \cdot \langle \lambda - \Lambda \rangle^{-1}) \quad (5.41)$$

$$\Rightarrow \| \langle \lambda \rangle^{b_+ - 1} A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{L^2_\lambda} \lesssim N_{max}^{3(b_+ - \frac{1}{2})} \cdot \| \langle \lambda \rangle^{b_+ - 1} \langle \lambda - \Lambda \rangle^{-\frac{3}{2}(b_+ - \frac{1}{2})} \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon}$$

Thus, (5.109) is reduced to

$$N_{max}^{\varepsilon} N_0^{-\frac{1}{2}} N_1^{-1} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \mathbb{E}\Big[\Big\| \sum_{n_1 \in \mathbb{Z}^3} h^{b,m}_{n_0n_1n_2n_3} g_{n_1} \Big\|^p_{\ell^2_{n_2} \times \ell^2_{n_3} \to \ell^2_{n_0}} \Big]^{1/p}$$

Additional remarks:

Consider

$$X^{-1,b} \times X^{-1,b} \to L^\infty_T C^{-\frac{1}{2}}_x, \quad (w_2, w_3) \mapsto P_{N_0}[\ell_{N_1} P_{N_{23}}(P_{N_2} w_2 P_{N_3} w_3)]$$

We have

$$\langle \nabla \rangle^{-\frac{1}{2}} P_{N_0}[\ell_{N_1} P_{N_{23}}(P_{N_2} w_2 P_{N_3} w_3)]$$
$$= \sum_{\pm_1, \pm_2, \pm_3} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{n_0, n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}\{n_0 = n_{123}\} \cdot \Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \Big) \mathbb{1}_{N_{23}}(n_{23}) \langle n_0 \rangle^{-\frac{1}{2}} e^{i\langle n_0, x \rangle}$$
$$\times \frac{g^{\pm_1}_{n_1}}{\langle n_1 \rangle} e^{i(\pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle + \lambda_2 + \lambda_3)t} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \Big] \, d\lambda_3 \, d\lambda_2 ,$$

so that

$$\| \langle \nabla \rangle^{-\frac{1}{2}} P_{N_0}[\ell_{N_1} P_{N_{23}}(P_{N_2} w_2 P_{N_3} w_3)] \|_{L^\infty_T L^\infty_x}$$
$$= \Big\| \sum_{\pm_1, \pm_2, \pm_3} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{n_0, n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}\{n_0 = n_{123}\} \cdot \Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \Big) \mathbb{1}_{N_{23}}(n_{23}) \langle n_0 \rangle^{-\frac{1}{2}} e^{i\langle n_0, x \rangle}$$
$$\times \frac{g^{\pm_1}_{n_1}}{\langle n_1 \rangle} e^{i(\pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle + \lambda_2 + \lambda_3)t} \cdot \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \Big] \, d\lambda_3 \, d\lambda_2 \Big\|_{L^\infty_T L^\infty_x}$$
$$\overset{\text{Hölder}}{\lesssim} \sup_{\pm_j} \sup_{n_2, n_3} \sup_{\lambda_2, \lambda_3} \Big\| \sum_{n_1 \in \mathbb{Z}^3} \mathbb{1}_{N_0}(n_{123}) \mathbb{1}_{N_1}(n_1) \langle n_{123} \rangle^{-\frac{1}{2}} \frac{g^{\pm_1}_{n_1}}{\langle n_1 \rangle} e^{i\langle n_{123}, x \rangle} e^{i(\pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle + \lambda_2 + \lambda_3)t} \Big\|_{L^\infty_T L^\infty_x}$$
$$\times \sup_{\pm_j} \Big\| \sum_{n_2, n_3 \in \mathbb{Z}^3} \mathbb{1}_{N_{23}}(n_{23}) \mathbb{1}_{N_2}(n_2) \mathbb{1}_{N_3}(n_3) \, \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \cdot \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \Big\|_{L^1_{\lambda_2} L^1_{\lambda_3}}$$

If $\max(N_2, N_3) \leq N_1^{\eta}$, the second term can be bounded by

$$N_1^{10\eta} \| \langle n_2 \rangle^{-1} \widetilde{w}_2^{\pm_2}(\lambda_2, n_2) \|_{L^1_{\lambda_2} \ell^2_{n_2}} \| \langle n_3 \rangle^{-1} \widetilde{w}_3^{\pm_3}(\lambda_3, n_3) \|_{L^1_{\lambda_3} \ell^2_{n_3}}$$

$$\lesssim N_1^{10\eta} \| w_2 \|_{X^{-1,b}} \| w_3 \|_{X^{-1,b}}$$

Thus, to estimate

$$\mathbb{E}\Big[\| (w_2, w_3) \mapsto P_{N_0}[\varrho_{N_1} P_{N_{23}}(P_{N_2} w_2 P_{N_3} w_3)] \|^p_{X^{-1,b} \times X^{-1,b} \to L^\infty_T C_x^{-\frac{1}{2}}} \Big],$$

by Sobolev embedding $W_t^{s,p} \hookrightarrow L_t^\infty$, $W_x^{s,p} \hookrightarrow L_x^\infty$ and Gaussian hypercontractivity, we only need to estimate

$$\sup_t \sup_{\pm_j} \sup_{n_2, n_3} \sup_{\lambda_2, \lambda_3} \mathbb{E}\Big[\Big| \sum_{n_1 \in \mathbb{Z}^3} \mathbb{1}_{N_0}(n_{123}) \mathbb{1}_{N_1}(n_1) \langle n_{123} \rangle^{-\frac{1}{2}} \frac{g_{n_1}^{\pm_1}}{\langle n_1 \rangle^{1-\varepsilon}} e^{i\langle n_{123}, x \rangle} e^{i(\pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle + \lambda_2 + \lambda_3)t} \Big|^2 \Big]$$

Reading session 8 : Bilinear random operators

- Bilinear random operators (Section 8)
  - Tensors (Subsection 5.2)

Definition 5.6 A tensor $h = h_{n_A} : (\mathbb{Z}^d)^A \to \mathbb{C}$.

For (B, C) a partition of A, we define

$$\|h\|^2_{n_B \to n_C} = \sup \Big\{ \sum_{n_C} \Big| \sum_{n_B} h_{n_A} \cdot z_{n_B} \Big|^2 : \sum_{n_B} |z_{n_B}|^2 = 1 \Big\}$$

$$\overset{\text{duality}}{=} \sup \Big\{ \Big| \sum_{n_B, n_C} h_{n_A} \cdot z_{n_B} \cdot y_{n_C} \Big| : \sum_{n_B} |z_{n_B}|^2 = \sum_{n_C} |y_{n_C}|^2 = 1 \Big\}$$

Notation : $\|h_{n_A}(\lambda)\|_{L^2_\lambda (n_B \to n_C)} = \big\| \|h_{n_A}(\lambda)\|_{n_B \to n_C} \big\|_{L^2_\lambda}$

Lemma B.1 (Merging estimates, Proposition 4.11 in Deng-Nahmod-Yue '22)

Let $h^{(1)}_{k_{A_1}}$ and $h^{(2)}_{k_{A_2}}$, with $A_1 \cap A_2 = C$, $A_1 \triangle A_2 = A$. Define the semi-product

$$H_{k_A} = \sum_{k_C} h^{(1)}_{k_{A_1}} h^{(2)}_{k_{A_2}} .$$

Then, for any partition (X, Y) of A with $X_1 = X \cap A_1$, $Y_1 = Y \cap A_1$,

$X_2 = X \cap A_2$, $Y_2 = Y \cap A_2$, we have

$$\|H\|_{k_X \to k_Y} \leq \|h^{(1)}\|_{k_{X_1 \cup C} \to k_{Y_1}} \|h^{(2)}\|_{k_{X_2} \to k_{C \cup Y_2}}$$

Proposition B.2 (Moment method, Proposition 4.14 in Deng-Nahmod-Yue '22)

A, $\mathcal{X}$, $\mathcal{Y}$ disjoint finite index sets. $h = h_{n_A n_{\mathcal{X}} n_{\mathcal{Y}}}$ deterministic tensor.

$|n_j| \leq N$ for all $j \in A \cup \mathcal{X} \cup \mathcal{Y}$, $(\pm_j)_{j \in A} \in \{+,-\}^A$, Define

$$H_{n_{\mathcal{X}} n_{\mathcal{Y}}} = \sum_{n_A} h_{n_A n_{\mathcal{X}} n_{\mathcal{Y}}} \, SI[n_j, \pm_j : j \in A]$$

SI : Subsection 2.4
(can be viewed as Gaussian)

Then, for all $\delta > 0$ and $p \geqslant 1$, we have

$$\big\| \|H_{n_{\mathcal{X}} n_{\mathcal{Y}}}\|_{n_{\mathcal{X}} \to n_{\mathcal{Y}}} \big\|_{L^p_\omega} \lesssim_\delta N^\delta p^{\#A/2} \max_{B, C} \|h_{n_A n_{\mathcal{X}} n_{\mathcal{Y}}}\|_{n_B n_{\mathcal{X}} \to n_C n_{\mathcal{Y}}} ,$$

where the max is over all partitions of A.

Recall from Subsection 5.7 (Note 7) that we care about

$$\mathbb{E}\Big[\Big\| \sum_{n_1} (h^b)_{n_0 n_1 n_2 n_3} g_{n_1} \Big\|^p_{\ell^2_{n_2} \times \ell^2_{n_3} \to \ell^2_{n_0}} \Big]^{1/p}$$

with $\quad h^b_{n_0 n_1 n_2 n_3} := \prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \cdot \mathbb{1}\{n_0 = n_{123}\} \cdot \mathbb{1}\{ |\Omega - m| \leq 1 \}$, $\Omega = \sum_{j=0}^{3} (\pm_j) \langle n_j \rangle$, $m \in \mathbb{R}$

Lemma 5.7 (Base tensors estimates)

(1) Let $J \subseteq \{0,1,2,3\}$ with $\# J = 3$. Then,

$$\|h^b\|^2_{n_0 n_1 n_2 n_3} \lesssim \Big(\operatorname*{med}_{j \in J} N_j \Big)^{-1} \prod_{j \in J} N_j^3 \tag{5.25}$$

$$\|h^b\|^2_{n_0 n_1 n_2 n_3} \lesssim (N_0 N_1 N_2 N_3)^2 \cdot \frac{\min(N_0, N_1, N_2, N_3)}{\max(N_0, N_1, N_2, N_3)} \tag{5.26}$$

(2) Let $\{j_1, j_2, j_3, j_4\} = \{0,1,2,3\}$, $J \subseteq \{j_1, j_2, j_3\}$ with $\# J = 2$. Then,

$$\|h^b\|^2_{n_{j_1} n_{j_2} n_{j_3} \to n_{j_4}} \lesssim \Big(\operatorname*{med}_{j \in J \cup \{j_4\}} N_j \Big)^{-1} \prod_{j \in J} N_j^3 \lesssim \Big(\min_{j \in J} N_j \Big)^{-1} \prod_{j \in J} N_j^3 \tag{5.27}$$

$$\|h^b\|^2_{n_{j_1} n_{j_2} n_{j_3} \to n_{j_4}} \lesssim (N_{j_1} N_{j_2} N_{j_3})^2 \cdot \Big(\max_{j \in \{j_1, j_2, j_3\}} N_j \Big)^{-1} \tag{5.28}$$

(3) Let $|n_{j_1 j_2}| = |n_{j_3 j_4}| \sim N_{j_1 j_2} = N_{j_3 j_4}$. Then

$$\|h^b\|^2_{n_{j_1} n_{j_2} \to n_{j_3} n_{j_4}} \lesssim \min(N_{j_1}, N_{j_1 j_2})^{-1} \min(N_{j_3}, N_{j_3 j_4})^{-1} (N_{j_1} N_{j_3})^3 \tag{5.29}$$

Proof: (5.25) follows from (5.13) $\Big\}$ Lemma 5.4 (Note 5)

(5.26) follows from (5.16)

Schur's test : $\|h^b\|^2_{n_{j_1} n_{j_2} n_{j_3} \to n_{j_4}} \leq \sup_{n_{j_4}} \|h^b\|_{n_{j_1} n_{j_2} n_{j_3}} \sup_{n_{j_1}, n_{j_2}, n_{j_3}} \|h^b\|_{n_{j_4}} \leq \sup_{n_{j_4}} \|h^b\|_{n_{j_1} n_{j_2} n_{j_3}}$

(5.27) follows from (5.12) $\Big\}$ Lemma 5.4 (Note 5)

(5.28) follows from (5.15)

To prove (5.29), WLOG $(j_1, j_2, j_3, j_4) = (1,2,3,0)$, $N_2 = \max(N_1, N_2) \leq \max(N_3, N_0) = N_0$

Since $x \mapsto \min(x,y)^{-1} x^3$ monotone increasing in $x \geq 1$, it suffices to prove

$$\| \mathbb{1}_{\{|n_{12}| \sim N_{12}\}} h^b \|^2_{n_1 n_2 \to n_3 n_0} \lesssim \min(N_1, N_{12})^{-1} \min(N_3, N_{03})^{-1} (N_1 N_3)^3$$

Schur's test :

$$\| \mathbb{1}_{\{|n_{12}| \sim N_{12}\}} h^b \|^2_{n_1 n_2 \to n_3 n_0} \lesssim \Big(\sup_{n_0, n_3} \sum_{\substack{n_1, n_2 \\ n_{0123} = 0}} \mathbb{1}_{\{|n_{03}| \sim N_{03}\}} h^b \Big) \quad ①$$

$$\times \Big(\sup_{n_1, n_2} \sum_{\substack{n_0, n_3 \\ n_{0123} = 0}} \mathbb{1}_{\{|n_{12}| \sim N_{12}\}} h^b \Big) \quad ②$$

$$① \lesssim \sup_{m' \in \mathbb{Z}} \sup_{\substack{n_0, n_3 \\ |n_{03}| \sim N_{03}}} \sum_{|n_1| \sim N_1} \mathbb{1}\{ | \pm_1 \langle n_1 \rangle \pm_2 \langle n_{013} \rangle - m' | \leq 1 \} \overset{\text{Lm 5.1 (Note 5)}}{\lesssim} \min(N_{12}, N_1)^{-1} N_1^3$$

$$② \lesssim \sup_{m' \in \mathbb{Z}} \sup_{\substack{n_1, n_2 \\ |n_{12}| \sim N_{12}}} \sum_{|n_3| \sim N_3} \mathbb{1}\{ | \pm_0 \langle n_{123} \rangle \pm_3 \langle n_3 \rangle - m' | \leq 1 \} \overset{\text{Lm 5.1 (Note 5)}}{\lesssim} \min(N_{03}, N_3)^{-1} N_3^3 \qquad \square$$

Lemma 5.2 (A box-counting lemma)

Given dyadic numbers A, N, $a \in \mathbb{Z}^3$ with $|a|_\infty \sim A$, and $\zeta \in \mathbb{Z}^3$, we have

$$\sup_{m \in \mathbb{Z}} \#\{n \in \mathbb{Z}^3 : |n|_\infty \sim N,\ |n-\zeta|_\infty \leq A,\ |\langle a+n\rangle \pm \langle n\rangle - m| \leq 1\} \lesssim N^2.$$

Proof: Same as the proof of Lemma 5.1 (Note 5), but with following

$$\mathrm{Leb}\left(\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ |\xi-\zeta| \lesssim A,\ \big||a+\xi| + |\xi| - m\big| \lesssim 1\}\right)$$

$$\lesssim \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ |m_1+m_2-m| \lesssim 1 \\ |m_2 - |\zeta|| \lesssim A}} \mathrm{Leb}\left(\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ |a+\xi| = m_1 + O(1),\ |\xi| = m_2 + O(1)\}\right)$$

$$\lesssim A \sup_{m_1, m_2 \in \mathbb{Z}} \mathrm{Leb}\left(\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ |a+\xi| = m_1 + O(1),\ |\xi| = m_2 + O(1)\}\right) \qquad \square$$

- Bilinear operator

$$(w_2, w_3) \in X^{\frac{1}{2},b} \times X^{\frac{1}{2},b} \mapsto P_{\leq N}[\rho_{\leq N} P_{\leq N} w_2 P_{\leq N} w_3] \in X^{-\frac{1}{2}, b_+ - 1} \quad (b = \tfrac{1}{2}+) \qquad (8.1)$$

Abstract setting:

$$B : \ell^2 \times \ell^2 \to \ell^2, \qquad B(v,w)_a = \sum_{b,c,d \in \mathbb{Z}^D} h_{abcd}\, g_b v_c w_d \quad (D \geq 1)$$

Remark: $\|B\|_{\ell^2 \times \ell^2 \to \ell^2} \leq \left\| \sum_{b \in \mathbb{Z}^D} h_{abcd}\, g_b \right\|_{a \to cd}$

Proposition B.2 =>

$$\mathbb{E}\left[\left\| \sum_{b \in \mathbb{Z}^D} h_{abcd}\, g_b \right\|_{a \to cd}^p\right]^{1/p} \lesssim (\#\,\mathrm{supp}\, h)^\varepsilon \max\Big(\underset{\underset{\text{power can be improved}}{\downarrow}}{\|h\|_{a \to bcd}},\ \underset{\underset{\text{good}}{\downarrow}}{\|h\|_{ab \to cd}}\Big)$$

Lemma 8.1 For all $\varepsilon > 0$ and $p \geq 2$,

$$\mathbb{E}\left[\|B\|_{\ell^2 \times \ell^2 \to \ell^2}^p\right]^{1/p} \lesssim_\varepsilon (\#\,\mathrm{supp}\, h)^\varepsilon \max\Big(\|h\|_{ad \to bc} \|h\|_{ac \to bd},\ \|h\|_{ad \to bc} \overset{\overset{\text{power } \frac{1}{2}}{\uparrow}}{\|h\|_{abc \to d}},$$
$$\|h\|_{ac \to bd} \overset{\overset{\text{power } \frac{1}{2}}{\uparrow}}{\|h\|_{abd \to c}},\ \|h\|_{ab \to cd}^2\Big)^{1/2} p^{1/2}$$

Proof:

$$\|B(v,w)\|_{\ell^2}^2 = \sum_a \Big| \sum_{b,c,d} h_{abcd}\, g_b v_c w_d \Big|^2$$

$$= \sum_{c,c',d,d'} \underbrace{\Big(\sum_{a,b,b'} h_{abcd} \overline{h_{ab'c'd'}}\, g_b \overline{g_{b'}} \Big)}_{=:\, B_{cc'dd'}} v_c \overline{v_{c'}}\, w_d \overline{w_{d'}}$$

Thus,

$$\begin{aligned} \|B\|^2_{\ell^2\times\ell^2\to\ell^2} &= \sup_{\|v\|_{\ell^2}\le 1}\ \sup_{\|w\|_{\ell^2}\le 1} \|B(v,w)\|^2_{\ell^2} \\ &= \sup_{\|v\|_{\ell^2}\le 1}\ \sup_{\|w\|_{\ell^2}\le 1} \sum_{c,c',d,d'} B_{cc'dd'}\, v_c \overline{v_{c'}}\, w_d \overline{w_{d'}} \end{aligned} \qquad (8.4)$$

We decompose

$$\begin{aligned} B_{cc'dd'} &= \sum_{a,b,b'} h_{abcd} \overline{h_{ab'c'd'}} \left(g_b \overline{g_{b'}} - \delta_{b=b'}\right) + \sum_{a,b,b'} h_{abcd} \overline{h_{ab'c'd'}}\, \delta_{b=b'} \\ &=: B^{(2)}_{cc'dd'} + B^{(0)}_{cc'dd'} \end{aligned}$$

For $B^{(2)}$:

$$\begin{aligned} \left| \sum_{c,c',d,d'} B^{(2)}_{cc'dd'}\, v_c \overline{v_{c'}}\, w_d \overline{w_{d'}} \right| &\le \|B^{(2)}_{cc'dd'}\|_{cc'\to dd'} \|v_c \overline{v_{c'}}\|_{cc'} \|w_d \overline{w_{d'}}\|_{dd'} \\ &\le \|B^{(2)}_{cc'dd'}\|_{cc'\to dd'} \end{aligned} \qquad (8.7)$$

By Proposition B.2,

$$\begin{aligned} \mathbb{E}\left[\|B^{(2)}_{cc'dd'}\|^p_{cc'\to dd'}\right]^{1/p} \lesssim (\#\operatorname{supp} h)^{\varepsilon} \max\Big(&\|\textstyle\sum_a h_{abcd}\overline{h_{ab'c'd'}}\|_{bb'cc'\to dd'}, && ① \\ &\|\textstyle\sum_a h_{abcd}\overline{h_{ab'c'd'}}\|_{bcc'\to b'dd'}, && ② \quad (8.8) \\ &\|\textstyle\sum_a h_{abcd}\overline{h_{ab'c'd'}}\|_{b'cc'\to bdd'}, && ③ \\ &\|\textstyle\sum_a h_{abcd}\overline{h_{ab'c'd'}}\|_{cc'\to bb'dd'} \Big)^{1/2} \cdot p^{1/2} && ④ \end{aligned}$$

$$\left.\begin{aligned} ① &\overset{\text{Lm B.1}}{\lesssim} \|h_{abcd}\|_{abc\to d} \|\overline{h_{ab'c'd'}}\|_{b'c'\to ad'} = \|h\|_{ad\to bc}\|h\|_{abc\to d} \\ ② &\lesssim \|h_{abcd}\|_{bc\to ad} \|\overline{h_{ab'c'd'}}\|_{ac'\to b'd'} = \|h\|_{ad\to bc}\|h\|_{ac\to bd} \\ ③ &\lesssim \|h_{abcd}\|_{ac\to bd} \|\overline{h_{ab'c'd'}}\|_{b'c'\to ad'} = \|h\|_{ad\to bc}\|h\|_{ac\to bd} \\ ④ &\lesssim \|h_{abcd}\|_{ac\to bd} \|\overline{h_{ab'c'd'}}\|_{c'\to ab'd'} = \|h\|_{ac\to bd}\|h\|_{abd\to c} \end{aligned}\right\}\ \text{good}$$

For $B^{(0)}$:

$$\begin{aligned} \left| \sum_{c,c',d,d'} B^{(0)}_{cc'dd'}\, v_c \overline{v_{c'}}\, w_d \overline{w_{d'}} \right| &\le \|B^{(0)}_{cc'dd'}\|_{cd\to c'd'} \|v_c w_d\|_{cd} \|\overline{v_{c'}}\,\overline{w_{d'}}\|_{c'd'} \\ &\le \|B^{(0)}_{cc'dd'}\|_{cd\to c'd'} \end{aligned} \qquad (8.12)$$

By Lemma B.1,

$$\begin{aligned} \|B^{(0)}_{cc'dd'}\|_{cd\to c'd'} &= \|\textstyle\sum_{a,b} h_{abcd}\overline{h_{abc'd'}}\|_{cd\to c'd'} \\ &\le \|h_{abcd}\|_{cd\to ab} \|\overline{h_{abc'd'}}\|_{ab\to c'd'} \\ &= \|h\|^2_{ab\to cd} \quad \to \text{good} \end{aligned}$$

$\square$

Proposition 8.3 (Bilinear random operator)

Let $p \geq 2$, N_0, N_1, N_2, N_3 frequency-scales with $N_2, N_3 \ll N_1$, $T \geq 1$, $0 \in J \subseteq [-T, T]$ closed interval. Then, we have

$$\mathbb{E}\Big[\sup_J \big\| (w_2, w_3) \mapsto P_{N_0}[\,?_{N_1} P_{N_2} w_2 P_{N_3} w_3] \big\|^p_{X^{\frac{1}{2},b}(J) \times X^{\frac{1}{2},b}(J) \to X^{-\frac{1}{2}, b_+ - 1}(J)}\Big]^{1/p}$$

$$\lesssim p^{\frac{1}{2}} T^{\alpha} N_{max}^{\varepsilon} \big(N_1^{-\frac{1}{4}} + \max(N_2, N_3)^{-\frac{1}{3}}\big).$$

Proof: Using the reduction in Subsection 5.7 (see Note 7), we need to show

$$N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \mathbb{E}\Big[\big\| (\hat{w}_2, \hat{w}_3) \mapsto \sum_{n_1, n_2, n_3} h_{n_0 n_1 n_2 n_3} g_{n_1} \hat{w}_2(n_2) \hat{w}_3(n_3) \big\|^p_{\ell^2 \times \ell^2 \to \ell^2}\Big]^{1/p}$$

$$\lesssim p^{\frac{1}{2}} N_{max}^{\varepsilon} \big(N_1^{-\frac{1}{4}} + \max(N_2, N_3)^{-\frac{1}{3}}\big)$$

with $h_{n_0 n_1 n_2 n_3} = \Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \mathbb{1}_{\{n_0 = n_{123}\}} \mathbb{1}_{\{|\Omega - m| \leq 1\}}$, $\Omega = \sum_{j=0}^{3} (\pm_j) \langle n_j \rangle$, $m \in \mathbb{Z}$

If $N_2 \sim N_3$, we further decompose $\mathbb{1}_{N_j}(n_j) = \sum_{Q_j} \mathbb{1}_{Q_j}(n_j)$ for $j = 2, 3$, Q_2 and Q_3 have radius $\sim N_{23} \lesssim N_2 \sim N_3$

Orthogonality: $\sum_{Q_2, Q_3} \| h\, P_{Q_2} w_2 P_{Q_3} w_3 \|_{\ell^2} \leq \sum_{Q_2, Q_3} \| P_{Q_2} w_2 \|_{\ell^2} \| P_{Q_3} w_3 \|_{\ell^2} \|h\|_{\ell^2 \times \ell^2 \to \ell^2}$

Q₃ corresponds to O(1) many Q₂

Cauchy-Schwarz in Q₂

$\lesssim \|h\|_{\ell^2 \times \ell^2 \to \ell^2}$

$$h_{n_0 n_1 n_2 n_3} = \Big(\mathbb{1}_{\{N_2 \not\sim N_3\}} + \mathbb{1}_{\{N_2 \sim N_3\}} \mathbb{1}_{N_{23}}(n_{23}) \mathbb{1}_{Q_2}(n_2) \mathbb{1}_{Q_3}(n_3)\Big)$$
$$\times \Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j)\Big) \mathbb{1}_{\{n_0 = n_{123}\}} \mathbb{1}_{\{|\Omega - m| \leq 1\}}$$

By Lemma 8.1, we only need to show

$$N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \max\Big(\overset{①}{\|h\|_{n_0 n_3 \to n_1 n_2} \|h\|_{n_0 n_2 \to n_1 n_3}},\ \overset{②}{\|h\|_{n_0 n_3 \to n_1 n_2} \|h\|_{n_0 n_1 n_2 \to n_3}},\ \overset{③}{\|h\|_{n_0 n_2 \to n_1 n_3} \|h\|_{n_0 n_1 n_3 \to n_2}},$$
$$\overset{④}{\|h\|^2_{n_0 n_1 \to n_2 n_3}}\Big)^{1/2} \lesssim N_1^{-\frac{1}{4}} + \max(N_2, N_3)^{-\frac{1}{3}}$$

$$\left.\begin{aligned}
&① \overset{(5.29)}{\lesssim} N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \cdot (N_2 N_3)^{\frac{1}{2}} \cdot (N_2 N_3)^{\frac{1}{2}} \lesssim N_1^{-\frac{1}{2}} \\
&② \overset{(5.27)+(5.29)}{\lesssim} N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \cdot (N_2 N_3)^{\frac{1}{2}} \cdot (N_1^{\frac{3}{2}} N_2)^{\frac{1}{2}} \lesssim N_1^{-\frac{1}{4}} \\
&③ \overset{(5.27)+(5.29)}{\lesssim} N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \cdot (N_2 N_3)^{\frac{1}{2}} \cdot (N_1^{\frac{3}{2}} N_3)^{\frac{1}{2}} \lesssim N_1^{-\frac{1}{4}}
\end{aligned}\right\} \text{no need to decompose } N_{23}$$

④ (i) $N_2 \not\sim N_3$

$$N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \|h\|_{n_0 n_1 \to n_2 n_3}$$

$$\overset{(5.29)}{\lesssim} N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \cdot \max(N_2, N_3)^{-\frac{1}{2}} N_1^{\frac{3}{2}} \min(N_2, N_3)$$

$$\lesssim \max(N_2, N_3)^{-\frac{1}{2}}$$

(ii) $N_2 \sim N_3$

Schur's test : $\|h\|_{n_0n_1\to n_2n_3} \le \Big(\sup_{n_2,n_3}\sum_{n_0,n_1}|h|\Big)^{1/2}\Big(\sup_{n_0,n_1}\sum_{n_2,n_3}|h|\Big)^{1/2}$

↓ (5.1) (Note 5) ↓ Lemma 5.2 and $n_2\in Q_2$

$\lesssim N_{23}^{-\frac{1}{2}}N_1^{\frac{3}{2}}$ $\lesssim \min(N_2, N_{23}^{\frac{3}{2}}) \le N_2^{\frac{2}{3}}N_{23}^{\frac{1}{2}}$

$\lesssim N_1^{\frac{3}{2}}N_2^{\frac{2}{3}}$

Thus, $N_1^{-\frac{3}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}\|h\|_{n_0n_1\to n_2n_3} \lesssim N_2^{-\frac{1}{3}}$ □

Lemma 8.5 (Variant of the bilinear estimate) (to be used in Corollary 10.9)

Let $T\ge1$, $p\ge2$, $N_0, N_1, N_2, N_3, N_{12}$ frequency scales satisfying

$$N_{max}\sim N_2,\quad N_{12}\gtrsim N_2^{1-\frac{1}{100}},\quad \text{and}\quad N_0\lesssim N_2^{\frac{1}{100}}. \tag{8.18}$$

Then, we have ($0\in J\subseteq[-T,T]$ closed interval)

$$\mathbb{E}\Big[\sup_J \big\|(w_2,w_3)\mapsto P_{N_0}\big[P_{N_{12}}(\varphi_{N_1}P_{N_2}w_2)P_{N_3}w_3\big]\big\|^p_{X^{\frac{1}{2},b}(J)\times X^{0,b}(J)\to X^{-\frac{1}{2},b_+-1}(J)}\Big]^{1/p} \lesssim p^{1/2}T^{\alpha}N_2^{-\frac{1}{8}} \tag{8.19}$$

Proof : As in the proof of Proposition 8.3, it suffices to show

$$N_0^{-1}N_1^{-2}N_2^{-1}\max\Big(\overset{①}{\|h\|_{n_0n_3\to n_1n_2}\|h\|_{n_0n_2\to n_1n_3}},\ \overset{②}{\|h\|_{n_0n_3\to n_1n_2}\|h\|_{n_0n_1n_2\to n_3}},\ \overset{③}{\|h\|_{n_0n_2\to n_1n_3}\|h\|_{n_0n_1n_3\to n_2}},\ \overset{④}{\|h\|^2_{n_0n_1\to n_2n_3}}\Big) \lesssim N_2^{-\frac{1}{4}-\varepsilon},$$

where $h_{n_0n_1n_2n_3} := \mathbb{1}_{N_{12}}(n_{12})\mathbb{1}_{N_{23}}(n_{23})\mathbb{1}_{N_{13}}(n_{13})\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\Big(\prod_{j=2}^{3}\mathbb{1}_{Q_j}(n_j)\Big)\mathbb{1}_{\{n_0=n_{123}\}}\mathbb{1}_{\{|\Omega-m|\le1\}}$

$\Omega := \sum_{j=0}^{3}(\pm_j)\langle n_j\rangle$, $m\in\mathbb{Z}$, Q_2, Q_3 boxes of sidelength $\sim N_{23}$

① : $\|h\|^2_{n_0n_3\to n_1n_2} \overset{(5.29)}{\lesssim} \min(N_0,N_{12})^{-1}N_0^3\min(N_1,N_{12})^{-1}N_1^3 \lesssim N_0^2N_1^{2+\frac{1}{100}}$ $(N_{12}\gtrsim N_2^{1-\frac{1}{100}}\gtrsim N_1^{1-\frac{1}{100}})$

$\|h\|^2_{n_0n_2\to n_1n_3} \overset{(5.29)}{\lesssim} \min(N_0,N_{13})^{-1}N_0^3\min(N_1,N_{13})^{-1}N_1^3 \lesssim N_0^2N_1^2$ $(N_{13}\sim|n_0-n_2|\sim N_2\gtrsim N_0,N_1)$

$\Rightarrow N_0^{-1}N_1^{-2}N_2^{-1}\times ① \lesssim N_0N_1^{\frac{1}{200}}N_2^{-1} \lesssim N_2^{-1+\frac{3}{200}}$

② : $\|h\|^2_{n_0n_3\to n_1n_2} \overset{\text{above}}{\lesssim} N_0^2N_1^{2+\frac{1}{100}}$

$\|h\|^2_{n_0n_1n_2\to n_3} \overset{(5.27)}{\lesssim} \text{med}(N_0,N_1,N_3)^{-1}N_0^3N_1^3 \lesssim N_0^3N_1^{2+\frac{1}{100}}$

$N_0 \ll N_{12} \Rightarrow N_{12}\sim N_3$

(i) $N_1\le N_0\le N_3$: $\lesssim N_0^2N_1^3 \lesssim N_0^3N_1^2$

(ii) $N_0\le N_1\le N_3$: $\lesssim N_0^3N_1^2$

(iii) $N_0\le N_3\le N_1$: $\lesssim N_0^3N_1^3N_{12}^{-1} \overset{\text{above}}{\lesssim} N_0^3N_1^3N_1^{-1+\frac{1}{100}}$

$\Rightarrow N_0^{-1}N_1^{-2}N_2^{-1}\times ② \lesssim N_0^{\frac{3}{2}}N_1^{\frac{1}{100}}N_2^{-1} \lesssim N_2^{-1+\frac{5}{200}}$

$$\textcircled{3} : \quad \|h\|^2_{n_0n_2\to n_1n_3} \overset{\text{above}}{\lesssim} N_0^2 N_1^2$$

$$\|h\|^2_{n_0n_1n_3\to n_2} \overset{(5.27)}{\lesssim} \operatorname{med}(N_0, N_1, N_2)^{-1} N_0^3 N_1^3 \lesssim N_0^3 N_1^2$$

$$\Rightarrow N_0^{-1} N_1^{-2} N_2^{-1} \times \textcircled{3} \lesssim N_0^{\frac{3}{2}} N_2^{-1} \lesssim N_2^{-1+\frac{3}{200}}$$

$$\textcircled{4} : \quad \|h\|^2_{n_0n_1\to n_2n_3} \overset{\text{Schur's test}}{\lesssim} \Big(\sup_{n_2,n_3}\sum_{n_0,n_1}|h|\Big)\Big(\sup_{n_0,n_1}\sum_{n_2,n_3}|h|\Big)$$

$$\begin{array}{ll} \downarrow (5.1)\ (\text{Note } 5) & \downarrow \text{Lemma 5.2 (Note 8) and } n_2\in Q_2 \\ \lesssim \min(N_0, N_{23})^{-1} N_0^3 & \lesssim \min(N_2^2, N_{23}^3) \end{array}$$

$$\lesssim N_0^3 \min(N_{23}^3, N_2^2)$$

Since $N_{23} \lesssim \max(N_0, N_1)$, we obtain

$$N_0^{-1} N_1^{-2} N_2^{-1} \times \textcircled{4} \lesssim N_0^2 N_1^{-2} N_2^{-1} \min\big(N_2^2, \max(N_0, N_1)^3\big)$$

$$\lesssim N_0^2 N_1^{-2} N_2^{-1} (N_2^2)^{\frac{1}{3}} \big(\max(N_0, N_1)^3\big)^{\frac{2}{3}}$$

$$\lesssim N_0^4 N_2^{-\frac{1}{3}} \lesssim N_2^{-\frac{1}{3}+\frac{1}{25}} \qquad \square$$

Reading session 9 : Regularity estimates of basic stochastic objects

- Analytic aspects of basic stochastic diagrams (Section 7)
  - The renormalization constant and multiplier (Subsection 7.1)

Recall from Definition 6.2 (Note 2) :

$$\Gamma_{\leq N}(n) = 6 \cdot \mathbb{1}_{\leq N}(n) \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3 \\ n_{123} = n}} \Big[\prod_{j=1}^{3} \mathbb{1}_{\leq N}(n_j) \langle n_j \rangle^{-2} \Big]$$

$$\gamma_{\leq N} = \Gamma_{\leq N}(0)$$

Lemma 7.1 (Estimate of $\gamma_{\leq N} - \Gamma_{\leq N}$)

For all frequency-scales N and all $n \in \mathbb{Z}^3$, we have

$$\big| \gamma_{\leq N} - \Gamma_{\leq N}(n) \big| \lesssim \langle n \rangle^{\varepsilon} . \qquad (7.1)$$

Proof: By convolution inequality,

$$\begin{aligned} \Gamma_{\leq N}(n) &= 6 \cdot \mathbb{1}_{\leq N}(n) \sum_{n_1, n_2 \in \mathbb{Z}^3} \frac{\mathbb{1}_{\leq N}(n_1)}{\langle n_1 \rangle^2} \frac{\mathbb{1}_{\leq N}(n_2)}{\langle n_2 \rangle^2} \frac{\mathbb{1}_{\leq N}(n-n_1-n_2)}{\langle n-n_1-n_2 \rangle^2} \\ &\lesssim \mathbb{1}_{\leq N}(n) \sum_{n_1 \in \mathbb{Z}^3} \frac{\mathbb{1}_{\leq N}(n_1)}{\langle n_1 \rangle^2} \frac{1}{\langle n - n_1 \rangle} \\ &\lesssim \log(N) \qquad \text{independent of } n \end{aligned}$$

Thus, (7.1) holds for $|n| \gtrsim N$.

We now consider $|n| \ll N$. We decompose

$$\begin{aligned} &\Gamma_{\leq N}(n) - \gamma_{\leq N} \\ &= 6 \sum_{N_1, N_2, N_3 \leq N} \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}\{n_{123} = n\} - \mathbb{1}\{n_{123} = 0\} \Big(\prod_{j=1}^{3} \mathbb{1}_{N_j}(n_j) \langle n_j \rangle^{-2} \Big) \Big] \\ &= 6 \sum_{N_1, N_2, N_3 \leq N} \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}\{n_{123} = 0\} \Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j) \langle n_j \rangle^{-2} \Big) \Big(\mathbb{1}_{N_3}(n_3+n) \langle n_3+n \rangle^{-2} - \mathbb{1}_{N_3}(n_3) \langle n_3 \rangle^{-2} \Big) \Big] \\ &\overset{\text{Young's convolution}}{\lesssim} \sum_{N_1, N_2, N_3 \in N} \big\| \mathbb{1}_{N_1}(n_1) \langle n_1 \rangle^{-2} \big\|_{\ell^\infty_{n_1}} \big\| \mathbb{1}_{N_2}(n_2) \langle n_2 \rangle^{-2} \big\|_{\ell^1_{n_2}} \big\| \mathbb{1}_{N_3}(n_3+n) \langle n_3+n \rangle^{-2} - \mathbb{1}_{N_3}(n_3) \langle n_3 \rangle^{-2} \big\|_{\ell^1_{n_3}} \\ &\overset{\text{Mean Value Thm}}{\lesssim} \sum_{N_1, N_2, N_3 \in N} N_1^{-2} N_2 \langle n \rangle \qquad (\text{WLOG assume } N_1 \geq N_2 \geq N_3) \\ &\lesssim \sum_{N_1 \in N} N_1^{-1+\varepsilon} \langle n \rangle \qquad (|n| = |n_{123}| \lesssim N_1) \\ &\lesssim \langle n \rangle^{\varepsilon} \end{aligned}$$

□

Recall from Definition 6.11 (Note 4):

$$\Gamma_{\leq N}(n,t) = 6 \cdot \mathbb{1}_{\leq N}(n) \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^3 \\ n_{123}=n}} \left[\prod_{j=1}^{3} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j \rangle^2} \cos(t\langle n_j \rangle) \right] \tag{6.47}$$

Definition 7.2 (Dyadic components of time-dependent renormalization multiplier)

For all frequency-scales N_0, N_1, N_2, N_3, we define

$$\Gamma[N_*](n_0,t) := 6 \cdot \mathbb{1}_{N_0}(n_0) \sum_{\substack{n_0,n_1,n_2,n_3 \in \mathbb{Z}^3 \\ n_0=n_{123}}} \left[\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle^2} \cos(t\langle n_j \rangle) \right] \tag{7.4}$$

From (6.47) and (7.4), we have

$$\Gamma_{\leq N}(n_0,t) = \sum_{N_0,N_1,N_2,N_3 \leq N} \Gamma[N_*](n_0,t)$$

Lemma 7.3 (Estimate of $\Gamma[N_*]$)

For all frequency-scales N_0, N_1, N_2, N_3, $n_0 \in \mathbb{Z}^3$, $t \in \mathbb{R}$, and $\lambda \in \mathbb{R}$, we have

$$\left| \int_0^t \Gamma[N_*](n_0, t-t') e^{i\lambda t'} dt' \right| \lesssim \langle t \rangle \log(N_{max}) \max(N_{max}, \langle \lambda \rangle)^{-1}. \tag{7.5}$$

Furthermore, for all $\chi \in C_c^\infty(\mathbb{R})$, we have

$$\left| \int_{\mathbb{R}} \chi(t) \Gamma[N_*](n_0,t) e^{i\lambda t} dt \right| \lesssim_\chi \log(N_{max}) \max(N_{max}, \langle \lambda \rangle)^{-1}. \tag{7.6}$$

Proof: We only prove (7.5), since (7.6) is similar.

By taking the t'-integral and using a level-set decomposition,

$$\begin{aligned} &\left| \int_0^t \Gamma[N_*](n_0, t-t') e^{i\lambda t'} dt' \right| \\ &\lesssim \langle t \rangle N_1^{-2} N_2^{-2} N_3^{-2} \sum_{m \in \mathbb{Z}} \sum_{\pm_1,\pm_2,\pm_3} \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^3: \\ n_0=n_{123}}} \left[\left(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \right) \mathbb{1}\left\{ \left| \sum_{j=1}^{3} (\pm_j) \langle n_j \rangle - m \right| \leq 1 \right\} (1+|\lambda-m|)^{-1} \right] \\ &\lesssim \langle t \rangle N_1^{-2} N_2^{-2} N_3^{-2} \left(\sum_{\substack{m \in \mathbb{Z}: \\ |m| \lesssim N_{max}}} (1+|\lambda-m|)^{-1} \right) \\ &\quad \times \sup_{m \in \mathbb{Z}} \sum_{\pm_1,\pm_2,\pm_3} \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^3: \\ n_0=n_{123}}} \left[\left(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \right) \mathbb{1}\left\{ \left| \sum_{j=1}^{3} (\pm_j) \langle n_j \rangle - m \right| \leq 1 \right\} \right] \end{aligned}$$

We can estimate

$$\sum_{\substack{m \in \mathbb{Z}: \\ |m| \lesssim N_{max}}} (1+|\lambda-m|)^{-1} \lesssim \begin{cases} \log(N_{max}) & \text{if } \langle \lambda \rangle \lesssim N_{max} \\ N_{max} \langle \lambda \rangle^{-1} & \text{if } \langle \lambda \rangle \gg N_{max} \end{cases} \lesssim \log(N_{max}) \min(1, N_{max} \langle \lambda \rangle^{-1}) \tag{7.5a}$$

By (5.12) in Lemma 5.4 (Note 5), we have

$$N_1^{-2} N_2^{-2} N_3^{-2} \sum_{\pm_1,\pm_2,\pm_3} \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^3: \\ n_0=n_{123}}} \left[\left(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \right) \mathbb{1}\left\{ \left| \sum_{j=1}^{3} (\pm_j) \langle n_j \rangle - m \right| \leq 1 \right\} \right] \lesssim N_{max}^{-1} \tag{7.5b}$$

(7.5a) x (7.5b) $\Longrightarrow$ (7.5) $\square$

- The linear and cubic stochastic objects (Subsection 7.2)

Lemma 7.4 (Regularity of linear evolution)

For any $T \geq 1$ and $p \geq 2$, we have

$$\mathbb{E}\Big[\sup_N \|\varrho_{\leq N}\|^p_{(L^\infty_t C_x^{-\frac{1}{2}-\varepsilon} \cap X^{-\frac{1}{2}-\varepsilon, b})([-T,T])} \Big]^{1/p} \lesssim p^{1/2} T^\alpha . \qquad (7.7)$$

Proof: By standard $X^{s,b}$-estimate, we have

$$\sup_N \|\varrho_{\leq N}\|_{X^{-\frac{1}{2}-\varepsilon, b}([-T,T])} \lesssim T^\alpha \|O\|_{H_x^{-\frac{1}{2}-\varepsilon} \times H_x^{-\frac{1}{2}-\varepsilon}} \qquad O : \text{initial data} \qquad (7.7a)$$

Gaussian hypercontractivity : $\mathbb{E}\Big[\|O\|^p_{H_x^{-\frac{1}{2}-\varepsilon} \times H_x^{-\frac{1}{2}-\varepsilon}} \Big]^{1/p} \lesssim p^{1/2} \, \mathbb{E}\Big[\|O\|^2_{H_x^{-\frac{1}{2}-\varepsilon} \times H_x^{-\frac{1}{2}-\varepsilon}} \Big]^{1/2}$

For the $L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}$ norm, we write

$$\sup_N \|\varrho_{\leq N}\|_{L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}} \leq \sup_N \sum_{K \leq N} \|\varrho_K\|_{L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}} \leq \sum_{K \text{ dyadic}} \|\varrho_K\|_{L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}}$$

Let $q = q(\varepsilon) \gg 1$ and use Sobolev embedding:

$$\begin{aligned} \|\varrho_K\|_{L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}([-T,T] \times \mathbb{T}^3)} &\lesssim \| \langle \nabla \rangle_t^{\frac{1}{q}} \langle \nabla \rangle_x^{-\frac{1}{2}-\varepsilon+\frac{3}{q}} \varrho_K \|_{L^q_t L^q_x([-T,T] \times \mathbb{T}^3)} \\ &\lesssim K^{\frac{4}{q}} \| \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \varrho_K \|_{L^q_t L^q_x([-T,T] \times \mathbb{T}^3)} \end{aligned}$$

Thus, for $p \geq q$

$$\begin{aligned} \mathbb{E}\Big[\|\varrho_K\|^p_{L^\infty_t C_x^{-\frac{1}{2}-\varepsilon}([-T,T] \times \mathbb{T}^3)} \Big]^{1/p} &\lesssim K^{\frac{4}{q}} \| \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \varrho_K \|_{L^q_t L^q_x L^p_\omega([-T,T] \times \mathbb{T}^3 \times \Omega)} && \text{(Minkowski)} \\ \text{(Gaussian hypercontractivity)} \quad &\lesssim p^{\frac{1}{2}} K^{\frac{4}{q}} \| \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \varrho_K \|_{L^q_t L^q_x L^2_\omega([-T,T] \times \mathbb{T}^3 \times \Omega)} \\ \text{(spatial translation-invariance)} \quad &= p^{\frac{1}{2}} K^{\frac{4}{q}} \| \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \varrho_K \|_{L^q_t L^2_x L^2_\omega([-T,T] \times \mathbb{T}^3 \times \Omega)} \\ \text{(Minkowski)} \quad &\lesssim p^{\frac{1}{2}} K^{\frac{4}{q}} \mathbb{E}\Big[\| \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \varrho_K \|^2_{L^q_t L^2_x([-T,T] \times \mathbb{T}^3)} \Big]^{1/2} \\ (X^{0,b} \hookrightarrow L^\infty_t L^2_x) \quad &\lesssim p^{\frac{1}{2}} T^{\frac{1}{q}} K^{\frac{4}{q}} \, \mathbb{E}\Big[\|\varrho_K\|^2_{X^{-\frac{1}{2}-\varepsilon, b}([-T,T])} \Big]^{1/2} \end{aligned}$$

We conclude by choosing $q = 8\varepsilon^{-1}$, (7.7a), and summing over dyadic K. $\square$

Lemma 7.5 (Regularity of cubic random object)

For any $T \geq 1$ and $p \geq 2$, we have

$$\mathbb{E}\Big[\sup_{N} \|\Psi_{\in N}\|^{p}_{(L^\infty_t C^{-\varepsilon}_x \cap X^{-\varepsilon, b_+})([-T,T])}\Big]^{p} \lesssim p^{\frac{3}{2}} T^{\alpha}.$$

Proof: We only consider $T=1$ (general case minor modification).

We only prove the $X^{-\varepsilon,b_+}$-estimate ($L^\infty_t C^{-\varepsilon}_x$ similar as in Lemma 7.4).

Similar to (6.40) (Note 4), we have

$$\Psi_{\in N} = \sum_{\substack{\varphi_1,\varphi_2,\varphi_3 \\ \in\{\cos,\sin\}}} \sum_{\substack{n_0,n_1,n_2,n_3 \in \mathbb{Z}^3 \\ n_{123}=n_0}} \Bigg[\Big(\prod_{j=0}^{3} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j\rangle}\Big) e^{i\langle n_0, x\rangle}$$

$$\times \Big(\int_0^t \sin((t-t')\langle n\rangle) \prod_{j=1}^{3} \varphi_j(t'\langle n_j\rangle)\, dt'\Big) \int_{[0,1]^3} 1 \bigotimes_{j=1}^{3} dW^{\varphi_j}_{s_j}(n_j)\Bigg]$$

Dyadic decomposition : $\Psi_{\leq N} = \sum_{N_0,N_1,N_2,N_3 \leq N} \Psi[N^*]$

with $$\Psi[N_*] = \sum_{\substack{\varphi_1,\varphi_2,\varphi_3 \\ \in\{\cos,\sin\}}} \sum_{n_0,n_1,n_2,n_3 \in \mathbb{Z}^3} \Big[H_{n_0n_1n_2n_3}[N_*,\varphi_*](t)\, e^{i\langle n_0,x\rangle} \int_{[0,1]^3} 1 \bigotimes_{j=1}^{3} dW^{\varphi_j}_{s_j}(n_j)\Big]$$

$$H_{n_0n_1n_2n_3}[N_*,\varphi_*](t) = \mathbb{1}_{\{n_0=n_{123}\}} \cdot \frac{\mathbb{1}_{N_0}(n_0)}{\langle n_0\rangle} \cdot \Big(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle}\Big) \Big(\int_0^t \chi(t)\chi(t') \sin((t-t')\langle n\rangle) \prod_{j=1}^{3} \varphi_j(t'\langle n_j\rangle)\, dt'\Big)$$

(χ smooth, $\chi \equiv 1$ on $[-1,1]$, $\chi=0$ outside $[-2,2]$)

By Corollary 5.10 (to be covered in the next note),

$$\mathbb{E}\Big[\|\Psi[N_*]\|^2_{X^{-\varepsilon,b_+}}\Big] \lesssim N_0^{-2\varepsilon} \max_{\pm} \sum_{\substack{\varphi_1,\varphi_2,\varphi_3 \\ \in\{\cos,\sin\}}} \sum_{n_0,n_1,n_2,n_3 \in \mathbb{Z}^3} \int_{\mathbb{R}} \langle\lambda\rangle^{2b_+} \big|\widetilde{H}^{\pm}_{n_0n_1n_2n_3}[N_*,\varphi_*](\lambda)\big|^2 d\lambda$$

(see (5.110) or Note 7)

$$\overset{\text{Cor 5.10}}{\lesssim} N_0^{-2\varepsilon} N_0 \max(N_1,N_2,N_3)^{-1+\varepsilon}$$

(if $N_{max} = N_0$, then $N_0 = \max(N_1,N_2,N_3)$)

$$\lesssim N_{max}^{-\varepsilon}$$

We conclude by Gaussian hypercontractivity. ▯

Reading session 10 : The cubic tensor estimates

For any $F : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{C}$, $n \in \mathbb{Z}^3$, and $\lambda \in \mathbb{R}$, we define

$$\widetilde{F}^{\pm}(n, \lambda) := \mathcal{F}_{x,t}[F](n, \lambda \pm \langle n \rangle)$$

If $h_n(t)$ is a function of $n \in \mathbb{Z}^3$ and $t \in \mathbb{R}$, we define

$$\widetilde{h}^{\pm}(\lambda) := \mathcal{F}_t[h](n, \lambda \pm \langle n \rangle)$$

Let χ be a smooth cutoff function with $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$.

The Duhamel integral :

$$IF(t) = \int_0^t \frac{\sin((t-s)\langle \nabla \rangle)}{\langle \nabla \rangle} F(s)\, ds, \qquad I_\chi F(t) = \chi(t) \cdot I(\chi(s) \cdot F(s))$$

Lemma 2.3 (Lemma 4.1 in Deng-Nahmod-Yue 22')

For all $F : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{C}$, we have

$$\widetilde{I_\chi F}^{\pm}(n, \lambda) = \int_{\mathbb{R}} K^{\pm}(\lambda, \sigma) \cdot \langle n \rangle^{-1} \cdot \widetilde{F}^{\pm}(n, \sigma)\, d\sigma, \tag{2.30}$$

where the kernels $K^{\pm}(\lambda, \sigma)$ satisfy

$$|K^{\pm}(\lambda, \sigma)| \lesssim_B \left(\frac{1}{\langle \lambda \rangle^B} + \frac{1}{\langle \lambda \mp \sigma \rangle^B} \right) \frac{1}{\langle \sigma \rangle} \lesssim \frac{1}{\langle \lambda \rangle \langle \lambda \mp \sigma \rangle} \tag{2.31}$$

for all $B \ge 1$ and $\lambda, \sigma \in \mathbb{R}$. Furthermore, $\partial_\lambda K^{\pm}(\lambda, \sigma)$ and $\partial_\sigma K^{\pm}(\lambda, \sigma)$ hold the same bound as (2.31).

- The cubic tensor

Lemma 5.9 (The cubic tensor estimates)

Let N_1, N_2, N_3, N_{123} be dyadic numbers and $\lambda_1, \lambda_2, \lambda_3$ be real numbers.

Let $N_{max} = \max(N_1, N_2, N_3)$, $N_{min} = \min(N_{123}, N_1, N_2, N_3)$ and $\chi(t)$ Schwartz.

Define

$$h_{nn_1n_2n_3}(t, \lambda_1, \lambda_2, \lambda_3) = \mathbb{1}_{\{n = n_{123}\}} \cdot \mathbb{1}_{N_{123}}(n) \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right) \cdot \chi(t) e^{i(\pm\langle n_1 \rangle \pm \langle n_2 \rangle \pm \langle n_3 \rangle + \lambda_1 + \lambda_2 + \lambda_3)t}$$

$$H_{nn_1n_2n_3}(t, \lambda_1, \lambda_2, \lambda_3) = \mathbb{1}_{\{n = n_{123}\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right)$$
$$\times \int_0^t \chi(t)\, \chi(t')\, \sin((t-t')\langle n \rangle)\, e^{i(\pm\langle n_1 \rangle \pm \langle n_2 \rangle \pm \langle n_3 \rangle + \lambda_1 + \lambda_2 + \lambda_3)t'}\, dt'.$$

Then, there exist $A_j = A_j(\lambda, \lambda_1, \lambda_2, \lambda_3)$ such that

$$\| \langle \lambda \rangle^{b_+ - 1} A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \tag{5.33a}$$

$$\| \langle \lambda \rangle^{b_+} A_2(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \tag{5.33b}$$

and the following bounds hold :

$$\| \tilde{h}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{nn_1n_2n_3} \lesssim A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) N_{123} \cdot \left(\frac{N_{min}}{N_{max}} \right)^{\frac{1}{2}} \tag{5.34}$$

$$\| \tilde{H}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{nn_1n_2n_3} \lesssim A_2(\lambda, \lambda_1, \lambda_2, \lambda_3) \cdot \left(\frac{N_{min}}{N_{max}} \right)^{\frac{1}{2}} \tag{5.35}$$

$$\| \tilde{h}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{nn_B \to n_C} \lesssim A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) N_{123} \cdot N_{max}^{-\frac{1}{2}} \tag{5.36}$$

$$\| \tilde{H}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{nn_B \to n_C} \lesssim A_2(\lambda, \lambda_1, \lambda_2, \lambda_3) \cdot N_{max}^{-\frac{1}{2}} \tag{5.37}$$

for all partitions (B, C) of $\{1,2,3\}$ with $C \neq \emptyset$. Furthermore, the ∂_λ and ∂_{λ_j} derivatives of $\tilde{h}$ and $\tilde{H}$ satisfy the same estimates (5.34) – (5.37).

Proof: For h, we have

$$\tilde{h}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) = \mathbb{1}_{\{n = n_{123}\}} \cdot \mathbb{1}_{N_{123}}(n) \cdot \hat{\chi}(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \Omega) \cdot \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right), \tag{5.38}$$

where $\Omega = \pm \langle n \rangle \pm \langle n_1 \rangle \pm \langle n_2 \rangle \pm \langle n_3 \rangle$

By Hölder's inequality,

$$\| \tilde{h}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{nn_1n_2n_3} \lesssim A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) \, (N_1 N_2 N_3)^{-1} \cdot \sup_{\substack{m \in \mathbb{Z} \\ |m| \lesssim N_{max}}} \| h^b \|_{nn_1n_2n_3}$$

$$h^b = \mathbb{1}_{N_{123}}(n) \cdot \prod_{j=1}^{3} \mathbb{1}_{N_j}(n_j) \cdot \mathbb{1}_{\{n = n_{123}\}} \cdot \mathbb{1}_{\{|\Omega - m| \leq 1\}}$$

$$A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) := \sum_{\substack{m \in \mathbb{Z} \\ |m| \lesssim N_{max}}} | \hat{\chi}(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - m) | \tag{5.40}$$

Define $\Lambda := \lambda_1 + \lambda_2 + \lambda_3$. Then, (5.40) implies

$$|A_1| \lesssim \min\left(1, \; N_{max} \cdot \langle \lambda - \Lambda \rangle^{-1}\right)$$

(if $\langle \lambda - \Lambda \rangle \gg N_{max}$, then $\langle \lambda - \Lambda - m \rangle \gtrsim \langle \lambda - \Lambda \rangle$ for $|m| \lesssim N_{max}$)

Thus,

$$\| \langle \lambda \rangle^{b_+ - 1} A_1(\lambda, \lambda_1, \lambda_2, \lambda_3) \|_{L^2_\lambda} \lesssim N_{max}^{\frac{3}{2}(b_+ - \frac{1}{2})} \cdot \| \langle \lambda \rangle^{b_+ - 1} \langle \lambda - \Lambda \rangle^{-\frac{3}{2}(b_+ - \frac{1}{2})} \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \tag{5.42}$$

$\Rightarrow$ (5.33a) holds

WLOG, $N_1 \geq N_2 \geq N_3$. By (5.25) in Lemma 5.7 (Note 8),

$$\|h^b\|^2_{nn_1n_2n_3} \lesssim \mathrm{med}(N_2, N_3, N_{123})^{-1} \cdot (N_2 N_3 N_{123})^3$$

$$\lesssim N_2^3 N_3^2 N_{123}^3 \leq N_{123} \cdot (N_{123} N_1 N_2 N_3)^2 \cdot N_{max}^{-1} \qquad (N_{max} = N_1)$$

By symmetry of (n, n_1, n_2, n_3) in h^b, we have

$$\|h^b\|^2_{nn_1n_2n_3} \overset{(*)}{\lesssim} N_{min} \cdot (N_{123} N_1 N_2 N_3)^2 \cdot N_{max}^{-1} \implies (5.34)$$

For H, we have

$$\tilde{H}_{nn_1n_2n_3}(\lambda, \lambda_1, \lambda_2, \lambda_3) \overset{Lm\,2.3}{=} \mathbb{1}_{\{n = n_{123}\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right)$$

$$\times \int_{\mathbb{R}} K(\lambda, \sigma) \left(\chi(t') e^{i(\pm\langle n_1 \rangle \pm \langle n_2 \rangle \pm \langle n_3 \rangle + \lambda_1 + \lambda_2 + \lambda_3) t'} \right)^{\wedge} (\sigma \pm \langle n \rangle) \, d\sigma$$

$$= \mathbb{1}_{\{n = n_{123}\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right)$$

$$\times \int_{\mathbb{R}} K(\lambda, \sigma + \lambda_1 + \lambda_2 + \lambda_3 + \Omega) \, \hat{\chi}(\sigma) \, d\sigma$$

$$= K(\lambda, \lambda_1 + \lambda_2 + \lambda_3 + \Omega) \cdot \mathbb{1}_{\{n = n_{123}\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \cdot \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right)$$

↳ satisfies (2.31)

By a level-set decomposition,

$$\|\tilde{H}\|_{\lambda n n_1 n_2 n_3} \lesssim \sum_{\substack{m \in \mathbb{Z} \\ |m| \lesssim N_{max}}} \frac{1}{\langle \lambda \rangle \langle \lambda \pm (\lambda_1 + \lambda_2 + \lambda_3 + m) \rangle}$$

$$\times \left\| \mathbb{1}_{\{n = n_{123}\}} \cdot \mathbb{1}_{\{|\Omega - m| \leq 1\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \cdot \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right) \right\|_{nn_1n_2n_3}$$

We define

$$A_2(\lambda, \lambda_1, \lambda_2, \lambda_3) := \sum_{\substack{m \in \mathbb{Z} \\ |m| \lesssim N_{max}}} \frac{1}{\langle \lambda \rangle \langle \lambda \pm (\lambda_1 + \lambda_2 + \lambda_3 + m) \rangle}$$

$$\lesssim \log(N_{max}) \cdot \langle \lambda \rangle^{-1} \cdot \min(1, N_{max} \langle \lambda \pm \Lambda \rangle^{-1})$$

(if $\langle \lambda \pm \Lambda \rangle \gg N_{max}$, then $\langle \lambda \pm \Lambda \pm m \rangle \gtrsim \langle \lambda \pm \Lambda \rangle$)

$$(5.42) \implies \|\langle \lambda \rangle^{b_4} A_2(\lambda, \lambda_1, \lambda_2, \lambda_3)\|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \implies (5.33b)$$

We also have

$$\|\tilde{H}\|_{nn_1n_2n_3} \lesssim |A_2(\lambda, \lambda_1, \lambda_2, \lambda_3)| \cdot \sup_{\substack{m \in \mathbb{Z} \\ |m| \lesssim N_{max}}} \|h^b\|_{nn_1n_2n_3} \cdot (N_{123} N_1 N_2 N_3)^{-1}$$

By (*) above, we obtain (5.35)

For (5.36) and (5.37), it suffices to show

$$\|h^b\|^2_{nn_B\to n_C} \lesssim (N_{123}N_1N_2N_3)^2 \cdot N_{max}^{-1}$$

$nn_1n_2 \to n_3$: $\|h^b\|^2 \overset{(5.28)}{\lesssim} (N_{123}N_1N_2)^2 \max(N_1, N_2)^{-1}$ ✓

$n \longrightarrow n_1n_2n_3$: $\|h^b\|^2 \overset{(5.28)}{\lesssim} (N_1N_2N_3)^2 \cdot N_{max}^{-1}$ ✓

$nn_1 \to n_2n_3$: if $N_{max} = N_{123} = N_1$, then $\|h^b\|^2 \overset{(5.29)}{\lesssim} N_{123}^2 N_1^2 N_{max}^{-1} \cdot \min(N_2, N_3)^3$ ✓

if $N_{max} = N_1 = N_2$, then $\|h^b\|^2 \overset{(5.29)}{\lesssim} N_{123}^2 N_1 N_2 N_3^2$ ✓

Other cases are similar.

Same bounds apply with ∂_λ or ∂_{λ_j} since χ is Schwartz and Lemma 2.3 holds for $\partial_\lambda K$ or $\partial_{\lambda_j} K$. □

Corollary 5.10 Suppose $\varphi_j \in \{\sin, \cos\}$ for $j = 1, 2, 3$, N_1, N_2, N_3, N_{123} dyadic, $N_{max} = \max(N_1, N_2, N_3)$. Let

$$\begin{aligned} H_{nn_1n_2n_3}(t) &= H_{nn_1n_2n_3}[N_1, N_2, N_3, N_{123}, \varphi_1, \varphi_2, \varphi_3](t) \\ &= \mathbb{1}_{\{n=n_{123}\}} \cdot \frac{\mathbb{1}_{N_{123}}(n)}{\langle n \rangle} \cdot \left(\prod_{j=1}^{3} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \right) \int_0^t \chi(t)\chi(t') \sin((t-t')\langle n \rangle) \prod_{j=1}^{3} \varphi_j(t'\langle n_j \rangle)\, dt' \end{aligned}$$

Then, we have

$$\sum_{\substack{\varphi_1,\varphi_2,\varphi_3 \\ \in\{\cos,\sin\}}} \|\langle\lambda\rangle^{b_+} \widetilde{H}_{nn_1n_2n_3}(\lambda)\|_{L^2_\lambda[nn_1n_2n_3]} \lesssim N_{123}^{\frac{1}{2}} \cdot N_{max}^{-\frac{1}{2}+\varepsilon} \tag{5.46}$$

$$\sum_{\substack{\varphi_1,\varphi_2,\varphi_3 \\ \in\{\cos,\sin\}}} \|\langle\lambda\rangle^{b_+} \widetilde{H}_{nn_1n_2n_3}(\lambda)\|_{L^2_\lambda[nn_B\to n_C]} \lesssim N_{max}^{-\frac{1}{2}+\varepsilon} \tag{5.47}$$

for all partitions (B, C) of $\{1,2,3\}$ with $C \neq \emptyset$.

Proof: Use Lemma 5.9 and choose $\lambda_1 = \lambda_2 = \lambda_3 = 0$. □

Reading session 11 : The quintic object (part 1)

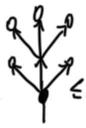
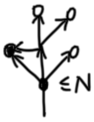
- The quintic stochastic objects

Proposition 7.7 (Regularity of the quintic stochastic object)

For all $T \geq 1$ and $p \geq 2$, we have

$\mathbb{E}\Big[\sup_N \Big\|$  $_{\le N}\Big\|^p_{(L^\infty_t C_x^{\frac{1}{2}-\varepsilon}\cap X^{\frac{1}{2}-\varepsilon,b})([-T,T])}\Big]^{1/p} \lesssim p^{\frac{5}{2}} T^{\alpha}.$

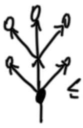
Recall from Subsection 6.3 (Note 4) :

3  $_{\le N} = 3$  $_{\le N} + 18$  $_{\le N} + 18$  $_{\le N}$

no pairing one pairing resistor

Lemma 7.8 (no pairing)

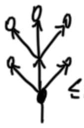
For all $T \geq 1$ and $p \geq 2$, we have

$\mathbb{E}\Big[\sup_N \Big\|$  $_{\le N}\Big\|^p_{(L^\infty_t C_x^{\frac{1}{2}-\varepsilon}\cap X^{\frac{1}{2}-\varepsilon,b})([-T,T])}\Big]^{1/p} \lesssim p^{\frac{5}{2}} T^{\alpha}.$

Proof : We only consider $T=1$ (general case minor modification).

We only prove the $X^{\frac{1}{2}-\varepsilon,b_+}$ - estimate ($L^\infty_t C_x^{\frac{1}{2}-\varepsilon}$ similar as in Lemma 7.4 (note 9)).

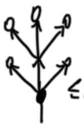
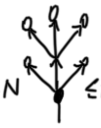
Recall (6.44), we write

 $_{\le N}(t,x) = \sum_{\substack{\varphi_1,\dots,\varphi_5\in\\ \{\cos,\sin\}}} \sum_{\substack{n_0,\dots,n_5\in\mathbb{Z}^3\\ n_0=n_{12345}}} \Big[\frac{\mathbb{1}_{\le N}(n_{234})}{\langle n_{234}\rangle}\Big(\prod_{j=0}^{5}\frac{\mathbb{1}_{\le N}(n_j)}{\langle n_j\rangle}\Big) e^{i\langle n_0,x\rangle}$

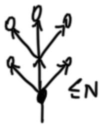
$$\times \int_0^t\int_0^{t'} \sin\big((t-t')\langle n_0\rangle\big)\sin\big((t'-t'')\langle n_{234}\rangle\big)\,\varphi_1(t'\langle n_1\rangle)$$

$$\times \Big(\prod_{j=2,3,4}\varphi_j(t''\langle n_j\rangle)\Big)\varphi_5(t'\langle n_5\rangle)\, dt''\, dt' \cdot \int_{[0,1]^5} 1 \bigotimes_{j=1}^{5} dW_{s_j}^{\varphi_j}(n_j)$$

By dyadic decomposition,

 $_{\le N} = \sum_{N_0,N_1,\dots,N_5,N_{234}\le N}$  $_{\le N}[N_*]$,

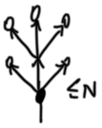
where



$$
{}_{\leq N}[N_*] = \sum_{\substack{\varphi_1,\dots,\varphi_5\in\\ \{\cos,\sin\}}} \sum_{n_0,\dots,n_5\in\mathbb{Z}^3} \left[H_{n_0n_1\cdots n_5}[N_*,\varphi_*](t)\, e^{i\langle n_0,x\rangle} \int_{[0,1]^5} \mathbb{1} \bigotimes_{j=1}^{5} dW_{\xi_j}^{\varphi_j}(n_j) \right],
$$

$$
H_{n_0n_1\cdots n_5}(t) = \mathbb{1}_{\{n_0=n_{12345}\}} \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \left(\prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right) \cdot \int_0^t \chi(t)\,\chi(t')\,\sin((t-t')\langle n_0\rangle)\,\varphi_1(t'\langle n_1\rangle)\,\varphi_5(t'\langle n_5\rangle)
$$
$$
\times \left(\int_0^{t'} \chi(t')\,\chi(t'')\,\sin((t'-t'')\langle n_{234}\rangle) \prod_{j=2}^{4} \varphi_j(t''\langle n_j\rangle)\, dt'' \right) dt'
$$

By Corollary 5.12 (5.70) (shown below),



$$
\mathbb{E}\left[\left\| {}_{\leq N}[N_*] \right\|^2_{X^{\frac{1}{2}-\varepsilon,b_4}} \right] \lesssim N_0^{1-2\varepsilon} N_{max}^{-1+\varepsilon} \lesssim N_{max}^{-\varepsilon}.
$$

We conclude by Gaussian hypercontractivity. □

- The quintic tensor

Lemma 5.11 (The quintic tensor estimates)

Let $N_0, \dots, N_5, N_{234}$ by dyadic numbers and $\lambda, \lambda_1, \dots, \lambda_5$ be real numbers. Let $N_{max} = \max(N_1, \dots, N_5)$. Define the tensors

$$
h_{n_0n_1\cdots n_5}(t,\lambda_1,\dots,\lambda_5) = \mathbb{1}_{\{n_0=n_{12345}\}}\, \mathbb{1}_{N_0}(n_0)\, \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \left(\prod_{j=1}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right) \cdot e^{it(\pm\langle n_1\rangle\pm\langle n_5\rangle+\lambda_1+\lambda_5)}
$$
$$
\times \int_0^t \chi(t)\,\chi(t')\,\sin((t-t')\langle n_{234}\rangle)\, e^{it'(\pm\langle n_2\rangle\pm\langle n_3\rangle\pm\langle n_4\rangle+\lambda_2+\lambda_3+\lambda_4)}\, dt'
$$

$$
H_{n_0n_1\cdots n_5}(t,\lambda_1,\dots,\lambda_5) = \mathbb{1}_{\{n_0=n_{12345}\}} \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \left(\prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \right)
$$
$$
\times \left(\int_0^t \chi(t)\,\chi(t')\,\sin((t-t')\langle n_0\rangle)\, e^{it'(\pm\langle n_1\rangle\pm\langle n_5\rangle+\lambda_1+\lambda_5)} \right.
$$
$$
\left. \times \left(\int_0^{t'} \chi(t')\,\chi(t'')\,\sin((t'-t'')\langle n_{234}\rangle)\, e^{it''(\pm\langle n_2\rangle\pm\langle n_3\rangle\pm\langle n_4\rangle+\lambda_2+\lambda_3+\lambda_4)}\, dt'' \right) dt' \right)
$$

Then, there exist two functions $B_j = B_j(\lambda,\lambda_1,\dots,\lambda_5)$, $j=1,2$ such that

$$
\| \langle\lambda\rangle^{b_4-1} B_1(\lambda,\lambda_1,\dots,\lambda_5) \|_{L^1_\lambda} \lesssim N_{max}^{\varepsilon}, \tag{5.50a}
$$

$$
\| \langle\lambda\rangle^{b_4} B_2(\lambda,\lambda_1,\dots,\lambda_5) \|_{L^1_\lambda} \lesssim N_{max}^{\varepsilon}, \tag{5.50b}
$$

and that we have

$$\|\tilde{h}(\lambda,\lambda_1,\dots,\lambda_5)\|_{n_0n_1\cdots n_5} \lesssim \frac{N_0 \cdot \min(N_2,N_3,N_4,N_{234})^{\frac{1}{2}}}{\max(N_0,N_1,N_5)^{\frac{1}{2}}\max(N_2,N_3,N_4)^{\frac{1}{2}}} B_1(\lambda,\lambda_1,\dots,\lambda_5)$$

$$\lesssim N_0 \cdot N_{max}^{-\frac{1}{2}} \cdot B_1(\lambda,\lambda_1,\dots,\lambda_5) \qquad (5.51)$$

$$\|\tilde{h}(\lambda,\lambda_1,\dots,\lambda_5)\|_{n_0n_1\cdots n_5} \lesssim \frac{N_0 \min(N_0,N_1,N_2,N_5)^{\frac{1}{2}}(N_3N_4)^{\frac{1}{2}}}{N_2\max(N_0,N_1,N_2,N_5)^{\frac{1}{2}}} B_1(\lambda,\lambda_1,\dots,\lambda_5) \qquad (5.52)$$

when $N_2 \sim N_{234}$

$$\|\tilde{H}(\lambda,\lambda_1,\dots,\lambda_5)\|_{n_0n_1\cdots n_5} \lesssim N_{max}^{-\frac{1}{2}} \cdot B_2(\lambda,\lambda_1,\dots,\lambda_5) \qquad (5.53)$$

$$\|\tilde{h}(\lambda,\lambda_1,\dots,\lambda_5)\|_{n_0n_A\to n_Bn_5} \lesssim N_0^{\frac{1}{2}}N_5^{-\frac{1}{2}} \cdot \{\max(N_0,N_2,N_3,N_4)^{-\frac{1}{2}} + \max(N_2,N_3,N_4,N_5)^{-\frac{1}{2}}\}$$
$$\times B_1(\lambda,\lambda_1,\dots,\lambda_5) \qquad (5.54)$$

$$\|\tilde{H}(\lambda,\lambda_1,\dots,\lambda_5)\|_{n_0n_A\to n_Bn_5} \lesssim (N_0N_5)^{-\frac{1}{2}} \cdot \{\max(N_0,N_2,N_3,N_4)^{-\frac{1}{2}} + \max(N_2,N_3,N_4,N_5)^{-\frac{1}{2}}\}$$
$$\times B_2(\lambda,\lambda_1,\dots,\lambda_5) \qquad (5.55)$$

for any partition (A,B) of $\{1,2,3,4\}$. The same bounds hold for all ∂_λ and ∂_{λ_j} derivatives of $\tilde{h}$ and $\tilde{H}$.

Proof: By Lemma 2.3, we have

$$\tilde{h}_{n_0n_1\cdots n_5}(\lambda,\lambda_1,\dots,\lambda_5) \overset{Lm2.3}{=} \mathbb{1}_{\{n_0=n_{12345}\}} \cdot \mathbb{1}_{N_0}(n_0) \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \cdot \left(\prod_{j=1}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle}\right)$$
$$\times \int_{\mathbb{R}} K(\lambda-\lambda_1-\lambda_5-\Omega',\sigma)\left(\chi(t')e^{it'(\pm\langle n_2\rangle\pm\langle n_3\rangle\pm\langle n_4\rangle+\lambda_2+\lambda_3+\lambda_4)}\right)^{\wedge}(\sigma\pm\langle n_{234}\rangle)\,d\sigma$$
$$= \mathbb{1}_{\{n_0=n_{12345}\}} \cdot \mathbb{1}_{N_0}(n_0) \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \cdot \left(\prod_{j=1}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle}\right)$$
$$\times \int_{\mathbb{R}} K(\lambda-\lambda_1-\lambda_5-\Omega',\sigma+\lambda_2+\lambda_3+\lambda_4+\Omega'')\,\hat{\chi}(\sigma)\,d\sigma$$
$$= \mathbb{1}_{\{n_0=n_{12345}\}} \cdot \mathbb{1}_{N_0}(n_0) \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234}\rangle} \cdot \left(\prod_{j=1}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle}\right)$$
$$\times \underbrace{K(\lambda-\lambda_1-\lambda_5-\Omega',\lambda_2+\lambda_3+\lambda_4+\Omega'')}_{\hookrightarrow\text{ satisfies (2.31)}},$$

$$\widetilde{H}_{n_0 n_1 \cdots n_5}(\lambda, \lambda_1, \dots, \lambda_5) \overset{\text{Lm 2.3}}{=} \mathbb{1}_{\{n_0 = n_{12345}\}} \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234} \rangle} \cdot \Big(\prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big)$$

$$\times \int_{\mathbb{R}} K(\lambda, \sigma) \Big[\chi(t') e^{it'(\pm\langle n_1 \rangle \pm \langle n_5 \rangle + \lambda_1 + \lambda_5)}$$

$$\times \Big(\int_0^{t'} \chi(t'') \sin\big((t'-t'')\langle n_{234} \rangle\big) e^{it''(\pm\langle n_2 \rangle \pm \langle n_3 \rangle \pm \langle n_4 \rangle + \lambda_2 + \lambda_3 + \lambda_4)} dt'' \Big) \Big]^{\wedge} (\sigma \pm \langle n_0 \rangle) \, d\sigma$$

$$\overset{\text{Lm 2.3}}{=} \mathbb{1}_{\{n_0 = n_{12345}\}} \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234} \rangle} \cdot \Big(\prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big)$$

$$\times \int_{\mathbb{R}} K(\lambda, \sigma) \int_{\mathbb{R}} K(\sigma - \lambda_1 - \lambda_5 - \Omega', \mu)$$

$$\times \Big(\chi(t'') e^{it''(\pm\langle n_2 \rangle \pm \langle n_3 \rangle \pm \langle n_4 \rangle + \lambda_2 + \lambda_3 + \lambda_4)} \Big)^{\wedge} (\mu \pm \langle n_{234} \rangle) \, d\mu \, d\sigma$$

$$= \mathbb{1}_{\{n_0 = n_{12345}\}} \cdot \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234} \rangle} \cdot \Big(\prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big)$$

$$\times \int_{\mathbb{R}} K(\lambda, \sigma) \cdot K(\sigma - \lambda_1 - \lambda_5 - \Omega', \lambda_2 + \lambda_3 + \lambda_4 + \Omega'') \, d\sigma ,$$

↳ satisfies (2.31)

where

$$\Omega' = \pm\langle n_0 \rangle \pm \langle n_1 \rangle \pm \langle n_5 \rangle \pm_{234} \langle n_{234} \rangle , \qquad \Omega'' = \mp_{234} \langle n_{234} \rangle \pm \langle n_2 \rangle \pm \langle n_3 \rangle \pm \langle n_4 \rangle .$$

By a level set decomposition and (2.31),

the sign of λ_j does not matter

$$\| \widetilde{h}(\lambda, \lambda_1, \dots, \lambda_5) \|_{n_0 n_1 \cdots n_5} \lesssim \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ |m_1|, |m_2| \lesssim N_{max}}} \frac{1}{\langle \lambda - \lambda_1 - \lambda_5 - m_1 \rangle \langle \lambda - \sum_{j=1}^{5} \lambda_j - m_2 \rangle} \quad =: B_1(\lambda, \lambda_1, \dots, \lambda_5)$$

$$\times \Big(\prod_{j=1}^{5} N_j \Big)^{-1} \cdot N_{234}^{-1} \, \Big\| \sum_{n_{234}} h^b_{n_0 n_1 n_{234} n_5} h^b_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 \cdots n_5} ,$$

where $h^b_{n_0 n_1 n_{234} n_5}$ and $h^b_{n_{234} n_2 n_3 n_4}$ are base tensors in (5.23) with $|m_1 - \Omega'| \leq 1$ and $|m_2 - m_1 - \Omega''| \leq 1$.

$$|B_1| \lesssim (\log N_{max})^2 \cdot \min\big(1, N_{max}^2 \langle \lambda - \lambda_1 - \lambda_5 \rangle^{-1}\big)$$

↳ $\langle \lambda - \lambda_1 - \lambda_5 - m_1 \rangle^{-1} \lesssim \langle m_1 \rangle \langle \lambda - \lambda_1 - \lambda_5 \rangle^{-1}$

$$\Rightarrow \| \langle \lambda \rangle^{b_+ - 1} B_1(\lambda, \lambda_1, \dots, \lambda_5) \|_{L^2_\lambda} \lesssim N_{max}^{3(b^+ - \frac{1}{2})} \cdot \| \langle \lambda \rangle^{b_+ - 1} \langle \lambda - \lambda_1 - \lambda_5 \rangle^{-\frac{3}{2}(b^+ - \frac{1}{2})} \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \Rightarrow \text{(5.50a)}$$

$$\| \widetilde{h} \|_{n_0 n_1 \cdots n_5} \lesssim |B_1(\lambda, \lambda_1, \dots, \lambda_5)| \cdot \Big(\prod_{j=1}^{5} N_j \Big)^{-1} \cdot N_{234}^{-1} \cdot \sup_{m_1, m_2} \Big\| \sum_{n_{234}} h^b_{n_0 n_1 n_{234} n_5} h^b_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 \cdots n_5}$$

By the merging estimate (Lemma B.1) and (5.26), (5.28) in Lemma 5.7,

$$\Big(\prod_{j=1}^{5} N_j \Big)^{-1} \cdot N_{234}^{-1} \cdot \Big\| \sum_{n_{234}} h^b_{n_0 n_1 n_{234} n_5} h^b_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 \cdots n_5}$$

$$\lesssim \Big(\prod_{j=1}^{5} N_j \Big)^{-1} \cdot N_{234}^{-1} \cdot \| h^b_{n_0 n_1 n_{234} n_5} \|_{n_0 n_1 n_5 \to n_{234}} \| h^b_{n_{234} n_2 n_3 n_4} \|_{n_2 n_3 n_4 n_{234}}$$

$$\lesssim \frac{N_0 \cdot \min(N_2, N_3, N_4, N_{234})^{1/2}}{\max(N_0, N_1, N_5)^{1/2} \max(N_2, N_3, N_4)^{1/2}} \lesssim N_0 \cdot \max(N_0, \dots, N_5)^{-1/2} \Rightarrow \text{(5.51)}$$

$|n_2 + n_3 + n_4| , |n_0 + n_1 + n_5| \sim N_{234}$

By the merging estimate (Lemma B.1), (5.26) in Lemma 5.7, and Schur's test,

$$
\begin{aligned}
&\Big(\prod_{j=1}^{5} N_j\Big)^{-1} \cdot N_{234}^{-1} \cdot \Big\| \sum_{n_{234}} h^{b}_{n_0 n_1 n_{234} n_5} h^{b}_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 \cdots n_5} \\
&\lesssim \Big(\prod_{j=1}^{5} N_j\Big)^{-1} \cdot N_{234}^{-1} \cdot \| h^{b}_{n_0 n_1 n_{234} n_5} \|_{n_0 n_1 n_{234} n_5} \| h^{b}_{n_{234} n_2 n_3 n_4} \|_{n_2 n_3 n_4 \to n_{234}} \\
&\lesssim N_0 N_3^{-1} N_4^{-1} N_{234}^{-1} \frac{\min(N_0, N_1, N_2, N_5)^{1/2}}{\max(N_0, N_1, N_2, N_5)^{1/2}} \cdot \sup_{n_{234}} \underset{\lesssim (N_3 N_4)^{3/2}}{\| h^{b}_{n_{234} n_2 n_3 n_4} \|^{1/2}_{n_2 n_3 n_4}} \\
&\lesssim \frac{N_0 \min(N_0, N_1, N_2, N_5)^{\frac{1}{2}} (N_3 N_4)^{\frac{1}{2}}}{N_2 \max(N_0, N_1, N_2, N_5)^{\frac{1}{2}}} \qquad \text{given } N_2 \sim N_{234} \qquad \Rightarrow \quad (5.52)
\end{aligned}
$$

For $\tilde{H}(\lambda, \lambda_1, \dots, \lambda_5)$, we have

$$
\|\tilde{H}\|_{n_0 n_1 \cdots n_5} \lesssim |B_2(\lambda, \lambda_1, \dots, \lambda_5)| \cdot \Big(\prod_{j=0}^{5} N_j\Big)^{-1} \cdot N_{234}^{-1} \cdot \sup_{m_1, m_2} \Big\| \sum_{n_{234}} h^{b}_{n_0 n_1 n_{234} n_5} h^{b}_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 \cdots n_5},
$$

where

$$
B_2(\lambda, \lambda_1, \dots, \lambda_5) \overset{(2.31)}{=} \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ |m_1|, |m_2| \lesssim N_{max}}} \int_{\mathbb{R}} \frac{d\sigma}{\langle \lambda \rangle \langle \lambda - \sigma \rangle \langle \sigma - \lambda_1 - \lambda_5 - m_1 \rangle \langle \sigma - \sum_{j=1}^{5} \lambda_j - m_2 \rangle}
$$

$$
|B_2| \lesssim (\log N_{max})^2 \langle \lambda \rangle^{-1} \min(1, N_{max}^2 \langle \lambda - \lambda_1 - \lambda_5 \rangle)
$$

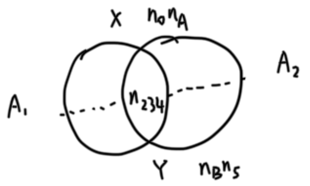
$$
\Rightarrow \| \langle \lambda \rangle^{b_+} B_2(\lambda, \lambda_1, \dots, \lambda_5) \|_{L^2_\lambda} \lesssim N_{max}^{3(b^+ - \frac{1}{2})} \cdot \| \langle \lambda \rangle^{b_+ - 1} \langle \lambda - \lambda_1 - \lambda_5 \rangle^{-\frac{3}{2}(b^+ - \frac{1}{2})} \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \quad \Rightarrow \quad (5.50b)
$$

Also, (5.53) follows from the same way as (5.51)

For (5.54) and (5.55), it suffices to show

$$
\Big\| \sum_{n_{234}} h^{b}_{n_0 n_1 n_{234} n_5} h^{b}_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 n_A \to n_B n_5} \lesssim \prod_{j=0}^{5} N_j \cdot N_{234} (N_0 N_5)^{-\frac{1}{2}} \times \Big\{ \max(N_0, N_2, N_3, N_4)^{-\frac{1}{2}} + \max(N_2, N_3, N_4, N_5)^{-\frac{1}{2}} \Big\} \qquad (5.62)
$$

By the merging estimate (Lemma B.1),



$$
\begin{aligned}
&\Big\| \sum_{n_{234}} h^{b}_{n_0 n_1 n_{234} n_5} h^{b}_{n_{234} n_2 n_3 n_4} \Big\|_{n_0 n_A \to n_B n_5} \\
&\lesssim \min \Big\{ \| h^{b}_{n_0 n_1 n_{234} n_5} \|_{n_X \to n_Y} \| h^{b}_{n_{234} n_2 n_3 n_4} \|_{n_Z \to n_W}, \ \| h^{b}_{n_0 n_1 n_{234} n_5} \|_{n_X \to n_Y} \| h^{b}_{n_{234} n_2 n_3 n_4} \|_{n_{Z'} \to n_{W'}} \Big\}
\end{aligned}
$$

with $0 \in X$, $5 \in Y$, $Z \in \{ \emptyset, \{2\}, \{3\}, \{4\} \}$, $Z' = Z \cup \{234\}$.

We first have

$$\| h^b_{n_0 n_1 n_{234} n_5} \|_{n_X \to n_Y} \lesssim N_0 N_1 N_{234} N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \tag{5.64}$$

If $\min(N_0, N_5) \gg \max(N_2, N_3, N_4)$, we have a stronger bound

$$\| h^b_{n_0 n_1 n_{234} n_5} \|_{n_X \to n_Y} \lesssim N_0 N_1 N_{234} N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \left(N_0^{-\frac{1}{2}} + N_5^{-\frac{1}{2}} \right) \tag{5.65}$$

Reasoning for (5.64) and (5.65) :

- If $|X| = 1$ or $|Y| = 1$,

  (5.64) follows from (5.28) in Lemma 5.7 e.g. $X = \{0\}$; $N_1 N_{234} N_5 \max(N_1, N_{234}, N_5)^{-\frac{1}{2}} \leq N_0^{\frac{1}{2}} N_1 N_{234} N_5^{\frac{1}{2}}$

  (5.65) follows from (5.28) in Lemma 5.7 e.g. $X = \{0\}$: $N_1 N_{234} N_5 \cdot N_5^{-\frac{1}{2}}$

- If $|X| = 2$ and $|Y| = 2$,

  (5.64) follows from (5.29) in Lemma 5.7 e.g. $X = \{0, 234\}$: $(N_0^3 N_1^3)^{1/3} (N_{234}^3 N_5^3)^{1/3} (N_{234}^3 N_1^1)^{1/3}$

  (5.65) follows from Schur's test and Lemma 5.1 e.g. $X = \{0, 234\}$ $\sup\limits_{n_1, n_5} \| h^b_{n_0 n_1 n_{234} n_5} \|_{\ell^1_{n_0 n_{234}}} \overset{\text{Lm 5.1}}{\lesssim} N_{234}^2$

For $h^b_{n_{234} n_2 n_3 n_4}$,

- If $Z = \phi$, then $Z' = \{234\}$. By (5.28) in Lemma 5.7,

  $$\| h^b_{n_{234} n_2 n_3 n_4} \|_{n_{Z'} \to n_{W'}} \lesssim N_2 N_3 N_4 \cdot \max(N_2, N_3, N_4)^{-\frac{1}{2}}$$

- If $Z = \{2\}$ (others similar) :
  - If $N_{234} \lesssim N_2$, by (5.28) in Lemma 5.7,

    $$\begin{aligned} \| h^b_{n_{234} n_2 n_3 n_4} \|_{n_Z \to n_W} &\lesssim N_{234} N_3 N_4 \cdot \max(N_{234}, N_3, N_4)^{-\frac{1}{2}} \\ &\lesssim N_2 N_3 N_4 \cdot \max(N_2, N_3, N_4)^{-\frac{1}{2}} \end{aligned}$$

  - If $N_{234} \gg N_2$, $Z' = \{2, 234\}$. By Schur's test and Lemma 5.1,

    $$\begin{aligned} \| h^b_{n_{234} n_2 n_3 n_4} \|_{n_{Z'} \to n_{W'}} &\lesssim \sup_{n_3, n_4} \| h^b_{n_{234} n_2 n_3 n_4} \|^{\frac{1}{2}}_{\ell^1_{n_2 n_{234}}} \cdot \sup_{n_2, n_{234}} \| h^b_{n_{234} n_2 n_3 n_4} \|^{\frac{1}{2}}_{\ell^1_{n_3 n_4}} \\ &\lesssim N_2 \cdot N_3 N_4 \max(N_3, N_4)^{-\frac{1}{2}} \rightsquigarrow \text{(5.15) in Lemma 5.4} \\ &\sim N_2 N_3 N_4 \cdot \max(N_2, N_3, N_4)^{-\frac{1}{2}} \end{aligned}$$

Thus,

$$\text{LHS of (5.62)} \lesssim \prod_{j=0}^{5} N_j \cdot N_{234} (N_0 N_5)^{-\frac{1}{2}} \cdot \max\left(\max(N_2, N_3, N_4), \min(N_0, N_5) \right)^{-\frac{1}{2}} \Rightarrow \text{(5.62)}$$

The ∂_λ and ∂_{λ_j} derivative estimates follow in the same way since Lemma 2.3 holds for $\partial_\lambda K$ and $\partial_{\lambda_j} K$ □

Corollary 5.12 Suppose $\varphi_j \in \{\sin, \cos\}$ for $j = 1, \dots, 5$ and $N_0, \dots, N_5, N_{234}$ dyadic. Let $N_{max} = \max(N_1, \dots, N_5)$. Consider the quintic tensor (arising from )

$$H_{n_0 n_1 \cdots n_5}(t) = \mathbb{1}_{\{n_0 = n_{12345}\}} \cdot \prod_{j=0}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \int_0^t \chi(t)\, \chi(t')\, \sin\big((t-t')\langle n_0 \rangle\big)\, \varphi_1(t'\langle n_1 \rangle)$$
$$\times \frac{\mathbb{1}_{N_{234}}(n_{234})}{\langle n_{234} \rangle} \varphi_5(t'\langle n_5 \rangle) \Big(\int_0^{t'} \chi(t')\, \chi(t'')\, \sin\big((t'-t'')\langle n_{234} \rangle\big) \prod_{j=2}^{4} \varphi_j(t''\langle n_j \rangle)\, dt'' \Big)\, dt'.$$

Then, we have

$$\big\| \langle \lambda \rangle^{b_+} \widetilde{H}_{n_0 \cdots n_5}(\lambda) \big\|_{L^2_\lambda [n_0 \cdots n_5]} \lesssim N_{max}^{-\frac{1}{2}+\varepsilon}$$
$$\big\| \langle \lambda \rangle^{b_+} \widetilde{H}_{n_0 \cdots n_5}(\lambda) \big\|_{L^2_\lambda [n_0 n_A \to n_B n_5]} \lesssim N_{max}^{\varepsilon} (N_0 N_5)^{-\frac{1}{2}} \cdot \Big(\max(N_0, N_2, N_3, N_4)^{-\frac{1}{2}}$$
$$+ \max(N_2, N_3, N_4, N_5)^{-\frac{1}{2}} \Big)$$

for any partition (A, B) of $\{1, 2, 3, 4\}$.

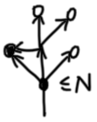
Proof: The estimates follow directly from Lemma 5.11 with $\lambda_1 = \cdots = \lambda_5 = 0$. $\square$

Reading session 12 : The quintic object (part 2)

- The quintic stochastic objects (continue)

Lemma 7.8 (one pairing)

For all $T \geq 1$ and $p \geq 2$, we have

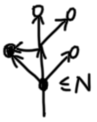


$$\mathbb{E}\left[\sup_N \left\| \cdot \right\|^p_{(L^\infty_t C^{\frac{1}{2}-\varepsilon}_x \cap X^{\frac{1}{2}-\varepsilon, b_+})([-T,T])} \right]^{1/p} \lesssim p^{\frac{3}{2}} T^\alpha.$$

Proof : We only consider $T=1$ (general case minor modification).

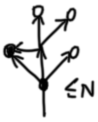
We only prove the $X^{\frac{1}{2}-\varepsilon, b_+}$-estimate ($L^\infty_t C^{\frac{1}{2}-\varepsilon}_x$ similar as in Lemma 7.4 (note 9)).

Recall (6.45), we write

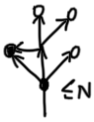


$$(t,x) = \sum_{\substack{\varphi_3,\varphi_4,\varphi_5 \in \\ \{\cos,\sin\}}} \sum_{\substack{n_0,n_1,\dots,n_5 \in \mathbb{Z}^3 \\ n_0 = n_{345}}} \Bigg[\mathbb{1}_{\{n_{12}=0\}} \frac{\mathbb{1}_{\leq N}(n_{234})}{\langle n_{234}\rangle} \left(\prod_{j=0}^{5} \frac{\mathbb{1}_{\leq N}(n_j)}{\langle n_j \rangle} \right) e^{i\langle n_0, x\rangle}$$
$$\times \left(\int_0^t \int_0^{t'} \sin((t-t')\langle n_0\rangle) \sin((t'-t'')\langle n_{234}\rangle) \cos((t'-t'')\langle n_2\rangle) \right.$$
$$\left. \times \left(\prod_{j=3,4} \varphi_j(t''\langle n_j\rangle) \right) dt'' \, \varphi_5(t'\langle n_5\rangle) \, dt' \right) \int_{[0,1]^3} 1 \bigotimes_{j=3}^{5} dW^{\varphi_j}_{s_j}(n_j) \Bigg]$$

By dyadic decomposition,

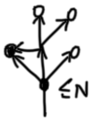


$$= \sum_{N_0, N_1, \dots, N_5, N_{234} \leq N}$$



$$[N_*],$$

where

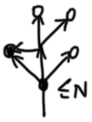


$$[N_*] = \sum_{\substack{\varphi_3,\varphi_4,\varphi_5 \in \\ \{\cos,\sin\}}} \sum_{\substack{n_0,n_3,n_4,n_5 \in \mathbb{Z}^3 \\ n_0 = n_{345}}} \left[H^{\text{sine}}_{n_0 n_3 n_4 n_5}[N_*, \varphi_*](t) \, e^{i\langle n_0, x\rangle} \int_{[0,1]^3} 1 \bigotimes_{j=3}^{5} dW^{\varphi_j}_{s_j}(n_j) \right],$$

$$H^{\text{sine}}_{n_0 n_3 n_4 n_5}[N_*, \varphi_*](t) = \mathbb{1}_{\{n_0 = n_{345}\}} \cdot \frac{\mathbb{1}_{N_0}(n_0)}{\langle n_0\rangle} \prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle} \int_0^t \chi(t)\chi(t') \sin((t-t')\langle n_0\rangle) \varphi_5(t'\langle n_5\rangle)$$
$$\times \int_0^{t'} \chi(t')\chi(t'') \cdot \text{Sine}[N_{234}, N_2](t'-t'', n_{34}) \, \varphi_3(t''\langle n_3\rangle) \, \varphi_4(t''\langle n_4\rangle) \, dt'' \, dt',$$

$$\text{Sine}[K, L](t,r) = \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12} = r}} \mathbb{1}_K(n_1) \mathbb{1}_L(n_2) \frac{\sin(t\langle n_1\rangle)}{\langle n_1\rangle} \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle}$$

By Corollary 5.19 (5.106) (shown in the next note),



$$\mathbb{E}\left[\left\| \cdot \, [N_*] \right\|^2_{X^{\frac{1}{2}-\varepsilon, b_+}} \right] \lesssim N_0^{1-2\varepsilon} N_{\max}^{-1+\varepsilon} \lesssim N_{\max}^{-\varepsilon}.$$

We conclude by Gaussian hypercontractivity. $\square$

- The sine-cancellation kernel and tensor

Definition 5.13 (The sine-cancellation kernel)

For any frequency-scales K and L and any $r \in \mathbb{Z}^3$, we define the sine-cancellation kernel $\mathrm{Sine} : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{Sine}(t,r) = \mathrm{Sine}[K,L](t,r) := \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3: \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \frac{\sin(t\langle n_1\rangle)}{\langle n_1\rangle} \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2} .$$

Lemma 5.15 (Symmetrization of the sine-cancellation kernel)

For any frequency-scales K and L and any $r \in \mathbb{Z}^3$, we have

$$\mathrm{Sine}[K,L](t,r) = \frac{1}{4} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \frac{\langle n_1\rangle - \langle n_2\rangle}{\langle n_1\rangle^2 \langle n_2\rangle^2} \sin\big(t(\langle n_1\rangle - \langle n_2\rangle)\big) \tag{5.73}$$

$$+ \frac{1}{4} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \frac{\langle n_1\rangle + \langle n_2\rangle}{\langle n_1\rangle^2 \langle n_2\rangle^2} \sin\big(t(\langle n_1\rangle + \langle n_2\rangle)\big) \tag{5.74}$$

$$- \frac{1}{2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \big(\mathbb{1}_K(n_1)\mathbb{1}_L(n_2) - \mathbb{1}_L(n_1)\,\mathbb{1}_K(n_2)\big) \frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2} \frac{\sin(t\langle n_2\rangle)}{\langle n_2\rangle} . \tag{5.75}$$

Furthermore, on the support of $\mathbb{1}_K(n_1)\mathbb{1}_L(n_2) - \mathbb{1}_L(n_1)\,\mathbb{1}_K(n_2)$, the vectors n_1 and n_2 satisfy the Γ-condition ; there exists $\Gamma \in \mathbb{R}$ such that

$$\text{either } \ |n_1|_\infty \leq \Gamma \leq |n_2|_\infty \ \text{ or } \ |n_2|_\infty \leq \Gamma \leq |n_1|_\infty \tag{5.5}$$

Proof: We have

$$\mathrm{Sine}(t,r) = \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \frac{\sin(t\langle n_1\rangle)}{\langle n_1\rangle} \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2}$$

$$= - \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \partial_t\Big(\frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2}\Big) \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2}$$

$$= -\frac{1}{2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \Big[\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \partial_t\Big(\frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2}\Big) \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2} + \mathbb{1}_L(n_1)\, \mathbb{1}_K(n_2) \frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2} \partial_t\Big(\frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2}\Big) \Big]$$

symmetrizing n_1 and n_2

$$= -\frac{1}{2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \Big[\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) \Big(\partial_t\Big(\frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2}\Big) \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2} + \frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2} \partial_t\Big(\frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2}\Big) \Big) \Big] \tag{5.77}$$

$$+ \frac{1}{2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \Big[\big(\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) - \mathbb{1}_L(n_1)\, \mathbb{1}_K(n_2) \big) \frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2} \partial_t\Big(\frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2}\Big) \Big] \tag{5.78}$$

Note that (5.78) = (5.75)

For (5.77), we have

$$(5.77) = -\frac{1}{2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ n_{12} = r}} \left[\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \partial_t \left(\frac{\cos(t\langle n_1\rangle)}{\langle n_1\rangle^2} \frac{\cos(t\langle n_2\rangle)}{\langle n_2\rangle^2} \right) \right]$$

$$= -\frac{1}{4} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ n_{12} = r}} \left[\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2)\, \frac{1}{\langle n_1\rangle^2 \langle n_2\rangle^2}\, \partial_t \Big(\cos\big(t(\langle n_1\rangle - \langle n_2\rangle)\big) + \cos\big(t(\langle n_1\rangle + \langle n_2\rangle)\big) \Big) \right]$$

$$= (5.73) + (5.74)$$

On the support of $\mathbb{1}_K(n_1)\mathbb{1}_L(n_2) - \mathbb{1}_L(n_1)\mathbb{1}_K(n_2)$:

If $K = L$: empty support

If $K > L$: $\left.\begin{array}{ll} \mathbb{1}_K(n_1)\mathbb{1}_L(n_2) \text{ supported in } & |n_1|_\infty \geq K \geq |n_2|_\infty \\ \mathbb{1}_L(n_1)\mathbb{1}_K(n_2) \text{ supported in } & |n_2|_\infty \geq K \geq |n_1|_\infty \end{array}\right\} \Rightarrow (5.5)$

If $K < L$: similar $\square$

Lemma 5.3 (The Γ-condition counting lemma)

Given $\Gamma \in \mathbb{R}$, dyadic $A, N \geq 1$, and $a \in \mathbb{Z}^3$ such that $|a|_\infty \sim A$, we have

$$\sup_{m \in \mathbb{Z}} \#\left\{ n \in \mathbb{Z}^3 : |n|_\infty \sim N,\ |\langle a+n\rangle - \langle n\rangle - m| \leq 1,\ |n|_\infty \geq \Gamma \geq |n+a|_\infty \right\} \lessapprox N^2 \log N \quad (5.6)$$

The same bound holds if one assumes $|n|_\infty \leq \Gamma \leq |n+a|_\infty$.

Proof: If $A \geq N/100$, then (5.6) follows from (5.1) in Lemma 5.1.

We assume $A \leq N/100 \Rightarrow \langle n\rangle \sim N \sim \langle a+n\rangle$

We write $n = (x, y, z)$ and $a = (x_a, y_a, z_a)$. WLOG, $|n|_\infty = |x|$.

$$|n|_\infty \geq \Gamma \geq |n+a|_\infty \Rightarrow |x| \geq \Gamma \geq |x + x_a| \geq |x| - |x_a|$$

Let $\delta \in 2^{\mathbb{Z}}$ be such that (assume $\delta \geq N^{-2}$ for now)

$$\left| \frac{n}{|n|} - \frac{n+a}{|n+a|} \right| = 2 \sin\left(\frac{\angle(n, n+a)}{2} \right) \in [\delta, 2\delta] \qquad (*)$$

Since $|a| \sim A \leq N/100 \sim |n|/100$, we have $\delta \leq \frac{1}{10}$.

Denote $\frac{n}{|n|} = n' = (x', y', z')$ and $\frac{a+n}{|a+n|} = n'' = (x'', y'', z'')$.

We claim: $\max\left(|y'-y''|, |z'-z''|\right) \geq \frac{\delta}{100}$ (5.7)

If not $\Rightarrow$ $|y'-y''| < \frac{\delta}{100}$ and $|z'-z''| < \frac{\delta}{100}$

Since $|n'-n''| \geq \delta$, we have $|x'-x''| \geq \frac{\delta}{2}$

$|x| = |n|_\infty > 10A \geq 5|x_a|$, $x' = \frac{x}{|n|}$, $x'' = \frac{x+x_a}{|n+a|}$ $\Rightarrow$ x' and x'' have the same sign

$|x| = |n|_\infty \geq \frac{|n|}{2}$, $x' = \frac{x}{|n|}$ $\Rightarrow$ $|x'| \geq \frac{1}{2}$

(x', y', z') and (x'', y'', z'') both unit $\Rightarrow$

$$\frac{\delta}{2} \leq |x'-x''| = \frac{|(x')^2-(x'')^2|}{|x'|+|x''|} = \frac{|(y')^2+(z')^2-(y'')^2-(z'')^2|}{|x'|+|x''|} \leq 4\left(|y'-y''|+|z'-z''|\right) \leq \frac{\delta}{25}$$

$\Rightarrow$ contradiction $\Rightarrow$ (5.7)

WLOG, $|y'-y''| \geq \frac{\delta}{100}$.

Let $f(n) = |n+a| - |n|$ $\Rightarrow$ $\nabla f(n) = \frac{n+a}{|n+a|} - \frac{n}{|n|} = n''-n' = (x''-x', y''-y', z''-z')$

By assumption, $|f(n)-m| \lesssim 1$ and $\left|\frac{\partial f(n)}{\partial y}\right| = |y''-y'| \geq \frac{\delta}{100}$

$\Rightarrow$ for any fixed x and z, # choices of $y \lesssim \frac{1}{\delta}$

It remains to count # choices of x and z.

Law of sine $\Rightarrow$

$$|xz_a - zx_a| \leq |n \times a| = |n|\cdot|a|\cdot\sin(\angle(n,a)) \sim N\cdot|a|\cdot\frac{|n+a|}{|a|}\sin(\angle(n,n+a)) \lesssim N^2\delta \quad (5.8)$$

By assumption $|x| \gtrsim \Gamma \geq |x| - |x_a|$

① If $x_a = 0$ $\Rightarrow$ # choices of $x \leq 2$, # choices of y and $z \lesssim N^2$

② If $x_a \neq 0$ $\Rightarrow$ # choices of $x \leq 2|x_a|$

When x is fixed, by (5.8), # choices of $z \lesssim \frac{N^2\delta}{|x_a|}$

$\Rightarrow$ # choices for $(x,y,z) \lesssim |x_a|\cdot\frac{N^2\delta}{|x_a|}\cdot\frac{1}{\delta} = N^2$

Sum up $N^{-2} < \delta \leq \frac{1}{10}$ $\Rightarrow$ (5.6)

If (*) becomes

$$\left|\frac{n}{|n|} - \frac{n+a}{|n+a|}\right| = 2\sin\left(\frac{\angle(n,n+a)}{2}\right) < N^{-2}$$

Same computation in (5.8) $\Rightarrow$

$$|xz_a - zx_a| \lesssim 1, \quad |xy_a - yx_a| \lesssim 1$$

$\Rightarrow$ # choices for $(x,y,z) \lesssim N^2$ $\Rightarrow$ (5.6) $\square$

Lemma 5.17 (Direct estimate of the Sine-kernel)

For all frequency scales K and L, all $r \in \mathbb{Z}^3$, all $t \neq 0$, and all $\lambda \in \mathbb{R}$, we have

$$\left| \int_0^t \mathrm{Sine}[K,L](t-t',r)\, e^{i\lambda t'}\, dt' \right| \lesssim \langle t \rangle \frac{\log^2(2+\max(K,L,|\lambda|))}{\max(K,L,|\lambda|)} \tag{5.80}$$

Proof: If $|\lambda| \gg \max(K,L)$, we use the definition of $\mathrm{Sine}[K,L]$ to obtain

$$\begin{aligned} \left| \int_0^t \mathrm{Sine}[K,L](t-t',r)\, e^{i\lambda t'}\, dt' \right| &\lesssim \langle t \rangle \langle \lambda \rangle^{-1} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) \frac{1}{\langle n_1 \rangle} \frac{1}{\langle n_2 \rangle^2} \\ &\lesssim \langle t \rangle \langle \lambda \rangle^{-1} K^{-1} L^{-2} \min(K,L)^3 \\ &\lesssim \langle t \rangle \langle \lambda \rangle^{-1} \qquad \Longrightarrow \quad (5.80) \end{aligned}$$

For $|\lambda| \lesssim \max(K,L)$, by Lemma 5.15 and performing the t'-integral,

$$\left| \int_0^t \mathrm{Sine}[K,L](t-t',r)\, e^{i\lambda t'}\, dt' \right|$$

$$\lesssim \langle t \rangle \langle r \rangle K^{-2} L^{-2} \sum_{\pm} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) \frac{1}{1+|\langle n_1 \rangle - \langle n_2 \rangle \pm \lambda|} \tag{5.81}$$

$$+ \langle t \rangle K^{-2} L^{-2} \max(K,L) \sum_{\pm} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) \frac{1}{1+|\langle n_1 \rangle + \langle n_2 \rangle \pm \lambda|} \tag{5.82}$$

$$+ \langle t \rangle \min(K,L)^{-1} K^{-1} L^{-1} \sum_{\pm_1,\pm_2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \left[\left| \mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) - \mathbb{1}_L(n_1)\, \mathbb{1}_K(n_2) \right| \frac{1}{1+|\langle n_1 \rangle \pm_1 \langle n_2 \rangle \pm_2 \lambda|} \right] \tag{5.83}$$

All summands (5.81), (5.82), (5.83) are symmetric in K and L.

WLOG, $K \geq L$

By level-set decomposition of $\langle n_1 \rangle - \langle n_2 \rangle$ and (5.1) in Lemma 5.1, we have

$$\begin{aligned} (5.81) &\lesssim \langle t \rangle \langle r \rangle \log(2+K) K^{-2} L^{-2} \min(\langle r \rangle, L)^{-1} L^3 \\ &\lesssim \langle t \rangle \log(2+K) \max(\langle r \rangle, L) K^{-2} \\ &\lesssim \langle t \rangle \log(2+K) K^{-1}. \end{aligned}$$

By level-set decomposition of $\langle n_1 \rangle + \langle n_2 \rangle$ and (5.2) in Lemma 5.1, we have

$$(5.82) \lesssim \langle t \rangle \log(2+K) K^{-1} L^{-2} L^2 \lesssim \langle t \rangle \log(2+K) K^{-1}.$$

To deal with (5.83), we discuss two subcases $\langle r \rangle \gtrsim L$ and $\langle r \rangle \ll L$.

If $\langle r \rangle \gtrsim L$, by level-set decomposition of $\langle n_1 \rangle \pm_1 \langle n_2 \rangle$ and (5.1) in Lemma 5.1,

$$\begin{aligned} (5.83) &\lesssim \langle t \rangle K^{-1} L^{-2} \sum_{\pm_1,\pm_2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n_{12}=r}} \left[\left(\mathbb{1}_K(n_1)\, \mathbb{1}_L(n_2) + \mathbb{1}_L(n_1)\, \mathbb{1}_K(n_2) \right) \frac{1}{1+|\langle n_1 \rangle \pm_1 \langle n_2 \rangle \pm_2 \lambda|} \right] \\ &\lesssim \langle t \rangle K^{-1} L^{-2} \log(2+K) \min(\langle r \rangle, L)^{-1} L^3 \\ &\lesssim \langle t \rangle \log(2+K) K^{-1}. \end{aligned}$$

If $\langle r \rangle \ll L$, we have $L \leq K \lesssim \max(L, \langle r \rangle) \lesssim L \Rightarrow K \sim L$

By Lemma 5.15 (Γ- condition), level-set decomposition, and Lemma 5.3,

$$(5.83) \lesssim \langle t \rangle K^{-3} \sum_{\pm_1, \pm_2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ n_{12} = r}} \mathbb{1}\{ |n_1|_\infty \geq \Gamma \geq |n_1 - r|_\infty \text{ or } |n_1|_\infty \leq \Gamma \leq |n_1 - r|_\infty \}$$

$$\times \frac{1}{1 + |\langle n_1 \rangle \pm_1 \langle n_2 \rangle \pm_2 \lambda|}$$

$$\lesssim \langle t \rangle \log^2(2+K) K^{-1} \qquad \Rightarrow \quad (5.80) \qquad \square$$

Reading session 13: The sine-cancellation tensor estimates and the resistor

Lemma 5.18 (The sine-cancellation tensor estimates)

Suppose $N_0, N_2, \dots, N_5, N_{234}$ are dyadic and $\lambda, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{R}$.

Let $N_{max} = \max(N_0, N_2, \dots, N_5)$. Define the tensors

$$h^{\text{sine}}_{n_0n_3n_4n_5}(t, \lambda_3, \lambda_4, \lambda_5) = \mathbb{1}_{\{n_0 = n_{345}\}} \cdot \mathbb{1}_{N_0}(n_0) \cdot \Big(\prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big) e^{it(\pm\langle n_5\rangle + \lambda_5)}$$
$$\times \Big(\int_0^t \chi(t)\chi(t') \cdot \mathrm{Sine}[N_{234}, N_2](t-t', n_{34})\, e^{it'(\pm\langle n_3\rangle \pm \langle n_4 \rangle + \lambda_3 + \lambda_4)} dt' \Big)$$

$$H^{\text{sine}}_{n_0n_3n_4n_5}(t, \lambda_3, \lambda_4, \lambda_5) = \mathbb{1}_{\{n_0 = n_{345}\}} \frac{\mathbb{1}_{N_0}(n_0)}{\langle n_0\rangle} \prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \int_0^t \chi(t)\chi(t') \sin\big((t-t')\langle n_0 \rangle\big)$$
$$\times e^{it'(\pm\langle n_5\rangle + \lambda_5)} \Big(\int_0^{t'} \chi(t')\chi(t'') \cdot \mathrm{Sine}[N_{234}, N_2](t'-t'', n_{34})\, e^{it''(\pm\langle n_3\rangle \pm\langle n_4\rangle + \lambda_3 + \lambda_4)} dt'' \Big) dt'.$$

Then, there exist two functions $C_j = C_j(\lambda, \lambda_3, \lambda_4, \lambda_5)$ for $j = 1, 2$ such that

$$\| \langle \lambda \rangle^{b_+ - 1} C_1(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon}, \qquad \| \langle \lambda \rangle^{b_+} C_2(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{L^2_\lambda} \lesssim N_{max}^{\varepsilon} \qquad (5.87)$$

and that we have

$$\| \widetilde{h^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_3n_4n_5} \lesssim N_0 N_{max}^{-\frac{1}{2}} \cdot C_1(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.88)$$

$$\| \widetilde{h^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_3n_4n_5} \lesssim \frac{\min(N_0, N_5)^{\frac{3}{2}} (N_3 N_4)^{\frac{1}{2}}}{N_5 \cdot \max(N_2, N_{234})} C_1(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.89)$$

$$\| \widetilde{H^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_3n_4n_5} \lesssim N_{max}^{-\frac{1}{2}} \cdot C_2(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.90)$$

$$\| \widetilde{h^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_A \to n_Bn_5} \lesssim N_0^{\frac{1}{2}} N_5^{-\frac{1}{2}} N_{max}^{-\frac{1}{2}} \cdot C_1(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.91)$$

$$\| \widetilde{H^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_A \to n_Bn_5} \lesssim N_0^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_{max}^{-\frac{1}{2}} \cdot C_2(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.92)$$

$$\| \widetilde{h^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_5 \to n_3n_4} \lesssim N_0 \big(\max(N_0, N_5) \cdot \max(N_3, N_4)\big)^{-\frac{1}{2}} N_2^{-1} \cdot C_1(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.93)$$

$$\| \widetilde{H^{\text{sine}}}(\lambda, \lambda_3, \lambda_4, \lambda_5) \|_{n_0n_5 \to n_3n_4} \lesssim \big(\max(N_0, N_5) \cdot \max(N_3, N_4)\big)^{-\frac{1}{2}} N_2^{-1} \cdot C_2(\lambda, \lambda_3, \lambda_4, \lambda_5), \qquad (5.94)$$

for any partition (A, B) of $\{3, 4\}$. The same bounds hold for all ∂_λ and ∂_{λ_j} derivatives of these tensors.

Proof: We also dyadically localize $|n_3 + n_4| \sim N_{34}$ by losing a factor of $\log(N_{max})$.

By Lemma 2.3 and Lemma 5.15, we decompose $\widetilde{h^{\text{sine}}}$ into $h^{(-)}$, $h^{(+)}$, and $h^{(0)}$:

$$h^{(-)}_{n_0n_3n_4n_5} = \sum_{n_2 \in \mathbb{Z}^3} K\Big(\lambda - \lambda_5 \pm \langle n_5 \rangle \pm \langle n_0 \rangle \mp_{234} (\langle n_{234} \rangle - \langle n_2 \rangle),\ \lambda_3 + \lambda_4 \pm_{234} (\langle n_{234}\rangle - \langle n_2 \rangle) \pm \langle n_3 \rangle \pm \langle n_4 \rangle \Big)$$
$$\times \mathbb{1}_{\{n_0 = n_{345}\}} \mathbb{1}_{N_0}(n_0) \mathbb{1}_{N_{34}}(n_3 + n_4) \mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_2}(n_2) \cdot \frac{\langle n_{234} \rangle - \langle n_2 \rangle}{\langle n_{234}\rangle^2 \langle n_2 \rangle^2} \Big(\prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big)$$

$$h^{(+)}_{n_0n_3n_4n_5} = \sum_{n_2 \in \mathbb{Z}^3} K\Big(\lambda - \lambda_5 \pm \langle n_5 \rangle \pm \langle n_0 \rangle \mp_{234} (\langle n_{234} \rangle + \langle n_2 \rangle),\ \lambda_3 + \lambda_4 \pm_{234} (\langle n_{234}\rangle + \langle n_2 \rangle) \pm \langle n_3 \rangle \pm \langle n_4 \rangle \Big)$$
$$\times \mathbb{1}_{\{n_0 = n_{345}\}} \mathbb{1}_{N_0}(n_0) \mathbb{1}_{N_{34}}(n_3 + n_4) \mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_2}(n_2) \cdot \frac{\langle n_{234} \rangle + \langle n_2 \rangle}{\langle n_{234}\rangle^2 \langle n_2 \rangle^2} \Big(\prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \Big)$$

$$h^{(0)}_{n_0n_3n_4n_5} = \sum_{n_2\in\mathbb{Z}^3} K\Big(\lambda - \lambda_5 \pm \langle n_5\rangle \pm \langle n_0\rangle \mp_{234} \big(\langle n_{234}\rangle \pm \langle n_2\rangle\big),\ \lambda_3 + \lambda_4 \pm_{234} \big(\langle n_{234}\rangle \pm \langle n_2\rangle\big) \pm \langle n_3\rangle \pm \langle n_4\rangle\Big)$$

$$\times \mathbb{1}_{\{n_0 = n_{345}\}} \mathbb{1}_{N_0}(n_0)\, \mathbb{1}_{N_{34}}(n_3+n_4)\, \frac{\mathbb{1}_{N_{234}}(n_{234})\mathbb{1}_{N_2}(n_2) - \mathbb{1}_{N_2}(n_{234})\mathbb{1}_{N_{234}}(n_2)}{\langle n_{234}\rangle^2\langle n_2\rangle} \Big(\prod_{j=3}^{5}\frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j\rangle}\Big)$$

Similarly, we can decompose $\widetilde{H}^{\text{sine}}$ into $H^{(-)}$, $H^{(+)}$, and $H^{(0)}$, with $K(\cdots)$ replaced by $\int_{\mathbb{R}} K(\lambda,\sigma)\cdot K(\sigma - \lambda_5 \pm \cdots,\ \lambda_3 + \lambda_4 \pm \cdots)\, d\sigma$

We focus on $h^{(-)}$ and $H^{(-)}$ in (1) - (3) below, and discuss the other two in (4)

(1) For $h^{(-)}$, we define (similar to Lemma 5.11)

$$C_1(\lambda,\lambda_3,\lambda_4,\lambda_5) = \sum_{\substack{m_1,m_2\in\mathbb{Z}\\ |m_1|,|m_2|\lesssim N_{\max}}} \frac{1}{\langle\lambda-\lambda_5-m_1\rangle\langle\lambda - \sum_{j=3}^{5}\lambda_j - m_2\rangle}$$

The bound (5.87) for C_1 then follows from the same way as (5.50)

By level-set decomposition and (2.31)

$$\|h^{(-)}\|_{n_0n_3n_4n_5} \lesssim N_3^{-1}N_4^{-1}N_5^{-1}\, C_1(\lambda,\lambda_3,\lambda_4,\lambda_5)\cdot \sup_{m_1,m_2\in\mathbb{Z}} \Big\| \mathbb{1}_{\{n_0=n_{345}\}}\mathbb{1}_{N_0}(n_0)\,\mathbb{1}_{N_{34}}(n_3+n_4)\prod_{j=3}^{5}\mathbb{1}_{N_j}(n_j)$$

$$\times\, \mathbb{1}_{\{|\Omega-m_2|\le 1\}}\cdot\Big(\sum_{n_2}\mathbb{1}_{N_2}(n_2)\,\mathbb{1}_{N_{234}}(n_{234})\,\frac{|\langle n_{234}\rangle - \langle n_2\rangle|}{\langle n_{234}\rangle^2\langle n_2\rangle^2}\cdot\mathbb{1}_{\{|\Omega'-m_1|\le 1\}}\Big)\Big\|_{n_0n_3n_4n_5}, \qquad (5.97)$$

where $\Omega = \pm\langle n_0\rangle \pm\langle n_3\rangle \pm\langle n_4\rangle \pm\langle n_5\rangle$, $\Omega' = \pm\langle n_0\rangle \pm \langle n_5\rangle \mp_{234}\big(\langle n_{234}\rangle - \langle n_2\rangle\big)$

Denote $\mathfrak{h}^{(-)}$ to be everything inside the $\|\cdot\|$ norm in (5.97)

By Lemma 5.1 (5.1) and $|\Omega'-m_1|\le 1$,

$$|\mathfrak{h}^{(-)}| \lesssim N_2^{-2}N_{234}^{-2}\ \overset{\substack{\text{from } |\langle n_{234}\rangle-\langle n_2\rangle| \\ \uparrow}}{N_{34}}\cdot\Big(\min(N_2,N_{234})^2 + \min(N_2,N_{234})^3 N_{34}^{-1}\Big) \lesssim \max(N_2,N_{234})^{-1} \qquad (5.99)$$

Thus, we have

$$\|h^{(-)}\|^2_{n_0n_3n_4n_5} \lesssim \Big((N_3N_4N_5)^{-2}C_1^2\max(N_2,N_{234})^{-2}\Big)\cdot \underbrace{\sum_{(n_0,n_3,n_4,n_5)} 1}_{\lesssim (N_3N_4\min(N_0,N_5))^3} \implies (5.89)$$

For (5.88), let $i\in\{0,3,4,5\}$ with $N_i = \min(N_0,N_3,N_4,N_5)$.

If $\max(N_2,N_{234}) \gtrsim N_i$, then by Lemma 5.4 (5.16) and (5.99),

$$\|h^{(-)}\|^2_{n_0n_3n_4n_5} \lesssim (N_3N_4N_5)^{-2}C_1^2\max(N_2,N_{234})^{-2}\cdot(N_0N_3N_4N_5)^2\frac{N_i}{\max(N_0,N_3,N_4,N_5)}$$

$$\lesssim N_0^2 N_{\max}^{-1}\cdot C_1^2 \implies (5.88)$$

If $\max(N_2, N_{234}) \ll N_i$, then fix (n_i, n_j) with $\{i,j\} \in \{\{0,5\},\{3,4\}\}$

By (5.99), $|n_i \pm n_j| \sim N_{34}$, and Lemma 5.1 (5.1),

$$\|h^{(-)}\|^2_{n_0 n_3 n_4 n_5} \lesssim (N_3 N_4 N_5)^{-2} C_1^2 \max(N_2, N_{234})^{-2} \cdot N_i^3 N_{34}^3 \cdot \max(N_0, N_3, N_4, N_5)^3 N_{34}^{-1}$$

$$\lesssim N_0^2 N_i^{-1} N_{\max}^{-1} \cdot C_1^2 \quad \Rightarrow \quad (5.88)$$

if $N_{\max} \neq N_2$, $N_{34}^2 \lesssim \max(N_2, N_{234})^2$

if $N_{\max} = N_2$, $N_{34} \lesssim \max(N_2, N_{234})$

(5.90) then follows from (5.88) and by defining

$$C_2(\lambda, \lambda_3, \lambda_4, \lambda_5) = \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ |m_1|, |m_2| \leq N_{\max}}} \int_{\mathbb{R}} \frac{d\sigma}{\langle\lambda\rangle\langle\lambda - \sigma\rangle\langle\sigma - \lambda_5 - m_1\rangle\langle\sigma - \sum_{j=3}^{5} \lambda_j - m_2\rangle}$$

The bound (5.87) for C_2 then follows from the same way as (5.50)

(2) We now turn to (5.91) ((5.92) follows similarly)

By the same reduction above, it suffices to show

$$\|h^{(-)}\|_{n_0 n_A \to n_B n_5} \lesssim N_0 N_3 N_4 N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \cdot N_{\max}^{-\frac{1}{2}} \tag{5.101}$$

If $A = \{3\}$ and $B = \{4\}$, by Schur's test and (5.99),

$$\|h^{(-)}\|_{n_0 n_3 \to n_4 n_5} \lesssim \max(N_2, N_{234})^{-1} \min(N_0, N_3, N_{34})^{\frac{3}{2}} \min(N_4, N_5, N_{34})^{\frac{3}{2}}$$

$$\lesssim \max(N_2, N_{234})^{-1} \min(N_0, N_3)^{\frac{1}{2}} N_3^{\frac{1}{2}} N_{34}^{\frac{1}{2}} \min(N_4, N_5)^{\frac{1}{2}} N_4^{\frac{1}{2}} N_{34}^{\frac{1}{2}}$$

$$= N_0 N_3 N_4 N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \cdot \max(N_0, N_3)^{-\frac{1}{2}} \max(N_4, N_5)^{-\frac{1}{2}} \frac{N_{34}}{\max(N_2, N_{234})}$$

$$\lesssim N_0 N_3 N_4 N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \cdot N_{\max}^{-\frac{1}{2}} \quad \Rightarrow \quad (5.101)$$

$N_{34}^{\frac{1}{2}} \lesssim \max(N_2, N_{234})^{\frac{1}{2}}$

If $A = \{4\}$ and $B = \{3\}$, similar as above

If $A = \emptyset$ and $B = \{3,4\}$:

- If $N_{\max} \sim N_0$ or N_2 or N_{234}, by Lemma 5.7 (5.28) and (5.99),

$$\|h^{(-)}\|_{n_0 \to n_3 n_4 n_5} \lesssim \max(N_2, N_{234})^{-1} N_0 N_3 N_4 N_5 \cdot N_0^{-1} N_5^{-\frac{1}{2}} \quad \Rightarrow \quad (5.101)$$

- If $N_0, N_2, N_{234} \ll N_{\max}$, then $N_{34} \ll N_{\max}$, so $N_3 \sim N_4 \sim N_{\max}$

  By (5.99), Schur's test, fixing n_5, and Lemma 5.1 (5.1),

$$\|h^{(-)}\|_{n_0 \to n_3 n_4 n_5} \lesssim N_{34}^{-1} \cdot N_{34}^{\frac{3}{2}} \cdot N_3^{\frac{3}{2}} N_{34}^{-\frac{1}{2}} = N_3^{\frac{3}{2}} \lesssim N_0 N_3 N_4 N_5 \cdot (N_0 N_5)^{-\frac{1}{2}} \cdot N_{\max}^{-\frac{1}{2}} \quad \Rightarrow \quad (5.101)$$

If $A = \{3,4\}$ and $B = \emptyset$, similar as above

(3) We now consider (5.93) ((5.94) follows similarly)

By the same reduction above, it suffices to show

$$\|\mathfrak{h}^{(-)}\|_{n_0 n_5 \to n_3 n_4} \lesssim N_0 N_3 N_4 N_5 \cdot \left(\max(N_0, N_5) \cdot \max(N_3, N_4) \right)^{-\frac{1}{2}} \cdot N_2^{-1} \qquad (5.102)$$

By (5.99) and Schur's test,

$$\begin{aligned}\|\mathfrak{h}^{(-)}\|_{n_0 n_5 \to n_3 n_4} &\lesssim \max(N_2, N_{234})^{-1} \cdot \min(N_0, N_5)^{\frac{3}{2}} \cdot \min(N_3, N_4)^{\frac{3}{2}} \\ &\lesssim N_2^{-1} \cdot N_0 N_5 \max(N_0, N_5)^{-\frac{1}{2}} \cdot N_3 N_4 \max(N_3, N_4)^{-\frac{1}{2}} \implies (5.102)\end{aligned}$$

(4) For $h^{(+)}$ and $h^{(0)}$, as in the case of $h^{(-)}$, we define

$$\begin{aligned}\mathfrak{h}^{(+)} = \;& \mathbb{1}_{\{n_0 = n_{345}\}} \mathbb{1}_{N_0}(n_0) \mathbb{1}_{N_{34}}(n_3 + n_4) \prod_{j=3}^{5} \mathbb{1}_{N_j}(n_j) \\ & \times \mathbb{1}_{\{|\Omega - m_2| \le 1\}} \cdot \left(\sum_{n_2} \mathbb{1}_{N_2}(n_2) \mathbb{1}_{N_{234}}(n_{234}) \frac{\langle n_{234}\rangle + \langle n_2 \rangle}{\langle n_{234}\rangle^2 \langle n_2\rangle^2} \cdot \mathbb{1}_{\{|\Omega^{(+)} - m_1| \le 1\}} \right) \\ \mathfrak{h}^{(-)} = \;& \mathbb{1}_{\{n_0 = n_{345}\}} \mathbb{1}_{N_0}(n_0) \mathbb{1}_{N_{34}}(n_3 + n_4) \prod_{j=3}^{5} \mathbb{1}_{N_j}(n_j) \cdot \mathbb{1}_{\{|\Omega - m_2| \le 1\}} \\ & \times \left(\sum_{n_2} \frac{\mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_2}(n_2) - \mathbb{1}_{N_2}(n_{234}) \mathbb{1}_{N_{234}}(n_2)}{\langle n_{234}\rangle^2 \langle n_2 \rangle} \cdot \mathbb{1}_{\{|\Omega^{(0)} - m_1| \le 1\}} \right)\end{aligned}$$

where

$$\Omega = \pm\langle n_0\rangle \pm \langle n_3 \rangle \pm \langle n_4 \rangle \pm \langle n_5 \rangle,$$

$$\Omega^{(+)} = \pm\langle n_0\rangle \pm \langle n_5 \rangle \mp_{234} \left(\langle n_{234}\rangle + \langle n_2 \rangle \right),$$

$$\Omega^{(0)} = \pm\langle n_0\rangle \pm \langle n_5 \rangle \pm_{234} \left(\langle n_{234}\rangle \pm \langle n_2 \rangle \right).$$

By parts (1), (2), (3), it suffices to show (5.99) for $\mathfrak{h}^{(+)}$ and $\mathfrak{h}^{(-)}$.

By Lemma 5.1 (5.2), we have

$$|\mathfrak{h}^{(+)}| \lesssim N_2^{-2} N_{234}^{-2} (N_2 + N_{234}) \cdot \min(N_2, N_{234})^2 \lesssim \max(N_2, N_{234})^{-1} \implies (5.99)$$

Recall from Lemma 5.15 that n_2 and n_{234} in $\mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_2}(n_2) - \mathbb{1}_{N_2}(n_{234}) \mathbb{1}_{N_{234}}(n_2)$ satisfy the Γ-condition (5.5).

By Lemma 5.3, we have

$$|\mathfrak{h}^{(0)}| \lesssim N_2^{-1} N_{234}^{-2} \cdot \min(N_2, N_{234})^2 \log(N_{\max}) \lesssim \max(N_2, N_{234})^{-1} \underset{\substack{\downarrow \\ \text{can be absorbed in } C_1 \text{ or } C_2}}{\log(N_{\max})}$$

□

Corollary 5.19 Suppose $\varphi_j \in \{\sin, \cos\}$ for $3 \leq j \leq 5$ and $N_0, N_2, \dots, N_5, N_{234}$ are dyadic. Let $N_{max} = \max(N_0, N_2, \dots, N_5)$. Consider the Sine tensor $H^{sine}_{n_0 n_3 n_4 n_5}(t)$ defined by

$$H^{sine}_{n_0 n_3 n_4 n_5}(t) = \mathbb{1}_{\{n_0 = n_{345}\}} \frac{\mathbb{1}_{N_0}(n_0)}{\langle n_0 \rangle} \prod_{j=3}^{5} \frac{\mathbb{1}_{N_j}(n_j)}{\langle n_j \rangle} \int_0^t \chi(t)\,\chi(t')\,\sin\big((t-t')\langle n_0 \rangle\big)\,\varphi_5(t'\langle n_5 \rangle)$$
$$\times \Big(\int_0^{t'} \chi(t')\,\chi(t'') \cdot \mathrm{Sine}[N_{234}, N_2](t'-t'', n_{34})\,\varphi_3(t''\langle n_3 \rangle)\,\varphi_4(t''\langle n_4 \rangle)\,dt'' \Big)\,dt'.$$

Then, we have

$$\| \langle \lambda \rangle^{b_+} \widetilde{H}_{n_0 n_3 n_4 n_5}(\lambda) \|_{L^2_\lambda [n_0 n_3 n_4 n_5]} \lesssim N_{max}^{-\frac{1}{2}+\varepsilon} \qquad (5.106)$$

$$\| \langle \lambda \rangle^{b_+} \widetilde{H}_{n_0 n_3 n_4 n_5}(\lambda) \|_{L^2_\lambda [n_0 n_A \to n_B n_5]} \lesssim (N_0 N_5)^{-\frac{1}{2}} \cdot N_{max}^{-\frac{1}{2}+\varepsilon} \qquad (5.107)$$

$$\| \langle \lambda \rangle^{b_+} \widetilde{H}_{n_0 n_3 n_4 n_5}(\lambda) \|_{L^2_\lambda [n_0 n_5 \to n_3 n_4]} \lesssim N_{max}^{\varepsilon} \cdot \big(\max(N_0, N_5) \cdot \max(N_3, N_4) \big)^{-\frac{1}{2}} \cdot N_2^{-1} \qquad (5.108)$$

for any partition (A, B) of $\{3, 4\}$.

Proof: The bounds follow directy from Lemma 5.18 with $\lambda_3 = \lambda_4 = \lambda_5 = 0$ □

- The resistor

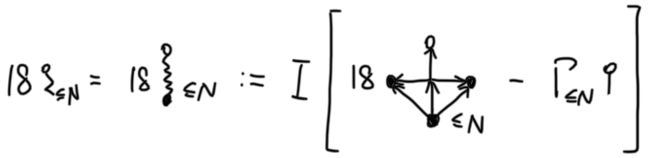
Lemma 7.10 For all $T \geq 1$ and $p \geq 2$, we have

$$\mathbb{E}\Big[\sup_N \| \xi_{\leq N} \|^p_{(L^\infty_t C^{\frac{1}{2}-\varepsilon}_x \cap X^{\frac{1}{2}-\varepsilon, b})([-T,T])} \Big]^{1/p} \lesssim p^{\frac{1}{2}} T^{\alpha}.$$

Proof: We only consider $T = 1$ (general case minor modification).

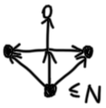
We only prove the $X^{\frac{1}{2}-\varepsilon, b_+}$-estimate ($L^\infty_t C^{\frac{1}{2}-\varepsilon}_x$ similar as in Lemma 7.4 (note 9)).

Recall Definition 6.13:



I = Duhamel

By Lemma 6.2 and Definition 7.2,

18 

$$- \Gamma_{\le N}\varphi = -\sum_{n_0\in\mathbb{Z}^3}\Big[\langle n_0\rangle^{-1}\Gamma_{\le N}(n_0,t)\,e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\cos}(n_0)\Big]$$

$$-\sum_{\varphi\in\{\cos,\sin\}}\sum_{n_0\in\mathbb{Z}^3}\Big[\Big(\int_0^t\Gamma_{\le N}(n_0,t-t')(\partial_t\varphi)(t'\langle n_0\rangle)\,dt'\Big)e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\varphi}(n_0)\Big]$$

$$= -\sum_{N_0,N_1,N_2,N_3\le N}\sum_{n_0\in\mathbb{Z}^3}\Big[\langle n_0\rangle^{-1}\Gamma[N_*](n_0,t)\,e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\cos}(n_0)\Big]$$

$$-\sum_{\varphi\in\{\cos,\sin\}}\sum_{N_0,N_1,N_2,N_3\le N}\sum_{n_0\in\mathbb{Z}^3}\Big[\Big(\int_0^t\Gamma[N_*](n_0,t-t')(\partial_t\varphi)(t'\langle n_0\rangle)\,dt'\Big)e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\varphi}(n_0)\Big]$$

By Gaussian hypercontractivity and Lemma 2.4, it suffices to show

$$\mathbb{E}\Big[\Big\|\sum_{n_0\in\mathbb{Z}^3}\Big[\langle n_0\rangle^{-1}\Gamma[N_*](n_0,t)\,e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\cos}(n_0)\Big]\Big\|^2_{X^{-\frac{1}{2}-\varepsilon,b_+-1}}\Big] \tag{7.34}$$

$$+\,\mathbb{E}\Big[\Big\|\sum_{n_0\in\mathbb{Z}^3}\Big[\Big(\int_0^t\Gamma[N_*](n_0,t-t')(\partial_t\varphi)(t'\langle n_0\rangle)\,dt'\Big)e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\varphi}(n_0)\Big]\Big\|^2_{X^{-\frac{1}{2}-\varepsilon,b_+-1}}\Big] \tag{7.35}$$

$$\lesssim N_{max}^{-\varepsilon}$$

By Lemma 7.3 (7.6),

$$(7.34) \lesssim \sum_{\pm_0}\sum_{n_0\in\mathbb{Z}^3}\langle n_0\rangle^{-3-2\varepsilon}\int_{\mathbb{R}}\langle\lambda\rangle^{2(b_+-1)}\Big|\int_{\mathbb{R}}\chi(t)\,\Gamma[N_*](n_0,t)\,e^{i(\pm_0\langle n_0\rangle+\lambda)t}\,dt\Big|^2 d\lambda$$

$$\lesssim \log^2(N_{max})\,N_{max}^{-2+\varepsilon}\sum_{\pm_0}\sum_{n_0\in\mathbb{Z}^3}\langle n_0\rangle^{-3-2\varepsilon}\int_{\mathbb{R}}\langle\lambda\rangle^{2(b_+-1)}\langle\lambda\pm_0\langle n_0\rangle\rangle^{-\varepsilon}\,d\lambda$$

$$\lesssim N_{max}^{-2+\frac{\varepsilon}{2}} \qquad \checkmark$$

By Lemma 7.3 (7.5),

$$(7.35) \lesssim \mathbb{E}\Big[\Big\|\sum_{n_0\in\mathbb{Z}^3}\Big[\Big(\int_0^t\Gamma[N_*](n_0,t-t')(\partial_t\varphi)(t'\langle n_0\rangle)\,dt'\Big)e^{i\langle n_0,x\rangle}\int_{[0,1]}1\,dW_s^{\varphi}(n)\Big]\Big\|^2_{L_t^2H_x^{-\frac{1}{2}-\varepsilon}}\Big]$$

$$\lesssim \sum_{n_0\in\mathbb{Z}^3}\langle n_0\rangle^{-1-2\varepsilon}\Big|\int_0^t\Gamma[N_*](n_0,t-t')(\partial_t\varphi)(t'\langle n_0\rangle)\,dt'\Big|^2$$

$$\lesssim N_0^{2-2\varepsilon}\log^2(N_{max})\,N_{max}^{-2}$$

$$\lesssim N_{max}^{-\varepsilon} \qquad \checkmark$$

□

Reading session 14 : Sextic stochastic objects

- Sextic stochastic objects (Subsection 7.6)

Proposition 7.15 For any dyadic $N_1 \leq N_2$, we have

$$\| P_{N_1} \Psi_{\leq N} \cdot P_{N_2} \Psi_{\leq N} - \mathcal{C}^{(3,3)}_{\leq N}[N_1, N_2] \|_{L^p_\omega(C^0_t C^{-\frac{1}{10}}_x([-T,T]))} \lesssim p^3 T^\alpha N_2^{-100\nu} \qquad (7.55)$$

$\curvearrowleft$ $\mathbb{E}[\cdot]$ $\qquad\qquad\qquad\qquad$ $\nu \ll 1$

Proof: Define $M_{3,3} = P_{N_1} \Psi_{\leq N} \cdot P_{N_2} \Psi_{\leq N} - \mathcal{C}^{(3,3)}_{\leq N}[N_1, N_2]$.

Consider $P_{\leq N_2^{\sqrt{\nu}}} M_{3,3}$ and $P_{\geq N_2^{\sqrt{\nu}}} M_{3,3}$.

By Lemma 7.5, we have

$$\| P_{N_j} \Psi_{\leq N} \|_{L^{2p}_\omega(C^0_t C^0_x([-T,T]))} \lesssim (2p)^{\frac{3}{2}} T^\alpha N_2^\varepsilon , \qquad j = 1, 2$$

so that

$$\| P_{N_1} \Psi_{\leq N} \cdot P_{N_2} \Psi_{\leq N} \|_{L^p_\omega(C^0_t C^0_x([-T,T]))} \lesssim p^3 T^\alpha N_2^\varepsilon \implies (7.55) \text{ for } P_{\geq N_2^{\sqrt{\nu}}} M_{3,3}$$

We consider $P_{\leq N_2^{\sqrt{\nu}}} M_{3,3}$ below.

By the embedding $W^{\varepsilon_1, p_1}_t W^{\varepsilon_2, p_2}_x \hookrightarrow C^0_t C^0_x$ for $0 < \varepsilon_1, \varepsilon_2 \ll 1$ and $1 < p_1, p_2 < \infty$, Minkowski's inequality, and Gaussian hypercontractivity, it suffice to show

$$\mathbb{E}[|M_{3,3}(t,x)|^2] \lesssim N_2^{-50\sqrt{\nu}} \quad \text{for fixed } (t,x)$$

(for W^{ε_1, p_1}_t, intepolate between $L^{p_1}_t$ and W^{1,p_1}_t and transfer time derivative to space derivative)

As in Lemma 7.5, we decompose $P_{N_1} \Psi_{\leq N}$ into $\Psi[M_1, M_2, M_3, M_{123} = N_1]$

$\cdots$ $P_{N_2} \Psi_{\leq N}$ into $\Psi[M_4, M_5, M_6, M_{456} = N_2]$

By (a dyadic version of) (6.40) and Corollary 5.10, we have

$$\Psi[M_1, M_2, M_3, M_{123} = N_1](t,x) = \sum_{n_0, n_1, n_2, n_3} H_{n_0 n_1 n_2 n_3}(t) e^{i n_0 \cdot x} \prod_{j=1}^{3} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j),$$

$$\Psi[M_4, M_5, M_6, M_{456} = N_2](t,x) = \sum_{n_0', n_4, n_5, n_6} (H')_{n_0' n_4 n_5 n_6}(t) e^{i n_0' \cdot x} \prod_{j=4}^{6} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j),$$

where $H_{n_0 n_1 n_2 n_3}(t)$ and $(H')_{n_0' n_4 n_5 n_6}(t)$ satisfy

$$\sup_t \|H\|_{n_0 n_1 n_2 n_3} \lesssim M_{123}^\varepsilon , \qquad \sup_t \|H\|_{n_A \to n_B} \lesssim \max(M_1, M_2, M_3)^{-\frac{1}{2}+\varepsilon}$$

$$\sup_t \|H'\|_{n_0' n_4 n_5 n_6} \lesssim M_{456}^\varepsilon , \qquad \sup_t \|H'\|_{n_A \to n_B} \lesssim \max(M_4, M_5, M_6)^{-\frac{1}{2}+\varepsilon}$$

for any partition (A, B) of $\{0,1,2,3\}$ or $\{0', 4, 5, 6\}$ with $A, B \neq \emptyset$.

Thus, by the product formula in Lemma 2.10,

$$P_{\leq N_2^{\sqrt{v}}}\left(P_{N_1}\Psi^{\curlyvee}_{\leq N}\cdot P_{N_2}\Psi^{\curlyvee}_{\leq N}\right)(t,x)$$

$$=\sum_{\mathcal{P}}\sum_{\substack{n_0,n_0',n_1,\dots,n_6\\ n_i+n_j=0\ \forall(i,j)\in\mathcal{P}}} e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}H_{n_0n_1n_2n_3}(t)\,(H')_{n_0'n_4n_5n_6}(t)\prod_{j\in\mathcal{O}}\int_0^1 1\,dW^{\varphi_j}_{s_j}(n_j), \qquad (7.60)$$

where $\mathcal{P}$ is a collection of pairings (disjoint two-element subsets $\{i,j\}$ of $\{1,\dots,6\}$) not containing any subset of $\{1,2,3\}$ or of $\{4,5,6\}$, and $\mathcal{O}$ is the set of indices in $\{1,\dots,6\}$ not appearing in $\mathcal{P}$.

The term $C^{(3,3)}_{\leq N}[N_1,N_2]$ exactly corresponds to the case where $\mathcal{P}$ contains three pairings

$\Rightarrow$ $P_{\leq N_2^{\sqrt{v}}}M_{3,3}$ can be written as in (7.60) with $\mathcal{P}$ containing at most two pairings

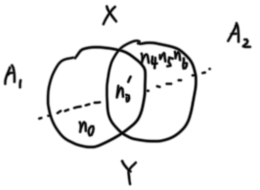
(1) $\mathcal{P}=\emptyset$.

$$P_{\leq N_2^{\sqrt{v}}}M_{3,3}(t,x)=\sum_{n_0,n_0',n_1,\dots,n_6}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}H_{n_0n_1n_2n_3}(t)\,(H')_{n_0'n_4n_5n_6}(t)\prod_{j=1}^{6}\int_0^1 1\,dW^{\varphi_j}_{s_j}(n_j)$$

By merging estimates (Lemma B.1), we have

$$\mathbb{E}\left[\left|P_{\leq N_2^{\sqrt{v}}}M_{3,3}(t,x)\right|^2\right]^{1/2}\lesssim\left\|\sum_{n_0,n_0'}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}H_{n_0n_1n_2n_3}(t)\,(H')_{n_0'n_4n_5n_6}(t)\right\|_{n_1\cdots n_6}$$

$$\lesssim\|H\|_{n_0n_1n_2n_3}\left\|\sum_{n_0'}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}(H')_{n_0'n_4n_5n_6}(t)\right\|_{n_0\to n_4n_5n_6}$$



Lm B.1

$$\lesssim\|H\|_{n_0n_1n_2n_3}\left\|e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}\right\|_{n_0'\to n_0}\|H'\|_{n_4n_5n_6\to n_0'}$$

Schur's test

$$\lesssim N_1^{\varepsilon}N_2^{3\sqrt{v}}\max(M_4,M_5,M_6)^{-\frac{1}{2}+\varepsilon}\lesssim N_2^{-50\sqrt{v}} \qquad \checkmark$$

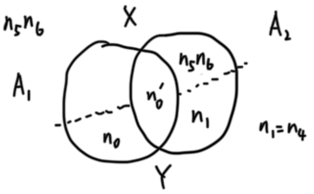
(2) $|\mathcal{P}|=1$, say $\mathcal{P}=\{\{1,4\}\}$ by symmetry

$$P_{\leq N_2^{\sqrt{v}}}M_{3,3}(t,x)=\sum_{n_0,n_0',n_1,n_2,n_3,n_5,n_6}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}H_{n_0n_1n_2n_3}(t)\,(H')_{n_0',-n_1,n_5n_6}(t)\prod_{j=2,3,5,6}\int_0^1 1\,dW^{\varphi_j}_{s_j}(n_j)$$

By merging estimates (Lemma B.1), we have

$$\mathbb{E}\left[\left|P_{\leq N_2^{\sqrt{v}}}M_{3,3}(t,x)\right|^2\right]^{1/2}\lesssim\left\|\sum_{n_0,n_0',n_1}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}H_{n_0n_1n_2n_3}(t)\,(H')_{n_0',-n_1,n_5n_6}(t)\right\|_{n_2n_3n_5n_6}$$

$$\lesssim\|H\|_{n_0n_1n_2n_3}\left\|\sum_{n_0'}e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}(H')_{n_0',-n_1,n_5n_6}(t)\right\|_{n_0n_1\to n_5n_6}$$



Lm B.1

$$\lesssim\|H\|_{n_0n_1n_2n_3}\left\|e^{i(n_0+n_0')\cdot x}\mathbb{1}_{\{|n_0+n_0'|\leq N_2^{\sqrt{v}}\}}\right\|_{n_0'\to n_0}\|H'\|_{n_5n_6\to n_0'n_4}$$

Schur's test

$$\lesssim N_1^{\varepsilon}N_2^{3\sqrt{v}}\max(M_4,M_5,M_6)^{-\frac{1}{2}+\varepsilon}\lesssim N_2^{-50\sqrt{v}} \qquad \checkmark$$

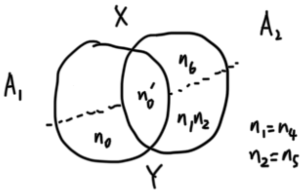
(3) $|P| = 2$, say $P = \{\{1,4\},\{2,5\}\}$ by symmetry

$$P_{\le N_2^{\sqrt{v}}} M_{3,3}(t,x) = \sum_{n_0, n_0', n_1, n_2, n_3, n_6} e^{i(n_0+n_0')\cdot x} \mathbb{1}_{\{|n_0+n_0'| \le N_2^{\sqrt{v}}\}} H_{n_0 n_1 n_2 n_3}(t) \, (H')_{n_0', -n_1, -n_2, n_6} \prod_{j=3,6} \int_0^1 1 \, dW_{s_j}^{\varphi_j}(n_j)$$

By merging estimates (Lemma B.1), we have

$$\mathbb{E}\left[\left| P_{\le N_2^{\sqrt{v}}} M_{3,3}(t,x) \right|^2 \right]^{1/2} \lesssim \Big\| \sum_{n_0, n_0', n_1, n_2} e^{i(n_0+n_0')\cdot x} \mathbb{1}_{\{|n_0+n_0'| \le N_2^{\sqrt{v}}\}} H_{n_0 n_1 n_2 n_3}(t) \, (H')_{n_0', -n_1, -n_2, n_6}(t) \Big\|_{n_3 n_6}$$

$$\lesssim \|H\|_{n_0 n_1 n_2 n_3} \Big\| \sum_{n_0'} e^{i(n_0+n_0')\cdot x} \mathbb{1}_{\{|n_0+n_0'| \le N_2^{\sqrt{v}}\}} (H')_{n_0', -n_1, -n_2, n_6}(t) \Big\|_{n_0 n_1 n_2 \to n_6}$$



$$\overset{\text{Lm B.1}}{\lesssim} \|H\|_{n_0 n_1 n_2 n_3} \underbrace{\left\| e^{i(n_0+n_0')\cdot x} \mathbb{1}_{\{|n_0+n_0'| \le N_2^{\sqrt{v}}\}} \right\|_{n_0' \to n_0}}_{\text{Schur's test}} \|H'\|_{n_6 \to n_0' n_4 n_5}$$

$$\lesssim N_1^{\varepsilon} N_2^{3\sqrt{v}} \max(M_4, M_5, M_6)^{-\frac{1}{2}+\varepsilon} \lesssim N_2^{-50\sqrt{v}} \quad \checkmark$$

$\square$

Proposition 7.16 For any dyadic $N_1 \le N_2$, we have

$$\Big\| P_{N_1} \varrho_{\le N} \cdot P_{N_2} \Psi_{\le N} - \underset{\mathbb{E}[\cdot]}{\mathcal{C}^{(1,5)}_{\le N}}[N_1, N_2] \Big\|_{L^p_\omega(C^0_t C_x^{-\frac{1}{10}}([-T,T]))} \lesssim p^3 T^{\alpha} N_2^{-100v} \tag{7.67}$$

Proof: Define $M_{1,5} = P_{N_1} \varrho_{\le N} \cdot P_{N_2} \Psi_{\le N} - \mathcal{C}^{(1,5)}_{\le N}[N_1, N_2]$

Consider $P_{\le N_2^{\sqrt{v}}} M_{1,5}$ and $P_{\ge N_2^{\sqrt{v}}} M_{1,5}$.

By Lemma 7.4 and Proposition 7.7, we have

$$\| P_{N_1} \varrho_{\le N} \|_{L^{2p}_\omega(C^0_t C^0_x([-T,T]))} \lesssim (2p)^{\frac{1}{2}} T^{\alpha} N_1^{\frac{1}{2}+\varepsilon}$$

$$\| P_{N_2} \Psi_{\le N} \|_{L^{2p}_\omega(C^0_t C^0_x([-T,T]))} \lesssim (2p)^{\frac{5}{2}} T^{\alpha} N_2^{-\frac{1}{2}+\varepsilon}$$

Thus,

$$\| P_{N_1} \varrho_{\le N} \cdot P_{N_2} \Psi_{\le N} \|_{L^p_\omega(C^0_t C^0_x([-T,T]))} \lesssim p^3 T^{\alpha} N_1^{\frac{1}{2}+\varepsilon} N_2^{-\frac{1}{2}+\varepsilon} \implies \text{(7.67) for } P_{\ge N_2^{\sqrt{v}}} M_{1,5}$$

We consider $P_{\le N_2^{\sqrt{v}}} M_{1,5}$ below (we can assume $N_1 \sim N_2$)

Recall the decomposition of $\Psi_{\le N}$ in (7.20):

$$\Psi_{\le N} = (\Psi_{\le N})_0 + (\Psi_{\le N})_1 + \xi$$

(1) We first consider $(\Psi_{\in N})_0$

As in Lemma 7.8, we decompose $P_{N_2}\Psi_{\in N}$ into

$$(\Psi_{\in N})_0\,[M_0 = N_2, M_1, \dots, M_5, M_{234}]$$

By (7.24), we can write

$$(\Psi_{\in N})_0\,[M_*] = \sum_{n_0', n_1, \dots, n_5} H_{n_0' n_1 \cdots n_5}(t)\, e^{i n_0' \cdot x} \prod_{j=1}^{5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j)$$

where H is a tensor satisfying (by Corollary 5.12)

$$\sup_t \|H\|_{n_0' n_1 \cdots n_5} \lesssim \max(M_1, \dots, M_5)^{-\frac{1}{2}+\varepsilon}$$

$$\sup_t \|H\|_{n_0' n_A \to n_B n_5} \lesssim (M_0 M_5)^{-\frac{1}{2}+\varepsilon} \left(M_0^{-\frac{1}{2}+\varepsilon} + \max(M_2, M_3, M_4, M_5)^{-\frac{1}{2}+\varepsilon}\right)$$

for any partition (A, B) of $\{1,2,3,4\}$

(i) If there is no pairing between ϙ and Ψ, we have

$$P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x) = \sum_{n_0, n_0', n_1, \dots, n_5} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \cdot H_{n_0' n_1 \cdots n_5}(t) \prod_{j=0}^{5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j)$$

Thus,

$$\mathbb{E}\left[\,|P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x)|^2\right]^{1/2} \lesssim \Big\| \sum_{n_0'} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \cdot H_{n_0' n_1 \cdots n_5}(t) \Big\|_{n_0 n_1 \cdots n_5}$$

$$\lesssim \Big\| \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \Big\|_{n_0 \to n_0'} \cdot \|H\|_{n_0' n_1 \cdots n_5}$$

Schur's test

$$\lesssim N_1^{-1} N_2^{3\sqrt{\upsilon}} \max(M_1, \dots, M_5)^{-\frac{1}{2}+\varepsilon} \implies (7.67) \qquad \max(M_1, \dots, M_5) \gtrsim N_2$$

(ii) If there is one pairing between ϙ and Ψ, assume $n_0 + n_1 = 0$ (others similar or simpler)

$$P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x) = \sum_{n_0, n_0', n_2, \dots, n_5} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \cdot H_{n_0', -n_0, n_2 \cdots n_5}(t) \prod_{j=2}^{5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j)$$

Thus,

$$\mathbb{E}\left[\,|P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x)|^2\right]^{1/2} \lesssim \Big\| \sum_{n_0, n_0'} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \cdot H_{n_0', -n_0, n_2 \cdots n_5}(t) \Big\|_{n_2 \cdots n_5}$$

$$\lesssim \Big\| \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0\rangle^{-1} \varphi_0(t\langle n_0\rangle) \Big\|_{n_0 n_0'} \cdot \|H\|_{n_2 \cdots n_5 \to n_0' n_1}$$

$$\lesssim N_1^{-1} (N_2^{\sqrt{\upsilon}})^{\frac{3}{2}} N_1^{\frac{3}{2}} \cdot (N_2 M_5)^{-\frac{1}{2}+\varepsilon} \left(N_2^{-\frac{1}{2}+\varepsilon} + \max(M_2, M_3, M_4, M_5)^{-\frac{1}{2}+\varepsilon}\right)$$

This implies (7.67) if $\max(M_2, M_3, M_4, M_5) \ge N_2^{\frac{1}{1000}}$

Suppose $\max(M_2, M_3, M_4, M_5) \le N_2^{\frac{1}{1000}}$ (only possible if pairing is (n_0, n_1))

By losing at most $N_2^{\frac{1}{50}}$, we can fix the values of n_2, n_3, n_4, n_5

We can thus write (summations on other φ_j's are omitted)

$$\mathcal{F}_x M_{1,5}(t,x) = \sum_{\varphi_0 \in \{\cos, \sin\}} \sum_{n_0, n_0'} \mathbb{1}_{\{n_0 + n_0' = k\}} \frac{\mathbb{1}_{M_1}(n_0)}{\langle n_0 \rangle^2} \varphi_0(t \langle n_0 \rangle) \cdot \frac{\mathbb{1}_{M_0}(n_0')}{\langle n_0' \rangle} \cdot \prod_{j=2}^{5} \frac{\mathbb{1}_{M_j}(n_j)}{\langle n_j \rangle} \int_0^t \sin((t-t')\langle n_0' \rangle)$$

$$\times \varphi_0(t' \langle n_0 \rangle) \frac{\mathbb{1}_{M_{234}}(n_{234})}{\langle n_{234} \rangle} \varphi_5(t' \langle n_5 \rangle) \int_0^{t'} \sin((t'-t'')\langle n_{234} \rangle) \prod_{j=2}^{4} \varphi_j(t'' \langle n_j \rangle) \, dt'' \, dt'$$

$\sim$ linear combination of

$$\sum_{n_0, n_0'} \mathbb{1}_{\{n_0 + n_0' = k\}} \frac{\mathbb{1}_{M_1}(n_0)}{\langle n_0 \rangle^2} \frac{\mathbb{1}_{M_0}(n_0')}{\langle n_0' \rangle} \int_0^t \chi(t) \chi(t') \cos((t-t')\langle n_0 \rangle) \sin((t-t')\langle n_0' \rangle) e^{i\lambda t'} dt'$$

$$= \int_0^t \mathrm{Sine}[M_0, M_1](t-t', k) e^{i\lambda t'} dt'$$ (see Definition 5.13 in Note 12)

with λ depending on n_2, n_3, n_4, n_5 and $|\lambda| \lesssim N_2^{\frac{1}{10}}$

By Lemma 5.17, we have

$$\sup_k |\mathcal{F}_x M_{1,5}(t,k)| \lesssim N_2^{-\frac{9}{10}}$$

for each $|k| \lesssim N_2^{\sqrt{\upsilon}}$ $\Longrightarrow$ (7.67)

(2) We now consider $(\Psi_{\in N})_1$

As in Lemma 7.9, we decompose

$$(\Psi_{\in N})_1 [M_*] = \sum_{n_0', n_3, n_4, n_5} H^{\mathrm{sine}}_{n_0' n_3 n_4 n_5}(t) e^{i n_0' \cdot x} \prod_{j=3}^{5} \int_0^1 1 \, dW_{s_j}^{\varphi_j}(n_j),$$

where $M_* = (M_0 = N_2, M_1, \ldots, M_5, M_{234})$ and H^{sine} satisfies (by Corollary 5.19)

$$\sup_t \|H^{\mathrm{sine}}\|_{n_0' n_3 n_4 n_5} \lesssim \max(M_3, M_4, M_5)^{-\frac{1}{2}+\varepsilon}$$

$$\sup_t \|H^{\mathrm{sine}}\|_{n_3 n_4 \to n_0' n_5} \lesssim M_0^{-\frac{1}{2}+\varepsilon} \max(M_3, M_4)^{-\frac{1}{2}+\varepsilon} M_1^{-1+\varepsilon}$$

(i) No pairing between φ and Ψ

$$P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x) = \sum_{n_0, n_0', n_3, n_4, n_5} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0 + n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t \langle n_0 \rangle) \cdot H^{\mathrm{sine}}_{n_0' n_3 n_4 n_5}(t) \prod_{j \in \{0,3,4,5\}} \int_0^1 1 \, dW_{s_j}^{\varphi_j}(n_j)$$

Thus,

$$\mathbb{E}\left[|P_{\le N_2^{\sqrt{\upsilon}}} M_{1,5}(t,x)|^2 \right]^{1/2} \lesssim \left\| \sum_{n_0'} \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0 + n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t \langle n_0 \rangle) \cdot H^{\mathrm{sine}}_{n_0' n_3 n_4 n_5}(t) \right\|_{n_0 n_3 n_4 n_5}$$

$$\lesssim \left\| \mathbb{1}_{\{|n_0 + n_0'| \le N_2^{\sqrt{\upsilon}}\}} e^{i(n_0 + n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t \langle n_0 \rangle) \right\|_{n_0 \to n_0'} \cdot \|H^{\mathrm{sine}}\|_{n_0' n_3 n_4 n_5}$$

Schur's test

$$\lesssim N_1^{-1} N_2^{3\sqrt{\upsilon}} \cdot \max(M_3, M_4, M_5)^{-\frac{1}{2}+\varepsilon} \Longrightarrow$$ (7.67) $\max(M_3, M_4, M_5) \gtrsim N_2$

(ii) One pairing between  and , assume $n_0 + n_5 = 0$ (others simpler)

$$P_{\leq N_2^{\sqrt{v}}} M_{1,5}(t,x) = \sum_{n_0, n_0', n_3, n_4} \mathbb{1}_{\{|n_0 + n_0'| \leq N_2^{\sqrt{v}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t\langle n_0 \rangle) \cdot H^{\text{sine}}_{n_0' n_3 n_4; -n_0}(t) \prod_{j \in \{3,4\}} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j)$$

Thus,

$$\mathbb{E}\left[\left| P_{\leq N_2^{\sqrt{v}}} M_{1,5}(t,x) \right|^2 \right]^{1/2} \lesssim \Big\| \sum_{n_0, n_0'} \mathbb{1}_{\{|n_0 + n_0'| \leq N_2^{\sqrt{v}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t\langle n_0 \rangle) \cdot H^{\text{sine}}_{n_0' n_3 n_4; -n_0}(t) \Big\|_{n_3 n_4}$$

$$\lesssim \Big\| \mathbb{1}_{\{|n_0 + n_0'| \leq N_2^{\sqrt{v}}\}} e^{i(n_0+n_0)\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t\langle n_0 \rangle) \Big\|_{n_0 n_0'} \cdot \| H^{\text{sine}} \|_{n_3 n_4 \to n_0' n_5}$$

$$\lesssim N_1^{-1} (N_2^{\sqrt{v}})^{\frac{3}{2}} N_1^{\frac{3}{2}} \cdot M_0^{-\frac{1}{2}+\varepsilon} \max(M_3, M_4)^{-\frac{1}{2}+\varepsilon} M_1^{-1+\varepsilon}$$

This implies (7.67) if $\max(M_1, M_3, M_4) \geq N_2^{\frac{1}{1000}}$

Suppose $\max(M_1, M_3, M_4) \leq N_2^{\frac{1}{1000}}$ (only possible if pairing is (n_0, n_1))

$\Rightarrow$ similar as in (1) by using the sine cancellation kernel

(3) We finally consider 

Note that $\mathcal{C}^{(1,5)}_{\leq N}[N_1, N_2]$ exactly corresponds to the case where  pairs with 

Thus, we can assume no pairing between  and 

By Corollary 6.15 and Lemma 7.10,

$$P_{\leq N_2^{\sqrt{v}}} M_{1,5}(t,x) = \sum_{n_0, n_0'} \mathbb{1}_{\{|n_0 + n_0'| \leq N_2^{\sqrt{v}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t\langle n_0 \rangle) \cdot F^{\varphi_0'}_{\leq N}(t, n_0') \int_0^1 \int_0^1 1 \, dW^{\varphi_0}_{s_0}(n_0) \, dW^{\varphi_0'}_{s_0'}(n_0')$$

where $F^{\varphi_0'}_{\leq N}$ satisfies

$$\sup_t \| F^{\varphi_0'}_{\leq N} \|_{\ell^2_{n_0'}} \lesssim \| F^{\varphi_0'}_{\leq N} \|_{X^{0,b}} \lesssim N_2^{-\frac{1}{2}+\varepsilon}$$

This implies

$$\mathbb{E}\left[\left| P_{\leq N_2^{\sqrt{v}}} M_{1,5}(t,x) \right|^2 \right] \lesssim \Big\| \mathbb{1}_{\{|n_0 + n_0'| \leq N_2^{\sqrt{v}}\}} e^{i(n_0+n_0')\cdot x} \langle n_0 \rangle^{-1} \varphi_0(t\langle n_0 \rangle) \cdot F^{\varphi_0'}_{\leq N}(t, n_0') \Big\|^2_{n_0 n_0'}$$

$$\lesssim N_2^{-1+2\varepsilon} \cdot \sup_{n_0'} \sum_{n_0 : |n_0 + n_0'| \leq N_2^{\sqrt{v}}} \langle n_0 \rangle^{-2}$$

$$\lesssim N_2^{-1+2\varepsilon} \cdot N_1^{-2} N_2^{3\sqrt{v}} \Rightarrow \text{(7.67)} \qquad \square$$

The following estimates will be useful for $\mathbb{X}^{(1)}_{\le N}$ and $\mathbb{X}^{(2)}_{\le N}$ in Section 10.

Lemma 7.17 (Crude estimates of $\mathcal{C}^{(1,5)}_{\le N}$ and $\mathcal{C}^{(3,3)}_{\le N}$)

Let $T \ge 1$, N_1, N_2, N dyadic, and $\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2]$, $\mathcal{C}^{(3,3)}_{\le N}[N_1,N_2]$ be as in Definition 3.13. Then, we have

$$\left|\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2](t)\right| + \left|\mathcal{C}^{(3,3)}_{\le N}[N_1,N_2](t)\right| \lesssim \max(N_1,N_2)^{2\varepsilon}\, T^{\alpha} \tag{7.70}$$

for all $t\in[-T,T]$. Furthermore, we have

$$\left\|\chi^2\!\left(\tfrac{t}{T}\right)\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2](t)\right\|_{H^b_t} + \left\|\chi^2\!\left(\tfrac{t}{T}\right)\mathcal{C}^{(3,3)}_{\le N}[N_1,N_2](t)\right\|_{H^b_t} \lesssim \max(N_1,N_2)^{2}\, T^{\alpha} \tag{7.71}$$

Proof: We only consider $\mathcal{C}^{(1,5)}_{\le N}$, since $\mathcal{C}^{(3,3)}_{\le N}$ is similar.

From the definition, $\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2]$ corresponds to the case where ϱ pairs with ξ

$\Rightarrow$ $\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2] \equiv 0$ unless $N_1 = N_2$

By translation invariance, Lemma 7.4, and Proposition 7.7,

$$\begin{aligned}
\left|\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2](t)\right| &= \left|\mathbb{E}\left[\int_{\mathbb{T}^3} \varrho_{N_1}(t,x)\, P_{N_2}\Psi_{\le N}(t,x)\,dx\right]\right| \\
&\lesssim \mathbb{E}\left[\|\varrho_{N_1}(t,x)\|^2_{L^2_x}\right]^{1/2}\mathbb{E}\left[\|P_{N_2}\Psi_{\le N}(t,x)\|^2_{L^2_x}\right]^{1/2} \\
&\lesssim N_1^{\frac12+\varepsilon} N_2^{-\frac12+\varepsilon}\, T^{\alpha} \\
&\lesssim \max(N_1,N_2)^{2\varepsilon}\, T^{\alpha} \quad \Rightarrow \quad (7.70)
\end{aligned}$$

Similarly, by the algebra property of H^b_t,

$$\begin{aligned}
&\left\|\chi^2\!\left(\tfrac{t}{T}\right)\mathcal{C}^{(1,5)}_{\le N}[N_1,N_2](t)\right\|_{H^b_t} \\
&\le \mathbb{E}\left[\int_{\mathbb{T}^3}\left\|\chi^2\!\left(\tfrac{t}{T}\right)\varrho_{N_1}(t,x)\,P_{N_2}\Psi_{\le N}(t,x)\right\|_{H^b_t}dx\right] \\
&\lesssim \mathbb{E}\left[\int_{\mathbb{T}^3}\left\|\chi\!\left(\tfrac{t}{T}\right)\varrho_{N_1}(t,x)\right\|^2_{H^b_t}dx\right]^{1/2}\mathbb{E}\left[\int_{\mathbb{T}^3}\left\|\chi\!\left(\tfrac{t}{T}\right)P_{N_2}\Psi_{\le N}(t,x)\right\|^2_{H^b_t}dx\right]^{1/2}
\end{aligned} \tag{7.72}$$

Since $\langle\lambda\rangle^b \lesssim \langle n\rangle^b\langle\lambda\mp n\rangle^b$, by Lemma 7.4 and Proposition 7.7,

$$\begin{aligned}
(7.72) &\lesssim \mathbb{E}\left[\left\|\chi\!\left(\tfrac{t}{T}\right)\varrho_{N_1}(t,x)\right\|^2_{X^{b,b}}\right]^{1/2}\mathbb{E}\left[\left\|\chi\!\left(\tfrac{t}{T}\right)P_{N_2}\Psi_{\le N}(t,x)\right\|^2_{X^{b,b}}\right]^{1/2} \\
&\lesssim N_1^{b+\frac12+\varepsilon} N_2^{b-\frac12+\varepsilon}\, T^{\alpha} \\
&\lesssim \max(N_1,N_2)^{2}\, T^{\alpha} \quad \Rightarrow \quad (7.71)
\end{aligned}$$

$\square$

Reading session 15 : Linear random operator involving the quadratic object

- Linear random operators (Section 9)
  - Quadratic object (Section 9.1)

$$P_{\leq N}\Big[V_{\leq N} Y_{\leq N} - \big(2\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,hi,lo}_{\leq N} + \Pi^{res}_{\leq N}\big)(\varrho_{\leq N}, \varrho_{\leq N}, Y_{\leq N})\Big] \qquad (9.5)$$

(Recall Note 3)

Definition 9.2 (The Quad-operators)

(1) For all $N \geq 1$, we define

$$\mathrm{Quad}_{\leq N}(Y) := P_{\leq N}\Big[V_{\leq N} Y_{\leq N} - \big(2\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,hi,lo}_{\leq N} + \Pi^{res}_{\leq N}\big)(\varrho_{\leq N}, \varrho_{\leq N}, Y_{\leq N})\Big]$$

(2) For all $N_0, N_1, N_2, N_3, N_{13}, N_{23} \geq 1$, we define

$$\mathrm{Quad}[N_*](Y) := \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3\\ n_0=n_{123}}} \Big[\mathbb{1}_{N_{13}}(n_{13})\,\mathbb{1}_{N_{23}}(n_{23})\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big) :\varrho_{N_1}(n_1)\,\varrho_{N_2}(n_2): \widehat{Y}(n_3)\, e^{i\langle n_0,x\rangle}\Big]$$

Lemma 9.3 (Estimate of Quad$[N_*]$)

Let $p \geq 2$, $T \geq 1$, and $N_0, N_1, N_2, N_3, N_{12}, N_{13}$ frequency scales satisfying

$$N_{13} \gtrsim N_2^{\eta} \quad \text{and} \quad N_{23} \gtrsim N_1^{\eta} \qquad (9.7)$$

Then, we have ($0 \in J \subseteq [-T, T]$ closed interval)

$$\mathbb{E}\Big[\sup_J \|\mathrm{Quad}[N_*]\|^p_{X^{\frac{1}{2},b}(J)\to X^{-\frac{1}{2},b_+-1}(J)}\Big]^{1/p} \lesssim N^{\varepsilon}_{max}\big(N_0^{-\frac{\eta}{2}} + N_3^{-\frac{1}{2}}\big)T^{\alpha}p \qquad (9.7a)$$

If instead we have

$$N_{13} \lesssim N_2^{\eta} \quad \text{and} \quad N_{23} \lesssim N_1^{\eta} \qquad (9.8)$$

Then, we have

$$\mathbb{E}\Big[\sup_J \|\mathrm{Quad}[N_*]\|^p_{X^{\frac{1}{2},b}(J)\to X^{-\frac{1}{2},b_+-1}(J)}\Big]^{1/p} \lesssim N_{max}^{-1+\varepsilon}T^{\alpha}p \qquad (9.8a)$$

Proof : Using the reduction in Subsection 5.7, (9.7a) follows from

$$N_0^{-\frac{1}{2}}N_1^{-1}N_2^{-1}N_3^{-\frac{1}{2}}\,\mathbb{E}\Big[\Big\|\widehat{Y}\mapsto\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} h_{n_0n_1,n_2n_3} :g_{n_1}g_{n_2}: \widehat{Y}(n_3)\Big\|^p_{\ell^2\to\ell^2}\Big]^{1/p} \qquad (9.9)$$

$$\lesssim N^{\varepsilon}_{max}\big(N_0^{-\eta/2} + N_3^{-1/2}\big)p.$$

where

$$h_{n_0n_1,n_2n_3} = \mathbb{1}_{N_{13}}(n_{13})\,\mathbb{1}_{N_{23}}(n_{23})\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\mathbb{1}_{\{n_0=n_{123}\}}\mathbb{1}_{\{|\Omega-m|\leq 1\}},$$

$$\Omega = \sum_{j=0}^{3}(\pm_j)\langle n_j\rangle, \quad \text{and} \quad m\in\mathbb{Z}$$

By Proposition B.2, it remains to show

$$N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \max\big(\underset{①}{\|h\|_{n_0n_1n_2\to n_3}}, \underset{②}{\|h\|_{n_0n_1\to n_2n_3}}, \underset{③}{\|h\|_{n_0n_2\to n_1n_3}}, \underset{④}{\|h\|_{n_0\to n_1n_2n_3}}\big) \qquad (9.10)$$
$$\lesssim N_0^{-\frac{\eta}{2}} + N_3^{-\frac12}$$

Use the base tensor estimate (Lemma 5.7 or Note 8):

①: $N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \|h\|_{n_0n_1n_2\to n_3}$

$$\overset{(5.27)}{\lesssim} N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \operatorname{med}(N_0, N_1, N_2)^{\frac32} \min(N_0, N_1, N_2) \lesssim N_3^{-\frac12}$$

②: $N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \|h\|_{n_0n_1\to n_2n_3}$

$$\overset{(5.29)}{\lesssim} N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \min(N_{23}, N_0, N_1)^{-\frac12} \min(N_0, N_1)^{\frac32} \qquad (9.11)$$
$$\times \min(N_{23}, N_2, N_3)^{-\frac12} \min(N_2, N_3)^{\frac32}$$

$$N_{23} \gtrsim N_1^{\eta} \quad \Rightarrow \quad \min(N_{23}, N_0, N_1) \gtrsim \min(N_0, N_1)^{\eta}$$

Thus,

$$(9.11) \lesssim N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \min(N_0, N_1)^{\frac32 - \frac{\eta}{2}} \min(N_2, N_3)^{\frac32} \lesssim N_0^{-\frac{\eta}{2}}$$

③: $N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \|h\|_{n_0n_2\to n_1n_3}$

$$\overset{(5.29)}{\lesssim} N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \min(N_{13}, N_0, N_2)^{-\frac12} \min(N_0, N_2)^{\frac32}$$
$$\times \min(N_{13}, N_1, N_3)^{-\frac12} \min(N_1, N_3)^{\frac32}$$

$$N_{13} \gtrsim N_2^{\eta} \quad \Rightarrow \quad \min(N_{13}, N_0, N_2) \gtrsim \min(N_0, N_2)^{\eta}$$

Thus,

$$(9.11) \lesssim N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \min(N_0, N_2)^{\frac32 - \frac{\eta}{2}} \min(N_1, N_3)^{\frac32} \lesssim N_0^{-\frac{\eta}{2}}$$

④: $N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \|h\|_{n_0n_1n_2\to n_3}$

$$\overset{(5.27)}{\lesssim} N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \operatorname{med}(N_1, N_2, N_3)^{\frac32} \min(N_1, N_2, N_3) \lesssim N_0^{-\frac12}$$

We now consider (9.8a), which follows from (using Subsection 5.7)

$$N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \mathbb{E}\Big[\Big\| \hat{Y} \mapsto \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} h_{n_0n_1n_2n_3} :g_{n_1} g_{n_2}: \hat{Y}(n_3) \Big\|_{\ell^2\to\ell^2}^p\Big]^{1/p} \lesssim N_{\max}^{-1} p \qquad (9.12)$$

By Cauchy-Schwarz in n_3, we have

$$\text{LHS of } (9.12) \lesssim N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \mathbb{E}\Big[\Big\| \sum_{n_1,n_2\in\mathbb{Z}^3} h_{n_0n_1n_2n_3} :g_{n_1} g_{n_2}: \Big\|_{n_0n_3}^p\Big]^{1/p}$$
$$\lesssim N_0^{-\frac12} N_1^{-1} N_2^{-1} N_3^{-\frac12} \|h\|_{n_0n_1n_2n_3}\, p$$

With $n_0 = n_{123}$, n_0, n_1, n_2, n_3 are uniquely determined by n_{13} and n_{23} and either one of n_0, n_1, n_2, n_3 .

$$\text{LHS of (9.12)} \lesssim N_0^{-\frac{1}{2}} N_1^{-1} N_2^{-1} N_3^{-\frac{1}{2}} \|h\|_{n_0 n_1 n_2 n_3} p$$

$$\lesssim N_0^{-\frac{1}{2}} N_1^{-1} N_2^{-1} N_3^{-\frac{1}{2}} N_{13}^{\frac{3}{2}} N_{23}^{\frac{3}{2}} \min(N_0, N_1, N_2, N_3)^{\frac{3}{2}} p$$

$$\overset{(9.8)}{\lesssim} N_0^{-\frac{1}{2}} N_1^{-1+\frac{3\eta}{2}} N_2^{-1+\frac{3\eta}{2}} N_3^{-\frac{1}{2}} \min(N_0, N_1, N_2, N_3)^{\frac{3}{2}} p$$

$$\lesssim N_{\max}^{-1} p \qquad \square$$

Proposition 9.1 (Random linear operator involving $V_{\leq N}$)

For all $p \geq 2$, $T \geq 1$, and $0 \in J \subseteq [-T, T]$ closed interval, we have

$$\mathbb{E}\Big[\sup_N \sup_J \Big\| Y \mapsto P_{\leq N}[V_{\leq N} P_{\leq N} Y - (2\Pi_{\leq N}^{\mathrm{hi,lo,lo}} + \Pi_{\leq N}^{\mathrm{hi,hi,lo}} + \Pi_{\leq N}^{\mathrm{res}})(\varphi_{\leq N}, \varphi_{\leq N}, P_{\leq N} Y)] \Big\|^p_{X^{\frac{1}{2}+\delta_2, b}(J) \to X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \Big]^{1/p} \lesssim T^{\alpha} p .$$

Proof: Definition 9.2 and Definition 3.14 (Note 3) $\Rightarrow$

$$P_{\leq N}[V_{\leq N} P_{\leq N} Y] = \sum_{N_0, N_1, N_2, N_3 \leq N} \sum_{N_{13}, N_{23}} \mathrm{Quad}[N_*](Y) \tag{9.13}$$

$$(2\Pi_{\leq N}^{\mathrm{hi,lo,lo}} + \Pi_{\leq N}^{\mathrm{hi,hi,lo}})(\varphi_{\leq N}, \varphi_{\leq N}, P_{\leq N} Y) = \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 \leq \max(N_1, N_2)^{\eta}}} \sum_{N_{13}, N_{23}} \mathrm{Quad}[N_*](Y) \tag{9.14}$$

Note : $\Pi_{\leq N}^{\mathrm{res}}$ no Wick-ordering , so we write

$$\Pi_{\leq N}^{\mathrm{res}}(\varphi_{\leq N}, \varphi_{\leq N}, P_{\leq N} Y) = \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 > \max(N_1, N_2)^{\eta}}} \sum_{N_{13}, N_{23}} \left(\mathbb{1}_{\{N_{13} \leq N_2^{\eta}\}} + \mathbb{1}_{\{N_{23} \leq N_1^{\eta}\}} \right) \mathrm{Quad}[N_*](Y) \tag{9.15}$$

$$+ \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 > \max(N_1, N_2)^{\eta}}} \sum_{N_{13}, N_{23}} \left(\mathbb{1}_{\{N_{13} \leq N_2^{\eta}\}} + \mathbb{1}_{\{N_{23} \leq N_1^{\eta}\}} \right) \mathrm{Quad}^{\circ}[N_*](Y) , \tag{9.16}$$

where $$\mathrm{Quad}^{\circ}[N_*](Y) := \sum_{\substack{n_0, n_1, n_2, n_3 \in \mathbb{Z}^3: \\ n_0 = n_3 , n_{12} = 0}} \left[\mathbb{1}_{N_{13}}(n_{13}) \mathbb{1}_{N_{23}}(n_{23}) \left(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \right) \frac{1}{\langle n_1 \rangle^2} \widehat{Y}(n_3) e^{i\langle n_0, x \rangle} \right] .$$

(9.13) - (9.14) - (9.15) - (9.16) yields

$$\mathrm{Quad}_{\leq N}(Y) = \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 > \max(N_1, N_2)^{\eta}}} \sum_{\substack{N_{13}: \\ N_{13} > N_2^{\eta}}} \sum_{\substack{N_{23}: \\ N_{23} > N_1^{\eta}}} \mathrm{Quad}[N_*](Y) \tag{9.18}$$

$$- \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 > \max(N_1, N_2)^{\eta}}} \sum_{\substack{N_{13}: \\ N_{13} \leq N_2^{\eta}}} \sum_{\substack{N_{23}: \\ N_{23} \leq N_1^{\eta}}} \mathrm{Quad}[N_*](Y) \tag{9.19}$$

$$- \sum_{\substack{N_0, N_1, N_2, N_3 \leq N; \\ N_3 > \max(N_1, N_2)^{\eta}}} \sum_{N_{13}, N_{23}} \left(\mathbb{1}_{\{N_{13} \leq N_2^{\eta}\}} + \mathbb{1}_{\{N_{23} \leq N_1^{\eta}\}} \right) \mathrm{Quad}^{\circ}[N_*](Y) \tag{9.20}$$

By Lemma 9.3 (9.7a), the contribution from (9.18) :

$$\sum_{\substack{N_0,N_1,N_2,N_3\le N;\\ N_3>\max(N_1,N_2)^\eta}} \sum_{\substack{N_{13}:\\ N_{13}>N_2^\eta}} \sum_{\substack{N_{23}:\\ N_{23}>N_1^\eta}} \mathbb{E}\Big[\sup_J \|\mathrm{Quad}[N_*]\|^p_{X^{\frac{1}{2}+\delta_2,b}(J)\to X^{-\frac{1}{2}+\delta_2,b_+-1}(J)}\Big]^{1/p}$$

$$\lesssim p T^\alpha \sum_{\substack{N_0,N_1,N_2,N_3\le N;\\ N_3>\max(N_1,N_2)^\eta}} \sum_{\substack{N_{13}:\\ N_{13}>N_2^\eta}} \sum_{\substack{N_{23}:\\ N_{23}>N_1^\eta}} \underbrace{N_{\max}^\varepsilon N_0^{\delta_2}\big(N_0^{-\frac{\eta}{2}}+N_3^{-\frac{1}{2}}\big)N_3^{-\delta_2}}_{\lesssim N_{\max}^{\varepsilon-\delta_2\eta}+N_{\max}^{\varepsilon+\delta_2-(\frac{1}{2}+\delta_2)\eta}\lesssim N_{\max}^{-\varepsilon}}$$

note: $N_3>N_1^\eta$, $N_3>N_2^\eta \Rightarrow N_3>N_0^\eta$

(if not, then $N_3\le N_0^\eta$, $N_0\sim N_1$ or $N_0\sim N_2$, contradiction)

By Lemma 9.3 (9.8a), (9.19) can be treated easily.

For (9.20), $\mathrm{Quad}^0(Y)$ is a Fourier-multiplier with symbol

$$n_3 \longmapsto \sum_{\substack{n_1,n_2\in\mathbb{Z}^3\\ n_{12}=0}}\Big[\mathbb{1}_{N_{13}}(n_{13})\,\mathbb{1}_{N_{23}}(n_{23})\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\frac{1}{\langle n_1\rangle^2}\Big],$$

which can be estimated by

$$\Big|\sum_{\substack{n_1,n_2\in\mathbb{Z}^3\\ n_{12}=0}}\Big[\mathbb{1}_{N_{13}}(n_{13})\,\mathbb{1}_{N_{23}}(n_{23})\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\frac{1}{\langle n_1\rangle^2}\Big]\Big|$$

$$\lesssim \mathbb{1}_{\{N_1=N_2\}}N_1^{-2}\min(N_{13},N_{23})^3$$

$$\lesssim \mathbb{1}_{\{N_1=N_2\}}N_1^{-2}N_3^{-1}\max(N_1,N_{13})\min(N_{13},N_{23})^3 \qquad \text{if } N_1\ll N_3 \text{, then } N_3\sim N_{13}$$

$$\lesssim \mathbb{1}_{\{N_1=N_2\}}N_1^{-1+3\eta}N_3^{-1} \qquad N_{13}\le N_2^\eta \text{ , } N_{23}\le N_1^\eta$$

$$\lesssim N_3^{-1} \Rightarrow \text{good for the estimate}$$

□

Reading session 16 : Linear random operator involving the linear and cubic objects

- Linear and cubic objects

$$w \mapsto P_{\leq N} \Pi^{*}_{\leq N} (\phi_{\leq N}, \Psi_{\leq N}, P_{\leq N} w)$$

$$w = \underbrace{\overset{\Psi}{\Psi}_{\leq N}, \ \mathbb{X}^{(1)}_{\leq N}, \ \mathbb{X}^{(2)}_{\leq N}}_{\text{we cover most (but not all) frequency-interactions here}}, \text{ or } Y_{\leq N}$$

Definition 9.5 (High × low × high - interaction)

For any $N \geq 1$ and $w : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we define

$$\Pi^{hi,lo,hi}_{\leq N} (\phi_{\leq N}, \Psi_{\leq N}, P_{\leq N} w) = \sum_{\substack{N_1, N_{234}, N_5 \leq N \\ N_{234} \leq N_1^{\nu} \\ N_1 \sim N_5 > N_1^{\eta}}} \phi_{N_1} P_{N_{234}} \Psi_{\leq N} P_{N_5} w .$$

$0 < \eta \ll \delta_1 \ll \nu \ll 1$

Definition 9.7 (The LinCub-operators)

For frequency scales $N_{234}, N_0, \ldots, N_5$, we define

$$\begin{aligned} \mathrm{LinCub}^{(5)}[N_*](w) := \sum_{n_0, \ldots, n_5 \in \mathbb{Z}^3} \sum_{\varphi_1, \ldots, \varphi_4 \in \{\cos, \sin\}} \Big[& \mathbb{1}_{\{n_0 = n_{12345}\}} \mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=0}^{5} \mathbb{1}_{N_j}(n_j) \Big) \\ & \times \langle n_{234} \rangle^{-1} \Big(\prod_{j=1}^{4} \langle n_j \rangle^{-1} \Big) \varphi_1(t \langle n_1 \rangle) \Big(\int_0^t \sin((t-t')\langle n_{234} \rangle) \prod_{j=2}^{4} \varphi_j(t' \langle n_j \rangle) \, dt' \Big) \\ & \times \widehat{w}(t, n_5) e^{i \langle n_0, x \rangle} . \prod_{j=1}^{4} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) \end{aligned}$$

Furthermore, we define

$$\begin{aligned} \mathrm{LinCub}^{\sin}[N_*](w) := \mathbb{1}_{\{N_1 = N_2\}} \sum_{n_0, n_3, n_4, n_5 \in \mathbb{Z}^3} \sum_{\varphi_3, \varphi_4 \in \{\cos, \sin\}} \Big[& \mathbb{1}_{\{n_0 = n_{345}\}} \Big(\prod_{j=0,3,4,5} \mathbb{1}_{N_j}(n_j) \Big) \langle n_3 \rangle^{-1} \langle n_4 \rangle^{-1} \\ & \times \Big(\int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34}) \prod_{j=3}^{4} \varphi_j(t' \langle n_j \rangle) \, dt' \Big) \widehat{w}(t, n_5) e^{i \langle n_0, x \rangle} \\ & \times \prod_{j=3}^{4} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) , \end{aligned}$$

where Sine is as in Definition 5.13 .

Lemma 9.9 (Decomposition using LinCub-operators)

For all frequency-scales $N \geq 1$ and $w : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we have

$$P_{\leq N}\Big[\stackrel{\circ}{|}_{\leq N} \Psi_{\leq N} P_{\leq N} w - \big(\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,lo,hi}_{\leq N} \big) \big(\stackrel{\circ}{|}_{\leq N}, \Psi_{\leq N}, P_{\leq N} w \big) \Big]$$

$$= \sum_{\substack{N_0,\dots,N_5,\\ N_{234} \leq N}} \Big[\Big(\underbrace{\mathbb{1}\{N_{234} > N_1^{\eta} \geq N_5\} + \mathbb{1}\{N_1 \not\sim N_5 > N_1^{\eta}\} + \mathbb{1}\{N_{234} > N_1^{\nu}, N_1 \sim N_5 > N_1^{\eta}\}}_{=} \Big) \times \mathrm{LinCub}^{(5)}[N_*](w) \tag{9.29}$$

$$+ 3 \sum_{\substack{N_0,\dots,N_5,\\ N_{234} \leq N}} \Big[\Big(\overbrace{1 - \mathbb{1}\{N_{234}, N_5 \leq N_1^{\eta}\} - \mathbb{1}\{N_{234} \leq N_1^{\nu}, N_1 \sim N_5 > N_1^{\eta}\}} \Big) \times \mathrm{LinCub}^{\sin}[N_*](w) \Big]. \tag{9.30}$$

Furthermore, for all functions $Y : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we have

$$P_{\leq N}\Big[\stackrel{\circ}{|}_{\leq N} \Psi_{\leq N} P_{\leq N} Y - \Pi^{hi,lo,lo}_{\leq N} \big(\stackrel{\circ}{|}_{\leq N}, \Psi_{\leq N}, P_{\leq N} Y \big) \Big]$$

$$= \sum_{\substack{N_0,\dots,N_5,\\ N_{234} \leq N}} \mathbb{1}\{\max(N_{234}, N_5) > N_1^{\eta}\} \, \mathrm{LinCub}^{(5)}[N_*](Y) \tag{9.31}$$

$$+ 3 \sum_{\substack{N_0,\dots,N_5,\\ N_{234} \leq N}} \mathbb{1}\{\max(N_{234}, N_5) > N_1^{\eta}\} \, \mathrm{LinCub}^{\sin}[N_*](Y) \tag{9.32}$$

Proof: Let $N_1, N_{234}, N_5 \leq N$.

By the definitions of $\stackrel{\circ}{|}$ and Ψ (Subsection 6.2),

$$P_{\leq N}\Big[\stackrel{\circ}{|}_{N_1} P_{N_{234}} \Psi_{\leq N} P_{N_5} w \Big]$$

$$= \sum_{n_0,\dots,n_5 \in \mathbb{Z}^3} \sum_{\varphi_1,\dots,\varphi_4 \in \{\cos,\sin\}} \Big[\mathbb{1}\{n_0 = n_{12345}\} \mathbb{1}_{N_1}(n_1) \mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_5}(n_5) \Big(\prod_{j=0,2,3,4} \mathbb{1}_{\leq N}(n_j) \Big)$$
$$\times \langle n_{234} \rangle^{-1} \Big(\prod_{j=1}^{4} \langle n_j \rangle^{-1} \Big) \varphi_1(t\langle n_1 \rangle) \Big(\int_0^t \sin((t-t')\langle n_{234} \rangle) \prod_{j=2}^{4} \varphi_j(t'\langle n_j \rangle) \, dt' \Big) \tag{9.33}$$
$$\times \widehat{w}(t,n_5) e^{i\langle n_0, x \rangle} \cdot \int_0^1 1 \, dW^{\varphi_1}_{s_1}(n_1) \times \prod_{j=2}^{4} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j)$$

By the product formula for multiple stochastic integrals,

$$(9.33) = \sum_{n_0,\dots,n_5 \in \mathbb{Z}^3} \sum_{\varphi_1,\dots,\varphi_4 \in \{\cos,\sin\}} \Big[\mathbb{1}\{n_0 = n_{12345}\} \mathbb{1}_{N_1}(n_1) \mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_5}(n_5) \Big(\prod_{j=0,2,3,4} \mathbb{1}_{\leq N}(n_j) \Big)$$
$$\times \langle n_{234} \rangle^{-1} \Big(\prod_{j=1}^{4} \langle n_j \rangle^{-1} \Big) \varphi_1(t\langle n_1 \rangle) \Big(\int_0^t \sin((t-t')\langle n_{234} \rangle) \prod_{j=2}^{4} \varphi_j(t'\langle n_j \rangle) \, dt' \Big) \tag{9.34}$$
$$\times \widehat{w}(t,n_5) e^{i\langle n_0, x \rangle} \cdot \prod_{j=1}^{4} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j)$$

$$+ 3 \sum_{n_0,\dots,n_5 \in \mathbb{Z}^3} \sum_{\varphi_1,\dots,\varphi_4 \in \{\cos,\sin\}} \Big[\mathbb{1}\{n_0 = n_{345}\} \mathbb{1}\{n_{12} = 0\} \mathbb{1}_{N_1}(n_1) \mathbb{1}_{N_{234}}(n_{234}) \mathbb{1}_{N_5}(n_5) \Big(\prod_{j=0,3,4} \mathbb{1}_{\leq N}(n_j) \Big)$$
$$\times \langle n_{234} \rangle^{-1} \Big(\prod_{j=1}^{4} \langle n_j \rangle^{-1} \Big) \mathbb{1}\{\varphi_1 = \varphi_2\} \varphi_1(t\langle n_1 \rangle) \Big(\int_0^t \sin((t-t')\langle n_{234} \rangle) \prod_{j=2}^{4} \varphi_j(t'\langle n_j \rangle) \, dt' \Big) \tag{9.35}$$
$$\times \widehat{w}(t,n_5) e^{i\langle n_0, x \rangle} \cdot \prod_{j=3}^{4} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j)$$

For the non-resonant component (9.34), by inserting dyadic decomposition in n_0, n_2, n_3, n_4 and by Definition 9.7,

$$(9.34) = \sum_{N_0, N_2, N_3, N_4 \leq N} \mathrm{LinCub}^{(5)}[N_*](w)$$

$\Rightarrow$ (9.29) and (9.31) ✓

For the resonant component (9.35), we have

$$\sum_{\varphi_1, \varphi_2 \in \{\cos, \sin\}} \mathbb{1}_{\{\varphi_1 = \varphi_2\}} \mathbb{1}_{\{n_{12} = 0\}} \varphi_1(t\langle n_1\rangle) \varphi_2(t'\langle n_2\rangle) = \cos((t-t')\langle n_1\rangle)$$

By Definition 5.13 (note 12), we have

$$\sum_{n_1, n_2 \in \mathbb{Z}^3} \left[\mathbb{1}_{\{n_{12}=0\}} \mathbb{1}_{N_1}(n_1) \mathbb{1}_{N_{234}}(n_{234}) \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \frac{\cos((t-t')\langle n_1\rangle)}{\langle n_1\rangle^2} \right]$$
$$= \sum_{\substack{n_1, n_{234} \in \mathbb{Z}^3 \\ n_1 + n_{234} = n_{34}}} \left[\mathbb{1}_{N_1}(n_1) \mathbb{1}_{N_{234}}(n_{234}) \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \frac{\cos((t-t')\langle n_1\rangle)}{\langle n_1\rangle^2} \right]$$
$$= \mathrm{Sine}[N_{234}, N_1](t-t', n_{34})$$

Thus,

$$(9.35) = 3 \cdot \mathbb{1}_{\{N_1 = N_2\}} \sum_{n_0, n_3, n_4, n_5 \in \mathbb{Z}^3} \sum_{\varphi_3, \varphi_4 \in \{\cos, \sin\}} \Big[\mathbb{1}_{\{n_0 = n_{345}\}} \Big(\prod_{j=0,3,4} \mathbb{1}_{\leq N}(n_j) \Big) \mathbb{1}_{N_5}(n_5) \langle n_3\rangle^{-1} \langle n_4\rangle^{-1}$$
$$\times \Big(\int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34}) \prod_{j=3}^{4} \varphi_j(t'\langle n_j\rangle)\, dt' \Big) \widehat{w}(t, n_5)\, e^{i\langle n_0, x\rangle} \cdot \prod_{j=3}^{4} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j)$$
$$= 3 \sum_{N_0, N_3, N_4 \leq N} \mathrm{LinCub}^{\sin}[N_*](w)$$

$\Rightarrow$ (9.30) and (9.32) $\square$

Proposition 9.6 (Linear random operator involving $\mathsf{I}_{\leq N} \Psi_{\leq N}$)

Let $T \geq 1$ and $p \geq 2$. Then, we have

$$\mathbb{E}\Big[\sup_N \sup_J \Big\| w \mapsto P_{\leq N} \big[\mathsf{I}_{\leq N} \Psi_{\leq N} P_{\leq N} w$$
$$- \big(\Pi_{\leq N}^{hi,lo,lo} + \Pi_{\leq N}^{hi,lo,hi}\big)\big(\mathsf{I}_{\leq N}, \Psi_{\leq N}, P_{\leq N} w\big)\big] \Big\|^p_{X^{\frac{1}{2}-\delta_1, b}(J) \to X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \Big]^{1/p} \quad (9.25)$$

$$\lesssim T^\alpha p^2,$$

where $0 \in J \subseteq [-T, T]$ closed interval. Furthermore,

$$\mathbb{E}\Big[\sup_N \sup_J \Big\| Y \mapsto P_{\leq N} \big[\mathsf{I}_{\leq N} \Psi_{\leq N} P_{\leq N} Y$$
$$- \big(\Pi_{\leq N}^{hi,lo,lo} + \Pi_{\leq N}^{hi,lo,hi}\big)\big(\mathsf{I}_{\leq N}, \Psi_{\leq N}, P_{\leq N} Y\big)\big] \Big\|^p_{X^{\frac{1}{2}+\delta_2, b}(J) \to X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \Big]^{1/p} \quad (9.26)$$

$$\lesssim T^\alpha p^2$$

Proof: By Lemma 9.9, we decompose (9.25) and (9.26) into $\mathrm{LinCub}^{(5)}$ and $\mathrm{LinCub}^{\sin}$ - terms.

We first consider (9.29).

By the reduction argument in Subsection 5.7 (note 7), the moment method (Proposition B.2 in note 8), and the quintic tensor estimate ($\underset{(5.54)}{\text{Lemma 5.11}}$ in note 11),

$$\mathbb{E}\Big[\sup_J \|w \mapsto (9.29)\|^p_{X^{\frac12-\delta_1,b}(J)\to X^{-\frac12+\delta_2,b_+-1}(J)}\Big]^{1/p}$$

$$\overset{\text{Sub 5.7}}{\lesssim} p^2 T^\alpha \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big[\Big(\mathbb{1}\{N_{234}>N_1^\eta \geq N_5\} + \mathbb{1}\{N_1 \not\sim N_5 > N_1^\eta\} + \mathbb{1}\{N_{234}>N_1^\nu,\ N_1\sim N_5>N_1^\eta\}\Big)$$

$$\times N_0^{-\frac12+\delta_2} N_5^{\frac12+\delta_1} \Big\| \langle\lambda_5\rangle^{-(b_--\frac12)} \Big\| \langle\lambda\rangle^{b_+-1} \sum_{n_1,n_2,n_3,n_4} \tilde{h}(\lambda,0,0,0,0,\lambda_5) \prod_{j=1}^{4} \int_0^1 1\, dW^{\varphi_j}_{s_j}(n_j) \Big\|_{L^2_\lambda(n_5\to n_0)} \Big\|_{L^\infty_{\lambda_5}}$$

$$\overset{\text{Prop B.2}}{\lesssim} p^2 T^\alpha \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big[\Big(\mathbb{1}\{N_{234}>N_1^\eta \geq N_5\} + \mathbb{1}\{N_1 \not\sim N_5 > N_1^\eta\} + \mathbb{1}\{N_{234}>N_1^\nu,\ N_1\sim N_5>N_1^\eta\}\Big)$$

$$\times N_0^{-\frac12+\delta_2} N_5^{\frac12+\delta_1} \max_{(A,B)} \| \langle\lambda\rangle^{b_+-1} \tilde{h}(\lambda,0,0,0,0,\lambda_5) \|_{L^2_\lambda(n_0 n_A \to n_B n_5)}$$

$$\overset{(5.54)}{\lesssim} p^2 T^\alpha \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big[\Big(\mathbb{1}\{N_{234}>N_1^\eta \geq N_5\} + \mathbb{1}\{N_1 \not\sim N_5 > N_1^\eta\} + \mathbb{1}\{N_{234}>N_1^\nu,\ N_1\sim N_5>N_1^\eta\}\Big)$$

$$\times N_0^{\delta_2} N_5^{\delta_1} N_{\max}^{\varepsilon} \Big(\max(N_0,N_2,N_3,N_4)^{-\frac12} + \max(N_2,N_3,N_4,N_5)^{-\frac12} \Big)\Big] \qquad (9.36)$$

where $h = h_{n_0 n_1 \dots n_5}$ is as in (5.48), and (A,B) is any partition of $\{1,2,3,4\}$.

For the first $\mathbb{1}$ in (9.36):

$$(9.36) \lesssim \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big[\mathbb{1}\{N_{234}>N_1^\eta \geq N_5\} N_{\max}^{\varepsilon+\eta\delta_1+\delta_2} \max(N_2,N_3,N_4)^{-\frac12} \Big]$$

$$\lesssim \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} N_{\max}^{\varepsilon+\eta\delta_1+\delta_2-\frac{\eta}{2}} \lesssim 1$$

$0<\varepsilon \ll \delta_2 \ll \eta \ll \delta_1 \ll 1$

worst : $N_1 = N_{\max}$

For the second $\mathbb{1}$ in (9.36):

$$N_1 \not\sim N_5 \Rightarrow \max(N_0,N_2,N_3,N_4) \gtrsim N_{\max}$$

$$N_5 > N_1^\eta \Rightarrow \max(N_2,N_3,N_4,N_5)^{-\frac12} N_5^{\delta_1} \lesssim \max(N_2,N_3,N_4,N_5)^{-\frac12+\delta_1} \lesssim N_{\max}^{-(\frac12-\delta_1)\eta}$$

$$(9.36) \lesssim \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big(N_{\max}^{\varepsilon+\delta_1+\delta_2-\frac12} + N_{\max}^{\varepsilon+\delta_2-(\frac12-\delta_1)\eta} \Big) \lesssim 1$$

For the third $\mathbb{1}$ in (9.36):

$$(9.36) \lesssim \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} \Big[\mathbb{1}\{N_{234}>N_1^\nu,\ N_1\sim N_5>N_1^\eta\} N_{\max}^{\varepsilon+\delta_1+\delta_2} \max(N_2,N_3,N_4)^{-\frac12} \Big]$$

$$\lesssim \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\le N}} N_{\max}^{\varepsilon+\delta_1+\delta_2-\frac{\nu}{2}} \lesssim 1$$

$0<\varepsilon \ll \delta_2 \ll \delta_1 \ll \nu \ll 1$

worst : $N_1 = N_{\max}$

This finishes (9.29)

We now consider (9.31).

Similar to (9.36), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_J \|Y \mapsto (9.31)\|^p_{X^{\frac{1}{2}+\delta_2,b}(J)\to X^{-\frac{1}{2}+\delta_2,b_+-1}(J)}\Big]^{1/p} \\
&\lesssim p^2 T^{\alpha} \sum_{\substack{N_0,\dots,N_5,\\ N_{234}\leq N}} \Big[\mathbb{1}_{\{\max(N_{234},N_5)>N_1^{\eta}\}} \\
&\qquad \times N_0^{\delta_2} N_5^{-\delta_2} N_{\max}^{\varepsilon}\Big(\max(N_0,N_2,N_3,N_4)^{-\frac{1}{2}}+\max(N_2,N_3,N_4,N_5)^{-\frac{1}{2}}\Big)\Big]
\end{aligned} \tag{9.37}$$

Since $\max(N_{234},N_5)>N_1^{\eta}$, we have

$$\max(N_2,N_3,N_4,N_5)\gtrsim \max(N_1,N_2,N_3,N_4,N_5)^{\eta}\gtrsim N_{\max}^{\eta}.$$

Thus,

$$(9.37)\lesssim p^2T^{\alpha}\sum_{\substack{N_0,\dots,N_5,\\ N_{234}\leq N}}\Big(N_{\max}^{\varepsilon}\max(N_0,N_2,N_3,N_4)^{-\frac{1}{2}+\delta_2}N_5^{-\delta_2}+N_{\max}^{\delta_2-\frac{\eta}{2}}\Big)\lesssim 1$$

This finishes (9.31).

$0<\varepsilon\ll\delta_2\ll\eta\ll 1$

For (9.30) and (9.32), we proceed as in (9.36), but now with $h=h^{\text{sine}}_{n_0n_3n_4n_5}$ as in (5.85) (note 13).

For the tensor estimate, we now apply (5.91) in Lemma 5.18.

Due to $N_{\max}^{-\frac{1}{2}}$ in (5.91), the situation is much simpler. □

Reading session 17 : Regularity estimates for $\mathbb{X}^{(1)}$ and $\mathbb{X}^{(2)}$

- Paracontrolled calculus (Section 10)

Definition 10.1 (Decomposition of $\mathbb{X}^{(1)}_{\leq N}$ and $\mathbb{X}^{(2)}_{\leq N}$)

Let $N, N_1, N_2, N_3, N_{23} \leq N$ be frequency-scales.

(i) The high × low × low-portion of $\mathbb{X}^{(1)}$: $0<\eta\ll 1$

$$\mathbb{X}^{(1),hi,lo,lo}[N_*, w_2, w_3] := \mathbb{1}\{N_2, N_3 \leq N_1^{\eta}\}\, P_{N_0} I\left[\<1>_{N_1} P_{N_2}w_2\, P_{N_3}w_3\right]$$

(ii) The resonant-portion of $\mathbb{X}^{(1)}$:

$$\mathbb{X}^{(1),res}[N_*, w_2, w_3] := \mathbb{1}\{N_3 > \max(N_1, N_2)^{\eta}\}\, \mathbb{1}\{N_{23} \leq N_1^{\eta}\}\, P_{N_0} I\left[\<1>_{N_1} P_{N_{23}}\left(P_{N_2}w_2\, P_{N_3}w_3\right)\right]$$

(iii) The explicit portion of $\mathbb{X}^{(1)}$:

$$\begin{aligned}\mathbb{X}^{(1),expl}_{\leq N} := I\Big[&\widetilde{A}_1(\<1>)(\gamma_{\leq N} - \Gamma_{\leq N})\<1>_{\leq N} + A_1(\<1>)\,\mathcal{C}_{\leq N}\<1>_{\leq N} \\ &+ 18 \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{max}\leq N \\ \max(N_2,N_3)\leq N_1^{\eta}}} P_{N_0}\left(\<1>_{N_1}\, \mathcal{C}^{(1,5)}_{\leq N}[N_2,N_3](t)\right) \\ &- A_3(\<1>, \<3>, \<3>) \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{max}\leq N \\ \max(N_2,N_3)\leq N_1^{\eta}}} P_{N_0}\left(\<1>_{N_1}\, \mathcal{C}^{(3,3)}_{\leq N}[N_2,N_3](t)\right)\Big]\end{aligned}$$

$\widetilde{A}_1$, A_1, A_3 : Lemma 3.12

$\mathcal{C}^{(1,5)}_{\leq N}[N_1, N_2] = \mathbb{E}\left[\<1>_{N_1} P_{N_2}\<5>_{\leq N}\right]$, $\mathcal{C}^{(3,3)}_{\leq N}[N_1, N_2] = \mathbb{E}\left[P_{N_1}\<3>_{\leq N} P_{N_2}\<3>_{\leq N}\right]$

(iv) The operator version of $\mathbb{X}^{(2)}$:

$$\mathbb{X}^{(2),op}[N_*, w] := -3 P_{N_0} I\left[:\<1>_{N_1}\<1>_{N_2}: P_{N_3}w\right]$$

Lemma 10.3 For all $N \geq 1$, we have

$$\begin{aligned}\mathbb{X}^{(1)}_{\leq N}[v_{\leq N}, Y_{\leq N}] = &-6 \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{max}\leq N}} \mathbb{X}^{(1),hi,lo,lo}\left[N_*, \<1>_{\leq N}, 3\<5>_{\leq N} + v_{\leq N}\right] && (10.1) \\ &+ \sum_{\varsigma^{(2)},\varsigma^{(3)}\in S^{b}_{0}} \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{max}\leq N}} A_3(\<1>, \varsigma^{(2)}, \varsigma^{(3)})\, \mathbb{X}^{(1),hi,lo,lo}\left[N_*, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}\right] && (10.2) \\ &- 6 \sum_{\substack{N_0,N_1,N_2,N_3,N_{23} \\ N_{max}\leq N}} \mathbb{X}^{(1),res}\left[N_*, \<1>_{\leq N}, Y_{\leq N}\right] && (10.3) \\ &+ \mathbb{X}^{(1),expl}_{\leq N} && (10.4)\end{aligned}$$

Furthermore, we have

$$\mathbb{X}^{(2)}_{\leq N}[v_{\leq N}] = \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{\max} \leq N \\ \min(N_1,N_2) > \max(N_1,N_2)^{\eta} \\ N_3 \leq \max(N_1,N_2)^{\eta}}} \mathbb{X}^{(2),op}\left[N_*,\ 3\Psi_{\leq N} + v_{\leq N}\right]$$

Proof : Recall that

$$\begin{aligned} \mathbb{X}^{(1)}_{\leq N}[v_{\leq N}, Y_{\leq N}] = I \Big[& -6\, P_{\leq N}\, \Pi^{hi,lo,lo}_{\leq N}(q_{\leq N}, q_{\leq N}, 3\Psi_{\leq N} + v_{\leq N}) && (3.46) \\ & + \sum_{\varsigma^{(2)},\varsigma^{(3)} \in S_0^b} A_3(q, \varsigma^{(2)}, \varsigma^{(3)})\, P_{\leq N}\, \Pi^{hi,lo,lo}_{\leq N}(q_{\leq N}, \varsigma^{(2)}_{\leq N}, \varsigma^{(3)}_{\leq N}) && (3.47) \\ & + A_1(q)\, \mathcal{C}_{\leq N}\, q_{\leq N} + \widetilde{A}_1(q)\, (\gamma_{\leq N} - \Gamma_{\leq N})\, q_{\leq N} && (3.48) \\ & - 3 P_{\leq N}\, \Pi^{res}_{\leq N}(q_{\leq N}, q_{\leq N}, Y_{\leq N}) \Big] && (3.49) \end{aligned}$$

Note that

$$(10.1) = (3.46) - 18 \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{\max} \leq N \\ \max(N_2,N_3) \leq N_1^{\eta}}} P_{N_0}\left(q_{N_1}\, \mathcal{C}^{(1,5)}_{\leq N}[N_2,N_3](t)\right)$$

$$(10.2) = (3.47) + A_3(q, \psi, \psi) \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{\max} \leq N \\ \max(N_2,N_3) \leq N_1^{\eta}}} P_{N_0}\left(q_{N_1}\, \mathcal{C}^{(3,3)}_{\leq N}[N_2,N_3](t)\right)$$

$$(10.3) = (3.49)$$

$$\begin{aligned} (10.4) = (3.48) & + 18 \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{\max} \leq N \\ \max(N_2,N_3) \leq N_1^{\eta}}} P_{N_0}\left(q_{N_1}\, \mathcal{C}^{(1,5)}_{\leq N}[N_2,N_3](t)\right) \\ & - A_3(q, \psi, \psi) \sum_{\substack{N_0,N_1,N_2,N_3 \\ N_{\max} \leq N \\ \max(N_2,N_3) \leq N_1^{\eta}}} P_{N_0}\left(q_{N_1}\, \mathcal{C}^{(3,3)}_{\leq N}[N_2,N_3](t)\right) \end{aligned}$$

$$\Rightarrow (10.1) + (10.2) + (10.3) + (10.4) = (3.46) + (3.47) + (3.48) + (3.49)$$

The formula for $\mathbb{X}^{(2)}_{\leq N}[v_{\leq N}]$ follows directly from (3.51) □

- Probabilistic Strichartz and regularity estimates (Section 10.1)

Lemma 10.4 (Probabilistic Strichartz and regularity estimates for $\mathbb{X}^{(1)}$):

$0 < \eta \ll 1$

$0 < b - \frac{1}{2} < b_+ - \frac{1}{2} \ll 1$

$0 < \delta_2 \ll \delta_1 \ll 1$

Let $T \geq 1$, $p \geq 2$, $0 \in J \subseteq [-T, T]$ closed intervals.

(i) For N_0, N_1, N_2, N_3 satisfying $N_2, N_3 \leq N_1^{\eta}$, we have

$$\mathbb{E}\Big[\sup_J |J|^{-(b_+-b)p} \big\| (w_2, w_3) \mapsto \mathbb{X}^{(1), hi, lo, lo}[N_*, w_2, w_3] \big\|^p_{X^{-1,b}(J) \times X^{-1,b}(J) \to X^{\frac{1}{2}-\delta_1, b}(J)} \Big]^{1/p}$$
$$\lesssim p^{\frac{1}{2}} T^{\alpha} N_{max}^{-\varepsilon}$$

Furthermore, we have

$$\mathbb{E}\Big[\sup_J |J|^{-(b_+-b)p} \big\| (w_2, w_3) \mapsto \mathbb{X}^{(1), hi, lo, lo}[N_*, w_2, w_3] \big\|^p_{X^{-1,b}(J) \times X^{-1,b}(J) \to L^{\infty}_t C^{\frac{1}{2}-\delta_1}_x(J)} \Big]^{1/p}$$
$$\lesssim p^{\frac{1}{2}} T^{\alpha} N_{max}^{-\varepsilon}$$

(ii) For $N_0, N_1, N_2, N_3, N_{23}$ satisfying $N_3 > \max(N_1, N_2)^{\eta}$ and $N_{23} \leq N_1^{\eta}$, we have

$$\mathbb{E}\Big[\sup_J |J|^{-(b_+-b)p} \big\| (w_2, w_3) \mapsto \mathbb{X}^{(1), res}[N_*, w_2, w_3] \big\|^p_{X^{-\frac{1}{2}-\varepsilon, b}(J) \times X^{\frac{1}{2}+\delta_2, b}(J) \to X^{\frac{1}{2}-\delta_1, b}(J)} \Big]^{1/p}$$
$$\lesssim p^{\frac{1}{2}} T^{\alpha} N_{max}^{-\varepsilon}$$

Furthermore, we have

$$\mathbb{E}\Big[\sup_J |J|^{-(b_+-b)p} \big\| (w_2, w_3) \mapsto \mathbb{X}^{(1), res}[N_*, w_2, w_3] \big\|^p_{X^{-\frac{1}{2}-\varepsilon, b}(J) \times X^{\frac{1}{2}+\delta_2, b}(J) \to L^{\infty}_t C^{\frac{1}{2}-\delta_1}_x(J)} \Big]^{1/p}$$
$$\lesssim p^{\frac{1}{2}} T^{\alpha} N_{max}^{-\varepsilon}$$

(iii) We have

$$\mathbb{E}\Big[\sup_N |J|^{-(b_+-b)p} \big\| \mathbb{X}^{(1), expl}_{\leq N} \big\|^p_{(X^{\frac{1}{2}-\delta_1, b} \cap L^{\infty}_t C^{\frac{1}{2}-\delta_1}_x)(J)} \Big]^{1/p} \lesssim p^{\frac{1}{2}} T^{\alpha}$$

Proof : (i) By $X^{s,b}$- linear estimates (Lemma 2.4), we have

$$\|\mathbb{X}^{(1),hi,lo,lo}[N_*,w_2,w_3]\|_{X^{\frac12-\delta_1,b}(J)} \lesssim \|\varphi_{N_1}P_{N_2}w_2P_{N_3}w_3\|_{X^{-\frac12-\delta_1,b-1}(J)}$$

$$\lesssim N_1^{-\frac12-\delta_1}\|\varphi_{N_1}P_{N_2}w_2P_{N_3}w_3\|_{L^2_tL^2_x(J)} \qquad N_2,N_3\le N_1^{\eta}$$

$$\lesssim |J|^{\frac12}N_1^{-\frac12-\delta_1}\|\varphi_{N_1}\|_{L^\infty_tL^\infty_x(J)}\prod_{j=2}^{3}\|P_{N_j}w_j\|_{L^\infty_tL^\infty_x(J)}$$

$$\lesssim |J|^{\frac12}N_1^{\varepsilon-\delta_1}\|\varphi_{N_1}\|_{L^\infty_tC_x^{-\frac12-\varepsilon}(J)}\prod_{j=2}^{3}\|\widehat{P_{N_j}w_j}\|_{L^\infty_t\ell^1_n(J)}$$

Cauchy-Schwarz

$$\lesssim |J|^{\frac12}N_1^{\varepsilon-\delta_1}N_2^{\frac32}N_3^{\frac32}\|\varphi_{N_1}\|_{L^\infty_tC_x^{-\frac12-\varepsilon}(J)}\prod_{j=2}^{3}\|P_{N_j}w_j\|_{L^\infty_tL^2_x(J)}$$

$b>\frac12$: $X^{s,b}\hookrightarrow C_tH^s$

$$\lesssim |J|^{\frac12}N_1^{\varepsilon-\delta_1}N_2^{\frac52}N_3^{\frac52}\|\varphi_{N_1}\|_{L^\infty_tC_x^{-\frac12-\varepsilon}(J)}\prod_{j=2}^{3}\|P_{N_j}w_j\|_{X^{-1,b}(J)}$$

Since $N_2,N_3\le N_1^{\eta}$, we have $N_1^{\varepsilon-\delta_1}N_2^{\frac52}N_3^{\frac52}\lesssim N_1^{\varepsilon+5\eta-\delta_1}\lesssim N_{max}^{-\frac{\delta_1}{2}}$ $\qquad \varepsilon\ll\eta\ll\delta_1$

Lemma 7.4 : $\mathbb{E}\Big[\|\varphi_{N_1}\|^p_{L^\infty_tC_x^{-\frac12-\varepsilon}([-T,T])}\Big]^{1/p}\lesssim p^{\frac12}T^{\alpha}$

Similarly, we have

$$\|\mathbb{X}^{(1),hi,lo,lo}[N_*,w_2,w_3]\|_{L^\infty_tC_x^{\frac12-\delta_1}(J)} \lesssim |J|\,\|\varphi_{N_1}P_{N_2}w_2P_{N_3}w_3\|_{L^\infty_tC_x^{-\frac12-\delta_1}(J)} \qquad I : \text{one degree of smoothing}$$

$$\lesssim |J|N_1^{-\frac12-\delta_1}\|\varphi_{N_1}P_{N_2}w_2P_{N_3}w_3\|_{L^\infty_tL^\infty_x(J)}$$

$$\lesssim |J|N_1^{-\frac12-\delta_1}\|\varphi_{N_1}\|_{L^\infty_tL^\infty_x(J)}\prod_{j=2}^{3}\|P_{N_j}w_j\|_{L^\infty_tL^\infty_x(J)}$$

above argument

$$\lesssim |J|N_{max}^{-\frac{\delta_1}{2}}\|\varphi_{N_1}\|_{L^\infty_tC_x^{-\frac12-\varepsilon}(J)}\prod_{j=2}^{3}\|P_{N_j}w_j\|_{X^{-1,b}}$$

(ii) Mostly similar to (i) with following additional considerations :

If $N_2\gg N_3$, we must have $N_3\ll N_2\sim N_{23}\le N_1^{\eta}$

If $N_2\ll N_3$, we must have $N_2\ll N_3\sim N_{23}\le N_1^{\eta}$

$$\|P_{N_2}w_2\|_{L^\infty_tL^\infty_x(J)}\lesssim N_2^{2+\varepsilon}\|P_{N_2}w_2\|_{X^{-\frac12-\varepsilon,b}(J)},\quad \|P_{N_3}w_3\|_{L^\infty_tL^\infty_x}\lesssim N_3^{1-\delta_2}\|P_{N_3}w_3\|_{X^{\frac12+\delta_2,b}(J)}$$

$\Rightarrow$ argue as in (i)

If $N_2\sim N_3$, orthogonality argument :

Decompose $\{|n_2|\sim N_2\}$ and $\{|n_3|\sim N_3\}$ into balls of radius $\sim N_{23}$

Set of balls : $\mathcal{J}_2$ $\qquad\qquad$ $\mathcal{J}_3$

Observation : for each $J_2\in\mathcal{J}_2$, at most $O(1)$ number of $J_3\in\mathcal{J}_3$ s.t.

$$P_{N_{23}}(P_{J_2}P_{N_2}w_2\,P_{J_3}P_{N_3}w_3) \text{ is nonzero}$$

$$\| P_{J_2} P_{N_2} w_2 \|_{L_t^\infty L_x^\infty(J)} \lesssim N_{23}^{\frac{3}{2}} N_2^{\frac{1}{2}+\varepsilon} \| P_{J_2} P_{N_2} w_2 \|_{X^{-\frac{1}{2}-\varepsilon,b}(J)}$$

$$\| P_{J_3} P_{N_3} w_3 \|_{L_t^\infty L_x^\infty(J)} \lesssim N_{23}^{\frac{3}{2}} N_3^{-\frac{1}{2}-\delta_2} \| P_{J_3} P_{N_3} w_3 \|_{X^{\frac{1}{2}+\delta_2,b}(J)}$$

Remains to estimate:

$$N_1^{\varepsilon-\delta_1} N_{23}^{3} N_2^{\frac{1}{2}+\varepsilon} N_3^{-\frac{1}{2}-\delta_2} \sum_{J_2, J_3} \| P_{J_2} P_{N_2} w_2 \|_{X^{-\frac{1}{2}-\varepsilon,b}(J)} \| P_{J_3} P_{N_3} w_3 \|_{X^{\frac{1}{2}+\delta_2,b}(J)}$$

$N_{23} \leq N_1^{\eta}$

$N_2^{\frac{1}{2}+\varepsilon} N_3^{-\frac{1}{2}-\delta_2} \sim N_2^{\varepsilon-\delta_2}$ $\quad \varepsilon \ll \delta_2$

Cauchy-Schwarz in $J_2 \in \mathcal{J}_2$

(iii) $(\gamma_{\leq N} - \Gamma_{\leq N}) \rho_{\leq N}$:

Lemma 7.1 $\Rightarrow$ $|\gamma_{\leq N} - \Gamma_{\leq N}(n)| \lesssim \langle n \rangle^{\varepsilon}$

$\mathcal{C}_{\leq N} \rho_{\leq N}$:

Lemma 6.23 $\Rightarrow$ $\| \mathcal{C}_{\leq N}(t) \|_{L_t^\infty(J)} \overset{\text{Sobolev}}{\lesssim} \| \chi(t) \mathcal{C}_{\leq N}(t) \|_{H_t^{\frac{1}{2}+}} \lesssim_\chi 1$ $\quad \chi \equiv 1$ on J

$$\sum_{\substack{N_0, N_1, N_2, N_3 : \\ N_{max} \leq N \\ \max(N_2, N_3) \leq N_1^{\eta}}} P_{N_0}\left(\rho_{N_1} \mathcal{C}_{\leq N}^{(1,5)}[N_2, N_3](t) \right) \text{ and } \sum_{\substack{N_0, N_1, N_2, N_3 : \\ N_{max} \leq N \\ \max(N_2, N_3) \leq N_1^{\eta}}} P_{N_0}\left(\rho_{N_1} \mathcal{C}_{\leq N}^{(3,3)}[N_2, N_3](t) \right) :$$

Lemma 7.17 $\Rightarrow$ $\left| \mathcal{C}_{\leq N}^{(1,5)}[N_2, N_3](t) \right| + \left| \mathcal{C}_{\leq N}^{(3,3)}[N_2, N_3](t) \right| \lesssim \max(N_2, N_3)^{2\varepsilon} T^{\alpha}$

for all $t \in [-T, T]$

Rest of the argument : similar to (and much easier) than (i) $\square$

Lemma 10.5 (Probabilistic Strichartz and regularity estimates for $\mathbb{X}^{(2)}$) :

Let $T \geq 1$, $p \geq 2$, $0 \in J \in [-T, T]$ closed intervals. For frequency-scales N_0, N_1, N_2, N_3 satisfying $N_3 \leq \max(N_1, N_2)^{\eta} < \min(N_1, N_2)$, we have

$$\mathbb{E}\left[\sup_J |J|^{-(b_+-b)p} \left\| w \mapsto \mathbb{X}^{(2),op}[N_*, w] \right\|^p_{X^{-1,b}(J) \to X^{\frac{1}{2}-\delta_1,b}(J)} \right]^{1/p} \lesssim p T^{\alpha} N_{max}^{-\varepsilon} \qquad (10.6)$$

$$\mathbb{E}\left[\sup_J |J|^{-(b_+-b)p} \left\| w \mapsto \mathbb{X}^{(2),op}[N_*, w] \right\|^p_{X^{-1,b}(J) \to L_t^\infty C_x^{\frac{1}{2}-\delta_1}(J)} \right]^{1/p} \lesssim p T^{\alpha} N_{max}^{-\varepsilon} \qquad (10.7)$$

Proof: For (10.6), by using the reduction in Section 5.7, we need to show

$$N_0^{-\frac{1}{2}-\delta_1} N_1^{-1} N_2^{-1} N_3 \, \mathbb{E}\Big[\Big\| \sum_{n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: \Big\|^p_{\ell^2_{n_3} \to \ell^2_{n_0}} \Big]^{1/p} \lesssim p N_{max}^{-\delta_1 + 4\eta} , \qquad (10.8)$$

where

$$h^{b,m}_{n_0 n_1 n_2 n_3} = \Big(\prod_{j=0}^{3} \mathbb{1}_{N_j}(n_j) \Big) \mathbb{1}\{n_0 = n_{123}\} \mathbb{1}\{|\Omega - m| \leq 1\} ,$$

$$\Omega = \sum_{j=0}^{3} (\pm_j) \langle n_j \rangle \quad \text{and} \quad m \in \mathbb{Z}$$

$$:g_{n_1} g_{n_2}: = g_{n_1} g_{n_2} - \mathbb{E}[g_{n_1} g_{n_2}]$$

Note that

$$\mathbb{E}\Big[\Big\| \sum_{n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: \Big\|^p_{\ell^2_{n_3} \to \ell^2_{n_0}} \Big]^{1/p} \lesssim \mathbb{E}\Big[\Big\| \sum_{n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: \Big\|^p_{n_0 n_3} \Big]^{1/p} \qquad \|\cdot\|_{n_0 n_3} = \|\cdot\|_{\ell^2_{n_0} \ell^2_{n_3}}$$

$$\lesssim \Big\| \mathbb{E}\Big[\Big| \sum_{n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: \Big|^p \Big]^{1/p} \Big\|_{n_0 n_3}$$

$$\overset{\text{Wiener-chaos}}{\lesssim} p \Big\| \mathbb{E}\Big[\Big| \sum_{n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: \Big|^2 \Big]^{1/2} \Big\|_{n_0 n_3}$$

$$\lesssim p \, \| h^{b,m}_{n_0 n_1 n_2 n_3} \|_{n_0 n_1 n_2 n_3}$$

Lemma 5.7 =>

$$N_0^{-\frac{1}{2}-\delta_1} N_1^{-1} N_2^{-1} N_3 \, \| h^{b,m}_{n_0 n_1 n_2 n_3} \|_{n_0 n_1 n_2 n_3}$$

$$\lesssim N_0^{-\frac{1}{2}-\delta_1} N_1^{-1} N_2^{-1} N_3 \cdot N_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_0 N_1 N_2 N_3$$

$$\lesssim N_{max}^{-\delta_1} N_3^{\frac{5}{2}} \lesssim N_{max}^{-\delta_1 + 3\eta} \qquad \Rightarrow \quad (10.8)$$

For (10.7), by Sobolev embedding, we have

$$\| P_{N_0} F(t,x) \|_{L^\infty_t C_x^{\frac{1}{2}-\delta_1}} \lesssim \| P_{N_0} F(t,x) \|_{H^b_t W_x^{\frac{1}{2}-\frac{\delta_1}{2}, q}} \quad \text{for some large } q < \infty$$

Reduction in Section 5.7 =>

$$\mathbb{E}\Big[\Big\| \sum_{n_0,n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: e^{i\langle n_0, x\rangle} \Big\|^p_{L^q_x \ell^2_{n_3}} \Big]^{1/p}$$

$$\overset{\overset{p>q}{\text{Wiener-chaos}}}{\lesssim} p \Big\| \mathbb{E}\Big[\Big| \sum_{n_0,n_1,n_2} h^{b,m}_{n_0 n_1 n_2 n_3} :g_{n_1} g_{n_2}: e^{i\langle n_0, x\rangle} \Big|^2 \Big]^{1/2} \Big\|_{L^q_x \ell^2_{n_3}}$$

↳ sum in n_1 and n_2, since n_0 is determined by n_1, n_2, n_3 : $n_0 = n_{123}$

(10.7) follows by same reasoning above, and $\|1\|_{L^q_x} \sim 1$ □

Reading session 18 : Paracontrolled calculus with one linear stochastic object

- Paracontrolled calculus (Section 10)
  - Interactions with one linear stochastic object

Lemma 10.7 (Product estimate for $\overset{\circ}{|}\, \mathbb{X}^{(1)}$) (recall Lm 10.3 in note 17)

Let $T \geq 1$ and $p \geq 2$. Let $0 \in J \subseteq [-T, T]$ be any closed interval.

(i) For all frequency-scales $N_1, N_2, N_3, N_4, N_{234}$ satisfying $\max(N_3, N_4) \leq N_2^{\eta}$,

$$\mathbb{E}\Big[\sup_J \big\| (w_3, w_4) \mapsto \overset{\circ}{|}_{N_1} \mathbb{X}^{(1),hi,lo,lo}[N_* ; w_3, w_4] \big\|^p_{X^{-1,b}(J) \times X^{-1,b}(J) \to L_t^{\infty} C_x^{-\frac{1}{2}-\varepsilon}(J)} \Big]^{1/p}$$

$$\lesssim p\, T^{\alpha} N_{234}^{-\frac{1}{2}+10\eta}$$

(ii) For all frequency-scales $N_1, N_2, N_3, N_4, N_{34}, N_{234}$ satisfying $N_4 > \max(N_2, N_3)^{\eta}$ and $N_{34} \leq N_2^{\eta}$,

$$\mathbb{E}\Big[\sup_J \big\| (w_3, w_4) \mapsto \overset{\circ}{|}_{N_1} \mathbb{X}^{(1),res}[N_* ; w_3, w_4] \big\|^p_{X^{-\frac{1}{2}-\varepsilon,b}(J) \times X^{\frac{1}{2}+\delta_2,b}(J) \to L_t^{\infty} C_x^{-\frac{1}{2}-\varepsilon}(J)} \Big]^{1/p}$$

$$\lesssim p\, T^{\alpha} N_{234}^{-\frac{1}{2}+10\eta}$$

(iii) For all frequency-scales K_1 and K_2,

$$\mathbb{E}\Big[\sup_N \big\| \overset{\circ}{|}_{K_1} P_{K_2} \mathbb{X}^{(1),expl}_{\leq N} \big\|^p_{L_t^{\infty} C_x^{-\frac{1}{2}-\varepsilon}([-T,T])} \Big]^{1/p} \lesssim p\, T^{\alpha} K_2^{-\frac{1}{2}+10\eta}$$

Proof: As before, we only consider $T = 1$ and $J = [-1, 1]$.

(i) By reduction in Subsection 5.7 (note 7), it suffices to show

$$\sup_{t \in [-1,1]} \sup_{\pm_3, \pm_4} \sup_{\substack{n_3, n_4 \in \mathbb{Z}^3 \\ |n_3|, |n_4| \leq N_2^{\eta}}} \sup_{\lambda_3, \lambda_4 \in \mathbb{R}} \mathbb{E}\Big[\Big| \sum_{n_1, n_2 \in \mathbb{Z}^3} 1\!\!1_{N_{234}}(n_{234}) \Big(\prod_{j=1}^{2} 1\!\!1_{N_j}(n_j) \Big) \langle n_{1234} \rangle^{-\frac{1}{2}-\varepsilon} e^{i\langle n_{1234}, x \rangle} \overset{\circ}{|}_{N_1}(t, n_1)$$

$$\times \int_0^t \frac{\sin((t-t')\langle n_{234} \rangle)}{\langle n_{234} \rangle} \overset{\circ}{|}_{N_2}(t', n_2) \Big(\prod_{j=3}^{4} e^{i(\pm_j \langle n_j \rangle + \lambda_j)t'} \Big) dt' \Big|^2 \Big] \lesssim N_1^{-\varepsilon} N_{234}^{-1} \qquad (10.11)$$

Using the definition of the Sine-kernel in Definition 5.13 (note 12), we have

$$\sum_{n_1,n_2\in\mathbb{Z}^3} \mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j)\Big) \langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon} e^{i\langle n_{1234},x\rangle} \varrho_{N_1}(t,n_1)$$
$$\times \int_0^t \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \varrho_{N_2}(t',n_2) \Big(\prod_{j=3}^{4} e^{i(\pm_j\langle n_j\rangle+\lambda_j)t'}\Big)\, dt'$$
$$= \sum_{\varphi_1,\varphi_2\in\{\cos,\sin\}} \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j)\Big) e^{i\langle n_{1234},x\rangle} \langle n_1\rangle^{-1}\langle n_{234}\rangle^{-1}\langle n_2\rangle^{-1}\langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon} \varphi_1(t\langle n_1\rangle)$$
$$\times \int_0^t \sin((t-t')\langle n_{234}\rangle)\, \varphi_2(t'\langle n_2\rangle) \Big(\prod_{j=3}^{4} e^{i(\pm_j\langle n_j\rangle+\lambda_j)t'}\Big)\, dt' \cdot \prod_{j=1}^{2}\int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \qquad (10.12)$$
$$+ e^{i\langle n_{34},x\rangle} \langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon} \mathbb{1}_{\{N_1=N_2\}} \Big(\int_0^t \mathrm{Sine}[N_{234},N_2](t-t',n_{34}) \Big(\prod_{j=3}^{4} e^{i(\pm_j\langle n_j\rangle+\lambda_j)t'}\Big)\Big). \qquad (10.13)$$

For the non-resonant part (10.12), we have

$$\mathbb{E}\big[|(10.12)|^2\big] \lesssim \sum_{n_1,n_2\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=1}^{2}\mathbb{1}_{N_j}(n_j)\Big) \langle n_1\rangle^{-2}\langle n_{234}\rangle^{-2}\langle n_2\rangle^{-2}\langle n_{1234}\rangle^{-1-2\varepsilon}\Big]$$
$$\lesssim N_1^{-\varepsilon} \sum_{n_2\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234})\, \mathbb{1}_{N_2}(n_2)\, \langle n_{234}\rangle^{-2}\langle n_2\rangle^{-2}\Big]$$
$$\lesssim N_1^{-\varepsilon} N_{234}^{-1} \qquad \checkmark$$

For the resonant part (10.13), by Lemma 5.17, we have

$$|(10.13)|^2 \lesssim \mathbb{1}_{\{N_1=N_2\}} N_{234}^{-2+\varepsilon} \qquad \checkmark \quad (\text{recall } N_3, N_4 \le N_2^{\eta})$$

(ii) Orthogonality argument: (see note 17)

- localize w_3 and w_4 to balls of radius N_{34}
- repeat part (i)
- sum up balls using the Cauchy-Schwarz inequality

(iii) Consider following cases for $\mathbb{X}^{(1),\mathrm{expl}}_{\le N}$

① $\mathcal{C}_{\le N}\, \varrho_{\le N}$:

Lemma 6.23 (note 6) $\Rightarrow$ $\|\mathcal{C}_{\le N}(t)\|_{L_t^\infty(J)} \overset{\text{Sobolev}}{\lesssim} \|\chi(t)\,\mathcal{C}_{\le N}(t)\|_{H_t^{\frac{1}{2}+}} \lesssim_\chi 1$ $\quad \chi\equiv 1$ on J

$\Rightarrow$ similar (and easier) to part (i)

② $\displaystyle\sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{\max}\le N\\ \max(N_2,N_3)\le N_1^{\eta}}} P_{N_0}\big(\varrho_{N_1}\, \mathcal{C}^{(1,5)}_{\le N}[N_2,N_3](t)\big)$ and $\displaystyle\sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{\max}\le N\\ \max(N_2,N_3)\le N_1^{\eta}}} P_{N_0}\big(\varrho_{N_1}\, \mathcal{C}^{(3,3)}_{\le N}[N_2,N_3](t)\big)$:

Lemma 7.17 (note 14) $\Rightarrow$ $\big|\mathcal{C}^{(1,5)}_{\le N}[N_2,N_3](t)\big| + \big|\mathcal{C}^{(3,3)}_{\le N}[N_2,N_3](t)\big| \lesssim \max(N_2,N_3)^{2\varepsilon}\, T^{\alpha}$

for all $t\in[-T,T]$

$\Rightarrow$ similar (and easier) to part (i)

③ $(\gamma_{\leq N} - \Gamma_{\leq N})\, \varrho_{\leq N}$:

Set $N_1 := K_1$ and $N_2 := K_2$. We have

$$\varrho_{N_1} (\gamma_{\leq N} - \Gamma_{\leq N})\, I[\varrho_{N_2}]$$
$$= \sum_{\varphi_1,\varphi_2 \in \{\cos,\sin\}} \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j) \Big) \langle n_1\rangle^{-1} \langle n_2\rangle^{-2} (\gamma_{\leq N} - \Gamma_{\leq N}(n_2))\, e^{i\langle n_{12}, x\rangle}$$
$$\times\, \varphi_1(t\langle n_1\rangle) \int_0^t \sin((t-t')\langle n_2\rangle)\, \varphi_2(t'\langle n_2\rangle)\, dt' \cdot \prod_{j=1}^{2} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \Big] \qquad (10.14)$$
$$+ \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\mathbb{1}_{\{n_{12}=0\}} \Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j) \Big) \langle n_1\rangle^{-1} \langle n_2\rangle^{-2} (\gamma_{\leq N} - \Gamma_{\leq N}(n_2))$$
$$\times \int_0^t \sin((t-t')\langle n_2\rangle) \cos((t-t')\langle n_2\rangle)\, dt' \Big] \qquad (10.15)$$

Using Lemma 7.1 (note 9), the non-resonant part (10.14) follows from the same way as part (i)

For (10.15), we have

$$(10.15) = \mathbb{1}_{\{N_1=N_2\}} \sum_{n\in\mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j) \Big) \langle n\rangle^{-3} (\gamma_{\leq N} - \Gamma_{\leq N}(n)) \int_0^t \sin((t-t')\langle n\rangle) \cos((t-t')\langle n\rangle)\, dt' \Big]$$
$$= \frac{1}{4} \mathbb{1}_{\{N_1=N_2\}} \sum_{n\in\mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{2} \mathbb{1}_{N_j}(n_j) \Big) \langle n\rangle^{-4} (\gamma_{\leq N} - \Gamma_{\leq N}(n)) (\cos(2t\langle n\rangle) - 1) \Big] \qquad (10.16)$$

By Lemma 7.1 (note 9), we obtain

$$|(10.16)| \lesssim \mathbb{1}_{\{N_1=N_2\}} N_2^{-1+\varepsilon} \qquad ✓ \qquad \square$$

Lemma 10.8 (Product estimate for $\varrho\, \mathbb{X}^{(2)}$)

Let $T \geq 1$ and $p \geq 2$. Let $0 \in J \subseteq [-T, T]$ be any closed interval.

Let $N_1, N_2, N_3, N_4, N_{234} \geq 1$ be dyadic with $N_4 \leq \max(N_2,N_3)^{\eta} < \min(N_2,N_3)$.

Then, we have

$$\mathbb{E}\Big[\sup_J \big\| w_4 \mapsto \varrho_{N_1} \mathbb{X}^{(2),\varphi}[N_4, w_4] \big\|^p_{X^{-1,b}(J) \to L_t^\infty C_x^{-\frac{1}{2}-\varepsilon}(J)} \Big]^{1/p} \lesssim p\, T^{\alpha} \max(N_2,N_3)^{-\frac{1}{2}+5\eta}.$$

Proof: As before, we only consider $T=1$ and $J=[-1,1]$.

By reduction in Subsection 5.7 (note 7), it suffices to show

$$\sup_{t\in[-1,1]} \sup_{\pm_4} \sup_{|n_4|\leq \max(N_2,N_3)^{\eta}} \sup_{\lambda_4\in\mathbb{R}} \mathbb{E}\Big[\Big| \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=1}^{3} \mathbb{1}_{N_j}(n_j) \Big) \langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon} e^{i\langle n_{1234}, x\rangle} \varrho_{N_1}(t,n_1)$$
$$\times \int_0^t \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} :\varrho_{N_2}(t',n_2)\, \varrho_{N_3}(t',n_3):\, e^{i(\pm_4\langle n_4\rangle + \lambda_4)t'}\, dt' \Big|^2 \Big] \lesssim N_1^{-\varepsilon} \max(N_2,N_3)^{-1}$$

Using the definition of the Sine-kernel in Definition 5.13 (note 12), we have

$$\begin{aligned}
&\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234})\Big(\prod_{j=1}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon}e^{i\langle n_{1234},x\rangle}\overset{\circ}{|}_{N_1}(t,n_1)\\
&\qquad\times\int_0^t\frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle}:\overset{\circ}{|}_{N_2}(t',n_2)\ \overset{\circ}{|}_{N_3}(t',n_3):e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\Big]\\
&=\sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234})\Big(\prod_{j=1}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_{234}\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-1}\langle n_{1234}\rangle^{-\frac{1}{2}-\varepsilon}e^{i\langle n_{1234},x\rangle}\varphi_1(t\langle n_1\rangle)\\
&\qquad\times\int_0^t\sin((t-t')\langle n_{234}\rangle)\,\varphi_2(t'\langle n_2\rangle)\,\varphi_3(t'\langle n_3\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\cdot\prod_{j=1}^{3}\int_0^1 1\,dW_{s_j}^{\varphi_j}(n_j)\Big] &&(10.18)\\
&\quad+\mathbb{1}_{\{N_1=N_2\}}\sum_{\varphi_3\in\{\cos,\sin\}}\sum_{n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_3}(n_3)\langle n_3\rangle^{-1}\langle n_{34}\rangle^{-\frac{1}{2}-\varepsilon}e^{i\langle n_{34},x\rangle}\\
&\qquad\times\int_0^t\mathrm{Sine}[N_{234},N_2](t-t',n_{34})\,\varphi_3(t'\langle n_3\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\cdot\int_0^1 1\,dW_{s_3}^{\varphi_3}(n_3)\Big] &&(10.19)\\
&\quad+\mathbb{1}_{\{N_1=N_3\}}\sum_{\varphi_2\in\{\cos,\sin\}}\sum_{n_2\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_2}(n_2)\langle n_2\rangle^{-1}\langle n_{24}\rangle^{-\frac{1}{2}-\varepsilon}e^{i\langle n_{24},x\rangle}\\
&\qquad\times\int_0^t\mathrm{Sine}[N_{234},N_3](t-t',n_{24})\,\varphi_2(t'\langle n_2\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\cdot\int_0^1 1\,dW_{s_2}^{\varphi_2}(n_2)\Big] &&(10.20)
\end{aligned}$$

By symmetry in n_2 and n_3, we only need to consider (10.18) and (10.19)

For the non-resonant part (10.18), we have

$$\begin{aligned}
\mathbb{E}\big[|(10.18)|^2\big]&\lesssim\sum_{\substack{\varphi_1,\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234})\Big(\prod_{j=1}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-2}\langle n_{234}\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2}\langle n_{1234}\rangle^{-1-2\varepsilon}\\
&\qquad\times\Big|\int_0^t\sin((t-t')\langle n_{234}\rangle)\,\varphi_2(t'\langle n_2\rangle)\,\varphi_3(t'\langle n_3\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\Big|^2\Big]\\
&\overset{\text{sum in } n_1}{\lesssim}N_1^{-\varepsilon}\sum_{\substack{\varphi_2,\varphi_3\\ \in\{\cos,\sin\}}}\sum_{n_2,n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_{234}}(n_{234})\Big(\prod_{j=1}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_{234}\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2}\\
&\qquad\times\Big|\int_0^t\sin((t-t')\langle n_{234}\rangle)\,\varphi_2(t'\langle n_2\rangle)\,\varphi_3(t'\langle n_3\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\Big|^2\Big] &&(10.21)
\end{aligned}$$

By performing the t'-integral and Lemma 5.4 (5.15) for $q=2$ (note 6)

$$(10.21)\lesssim N_1^{-\varepsilon}\max(N_{234},N_2,N_3)^{-1}\qquad\checkmark$$

For the resonant part (10.19), by Lemma 5.17 (note 12), we have

$$\begin{aligned}
\mathbb{E}\big[|(10.19)|^2\big]&\lesssim\sum_{\varphi_3\in\{\cos,\sin\}}\sum_{n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_3}(n_3)\langle n_3\rangle^{-2}\langle n_{34}\rangle^{-1-2\varepsilon}\\
&\qquad\times\Big|\int_0^t\mathrm{Sine}[N_{234},N_2](t-t',n_{34})\,\varphi_3(t'\langle n_3\rangle)\,e^{i(\pm_4\langle n_4\rangle+\lambda_4)t'}dt'\Big|^2\Big]\\
&\lesssim\max(N_2,N_{234})^{-2+\varepsilon}\sum_{n_3\in\mathbb{Z}^3}\Big[\mathbb{1}_{N_3}(n_3)\langle n_3\rangle^{-2}\langle n_{34}\rangle^{-1-2\varepsilon}\Big]\\
&\lesssim\max(N_2,N_{234})^{-2+\varepsilon}
\end{aligned}$$

Since $N_4\leq\max(N_2,N_3)^{\eta}$, we have $\max(N_2,N_{234})\sim\max(N_2,N_3)$, so

$$\mathbb{E}\big[|(10.19)|^2\big]\lesssim\max(N_2,N_3)^{-2+\varepsilon}\qquad\checkmark\qquad\qquad\square$$

Proposition 10.6 (Product estimates for $\rho\mathbb{X}^{(1)}$ and $\rho\mathbb{X}^{(2)}$)

For all $A \geqslant 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geqslant 1 - c^{-1}\exp(-cA^{\zeta})$ such that :

For all frequency-scales N, K_1, and K_2, all $T \geqslant 1$, closed intervals $0 \in J \subseteq [-T, T]$, and $v_{\leq N}, Y_{\leq N} : J \times \mathbb{T}^3 \longrightarrow \mathbb{R}$, we have

$$\| \rho_{K_1} P_{K_2} \mathbb{X}^{(1)}_{\leq N}[v_{\leq N}, Y_{\leq N}] \|_{L_t^\infty C_x^{-\frac{1}{2}-\varepsilon}(J)} + \| \rho_{K_1} P_{K_2} \mathbb{X}^{(2)}_{\leq N}[v_{\leq N}] \|_{L_t^\infty C_x^{-\frac{1}{2}-\varepsilon}(J)}$$
$$\leq A\, T^{\alpha} K_2^{-\frac{1}{2}+10\eta} \left(1 + \| v_{\leq N} \|^2_{X^{-\frac{1}{2}-\varepsilon, b}(J)} + \| Y_{\leq N} \|_{X^{\frac{1}{2}+\delta_2, b}(J)} \right)$$

Proof: By Lemma 10.3 (note 17), $\mathbb{X}^{(1)}_{\leq N}$ and $\mathbb{X}^{(2)}_{\leq N}$ can be rewritten in terms of $\mathbb{X}^{(1),hi,lo,lo}[N_*]$, $\mathbb{X}^{(1),res}[N_*]$, $\mathbb{X}^{(1),expl}$, and $\mathbb{X}^{(2),op}[N_*]$.

The desired estimate follows from Lemma 10.7 and Lemma 10.8 □

Corollary 10.9 (The $\rho_{\leq N}\mathbb{X}^{(j)}_{\leq N}$ - operator)

For all $A \geqslant 1$, there exists $E_A \subseteq \Omega$ with $\mathbb{P}(E_A) \geqslant 1 - c^{-1}\exp(-cA^{\zeta})$ such that :

Let $T \geqslant 1$, $0 \in J \subset [-T, T]$ closed interval, and N, N_1, N_2, N_3 dyadic with

$$N_1, N_2, N_3 \leq N, \quad N_2 \gtrsim N_1^{\eta}, \quad \text{and} \quad N_2 \gtrsim N_3 .$$

Furthermore, let $v_{\leq N}, w_{\leq N}, Y_{\leq N} : J \times \mathbb{T}^3 \longrightarrow \mathbb{R}$. Then,

$$\max_{j=1,2} \| \rho_{N_1} P_{N_2} \mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}] P_{N_3} w_{\leq N} \|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}$$
$$\lesssim A\, T^{\alpha} \max(N_1, N_2)^{-\varepsilon} \left(1 + \| v_{\leq N} \|^2_{X^{-\frac{1}{2}-\varepsilon, b}(J)} + \| Y_{\leq N} \|_{X^{\frac{1}{2}+\delta_2, b}(J)} \right) \| w_{\leq N} \|_{X^{-\varepsilon, b}(J)}$$

Proof: As before, we only consider $T = 1$ and $J = [-1, 1]$.

We first decompose

$$\rho_{N_1} P_{N_2} \mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}] P_{N_3} w_{\leq N}$$
$$= \sum_{N_0, N_{12}} P_{N_0} \left[P_{N_{12}} \left(\rho_{N_1} P_{N_2} \mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}] \right) P_{N_3} w_{\leq N} \right]$$

By Proposition 10.6, we have

$$\Big\| P_{N_0}\Big[P_{N_{12}}\big(q_{N_1}, P_{N_2}\mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}]\big) P_{N_3} w_{\leq N}\Big]\Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}}$$

$$\lesssim \Big\| P_{N_0}\Big[P_{N_{12}}\big(q_{N_1}, P_{N_2}\mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}]\big) P_{N_3} w_{\leq N}\Big]\Big\|_{L^2_t H^{-\frac{1}{2}+\delta_2}_x}$$

$$\lesssim N_0^{-\frac{1}{2}+\delta_2} \big\| P_{N_{12}}\big(q_{N_1}, P_{N_2}\mathbb{X}^{(j)}_{\leq N}[v_{\leq N}, Y_{\leq N}]\big)\big\|_{L^\infty_t L^\infty_x} \| P_{N_3} w_{\leq N}\|_{L^2_t L^2_x}$$

$$\lesssim N_0^{-\frac{1}{2}+\delta_2} N_{12}^{\frac{1}{2}+\varepsilon} N_2^{-\frac{1}{2}+10\eta} N_3^{\varepsilon} \Big(1 + \|v_{\leq N}\|^2_{X^{-1,b}(J)} + \|Y_{\leq N}\|_{X^{\frac{1}{2}+\delta_2, b}(J)}\Big) \|w_{\leq N}\|_{X^{-\varepsilon, b}(J)}$$

If $N_1 \gg N_2$, then $N_0 \sim N_{12} \sim N_1$ (recall $N_3 \leq N_2 \ll N_1$).

Thus, since $N_2 \geq N_1^{\eta}$

$$N_0^{-\frac{1}{2}+\delta_2} N_{12}^{\frac{1}{2}+\varepsilon} N_2^{-\frac{1}{2}+10\eta} N_3^{\varepsilon} \lesssim N_1^{\varepsilon+\delta_2} N_2^{-\frac{1}{2}+10\eta+\varepsilon} \lesssim N_1^{-\varepsilon} \quad \checkmark \quad 0<\varepsilon \ll \delta_2 \ll \eta \ll 1$$

If $N_1 \lesssim N_2$, we write

$$N_0^{-\frac{1}{2}+\delta_2} N_{12}^{\frac{1}{2}+\varepsilon} N_2^{-\frac{1}{2}+10\eta} N_3^{\varepsilon} \leq N_0^{-\frac{1}{2}+\delta_2} \Big(\frac{N_{12}}{N_2}\Big)^{\frac{1}{2}+\varepsilon} N_2^{2\varepsilon+10\eta}$$

If either $N_0 \geq N_2^{\frac{1}{100}}$ or $N_{12} \leq N_2^{1-\frac{1}{100}}$, then above acceptable

If not, we have

$$N_2 \gtrsim N_1, N_3, \quad N_0 \leq N_2^{\frac{1}{100}}, \quad \text{and} \quad N_{12} \geq N_2^{1-\frac{1}{100}}.$$

This regime follows directly from Lemma 8.5, Lemma 10,4, Lemma 10.5. $\square$

Reading session 19 : Paracontrolled calculus with the quadratic stochastic object

- Paracontrolled calculus (Section 10)
  - Interactions with the quadratic stochastic object

We want to estimate the followings :

$$\overset{\circ\,\circ}{\vee}_{\leq N}\, \mathbb{X}^{(1)}_{\leq N} - \left(2\,\Pi^{hi,lo,lo}_{\leq N} + \Pi^{hi,hi,lo}_{\leq N}\right)\left(\overset{\circ}{|}_{\leq N},\, \overset{\circ}{|}_{\leq N},\, \mathbb{X}^{(1)}_{\leq N}\right),$$

$$3\,\overset{\circ\,\circ}{\vee}_{\leq N}\, \mathbb{X}^{(2)}_{\leq N} - \left(6\,\Pi^{hi,lo,lo}_{\leq N} + 3\,\Pi^{hi,hi,lo}_{\leq N}\right)\left(\overset{\circ}{|}_{\leq N},\, \overset{\circ}{|}_{\leq N},\, \mathbb{X}^{(2)}_{\leq N}\right) + \Gamma_{\leq N}\left(3\,\overset{\circ\circ\circ}{\Psi}_{\leq N} + v_{\leq N}\right)$$

Lemma 10.11 (Estimate of $\overset{\circ\,\circ}{\vee}_{\leq N}\, \mathbb{X}^{(1)}_{\leq N}$-terms)

Let $T \geq 1$, $p \geqslant 2$, and $0 \in J \subseteq [-T, T]$ be any closed interval.

(i) For all frequency-scales $N_0, \cdots, N_5, N_{234}$ satisfying

$$N_{234} > \max(N_1, N_5)^{\eta} \quad \text{and} \quad N_3, N_4 \leq N_2^{\eta},$$

we have

$$\mathbb{E}\left[\sup_J \left\| P_{N_0}\left[:\overset{\circ}{|}_{N_1} \overset{\circ}{|}_{N_5} : \mathbb{X}^{(1),hi,lo,lo}[N_*, w_3, w_4]\right]\right\|^p_{X^{-1,b}(J)\times X^{-1,b}(J)\to X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}\right]^{1/p} \lesssim p^{3/2}\, T^{\alpha}\, N_{max}^{-\varepsilon}$$

(ii) For all frequency-scales $N_0, \ldots, N_5, N_{34}, N_{234}$ satisfying

$$N_{234} > \max(N_1, N_5)^{\eta}, \quad N_4 > N_2^{\eta}, \quad \text{and} \quad N_{34} \leq N_2^{\eta},$$

we have

$$\mathbb{E}\left[\sup_J \left\| P_{N_0}\left[:\overset{\circ}{|}_{N_1} \overset{\circ}{|}_{N_5} : \mathbb{X}^{(1),res}[N_*, w_3, w_4]\right]\right\|^p_{X^{-\frac{1}{2}-\varepsilon,b}(J)\times X^{\frac{1}{2}+\delta_2,b}(J)\to X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}\right]^{1/p} \lesssim p^{3/2}\, T^{\alpha}\, N_{max}^{-\varepsilon}$$

(iii) For all frequency-scales K_0, K_1, K_2, K_3 satisfying $K_3 > \max(K_1, K_2)^{\eta}$,

$$\mathbb{E}\left[\sup_N \left\| P_{K_0}\left[:\overset{\circ}{|}_{K_1} \overset{\circ}{|}_{K_2} : P_{K_3}\, \mathbb{X}^{(1),expl}_{\leq N}\right]\right\|^p_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)}\right]^{1/p} \lesssim p^{3/2}\, T^{\alpha}\, K_{max}^{-\varepsilon}$$

Proof: As before, we only consider $T=1$ and $J = [-1, 1]$.

(i) By the definition of the Sine-kernel in Definition 5.13 (note 12),

$$P_{N_0}\left[:\overset{\circ}{|}_{N_1} \overset{\circ}{|}_{N_5} : \mathbb{X}^{(1),hi,lo,lo}[N_*, w_3, w_4]\right]$$

$$= \sum_{\substack{n_0,\ldots,n_5 \in \mathbb{Z}^3 \\ n_0 = n_{12345}}} \Bigg[\mathbb{1}_{N_{234}}(n_{234}) \left(\prod_{j=0}^{5} \mathbb{1}_{N_j}(n_j)\right) e^{i\langle n_0, x\rangle} :\overset{\circ}{|}_{N_1}(t, n_1)\, \overset{\circ}{|}_{N_5}(t, n_5):$$

$$\times \int_0^t \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle}\, \overset{\circ}{|}_{N_2}(t', n_2)\, \widehat{w_3}(t', n_3)\, \widehat{w_4}(t', n_4)\, dt' \Bigg]$$

$$= \sum_{\substack{\varphi_1,\varphi_2,\varphi_5 \in \\ \{\cos,\sin\}}} \sum_{\substack{n_0,\dots,n_5 \in \mathbb{Z}^3 \\ n_0 = n_{12345}}} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=0}^{5} \mathbb{1}_{N_j}(n_j) \Big) \langle n_{234}\rangle^{-1} \langle n_1\rangle^{-1} \langle n_2\rangle^{-1} \langle n_5\rangle^{-1} e^{i\langle n_0, x\rangle} \varphi_1(t\langle n_1\rangle)\, \varphi_5(t\langle n_5\rangle)$$

$$\times \int_0^t \sin((t-t')\langle n_{234}\rangle)\, \varphi_2(t'\langle n_2\rangle)\, \widehat{w_3}(t', n_3)\, \widehat{w_4}(t', n_4)\, dt' \cdot \prod_{j=1,2,5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \Big] \qquad (10.28)$$

$$+ \mathbb{1}_{\{N_1 = N_2\}} \sum_{\varphi_5 \in \{\cos,\sin\}} \sum_{\substack{n_0,n_3,n_4,n_5 \in \mathbb{Z}^3 \\ n_0 = n_{345}}} \Big[\Big(\prod_{j=0,3,4,5} \mathbb{1}_{N_j}(n_j) \Big) \langle n_5\rangle^{-1} e^{i\langle n_0, x\rangle} \varphi_5(t\langle n_5\rangle)$$

$$\times \int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34})\, \widehat{w_3}(t', n_3)\, \widehat{w_4}(t', n_4)\, dt' \cdot \int_0^1 1\, dW_{s_5}^{\varphi_5}(n_5) \Big] \qquad (10.29)$$

$$+ \mathbb{1}_{\{N_2 = N_5\}} \sum_{\varphi_1 \in \{\cos,\sin\}} \sum_{\substack{n_0,n_1,n_3,n_4 \in \mathbb{Z}^3 \\ n_0 = n_{134}}} \Big[\Big(\prod_{j=0,1,3,4} \mathbb{1}_{N_j}(n_j) \Big) \langle n_1\rangle^{-1} e^{i\langle n_0, x\rangle} \varphi_1(t\langle n_1\rangle)$$

$$\times \int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34})\, \widehat{w_3}(t', n_3)\, \widehat{w_4}(t', n_4)\, dt' \cdot \int_0^1 1\, dW_{s_1}^{\varphi_1}(n_1) \Big] \qquad (10.30)$$

By symmetry in n_1 and n_5, it suffices to treat (10.28) and (10.29)

For the non-resonant part (10.28) :

We use the quintic tensor from Lemma 5.11 (note 11) :

$$h_{n_0 n_1 \cdots n_5}(t, \lambda_3, \lambda_4) = h_{n_0 n_1 \cdots n_5}[N_0, \dots, N_5, N_{234}, \pm_1, \dots, \pm_5](t, \lambda_3, \lambda_4)$$

where we set $\lambda_1 = \lambda_2 = \lambda_5 = 0$.

Thus, we can write

$$(10.28) = \sum_{\pm_j} \int_{\mathbb{R}^2} \sum_{n_0,\dots,n_5} \Big[e^{i\langle n_0, x\rangle} h_{n_0 n_1 \dots n_5}(t, \lambda_3, \lambda_4)$$

$$\times \widetilde{\langle\nabla\rangle w_3^{\pm_3}}(n_3, \lambda_3)\ \widetilde{\langle\nabla\rangle w_4^{\pm_4}}(n_4, \lambda_4) \cdot \prod_{j=1,2,5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \Big] d\lambda_3\, d\lambda_4$$

Since $N_3, N_4 \le N_2^{\eta}$, the $\langle\nabla\rangle$-multipliers are essentially irrelevant

Using the reduction argument in Subsection 5.7, we have

$$\| (10.28) \|_{X^{-1,b} \times X^{-1,b} \to X^{-\frac{1}{2}+\delta_2, b_+ - 1}}$$

$$\lesssim N_0^{-\frac{1}{2}+\delta_2} N_3^2 N_4^2 \max_{\pm_j} \sup_{\lambda_3,\lambda_4 \in \mathbb{R}} \Big[\langle\lambda_3\rangle^{-(b_- - \frac{1}{2})} \langle\lambda_4\rangle^{-(b_- - \frac{1}{2})} \qquad (10.33)$$

$$\times \Big\| \langle\lambda\rangle^{b_+ - 1} \Big\| \sum_{n_1,n_2,n_5} \tilde{h}_{n_0 n_1 \dots n_5}(\lambda, \lambda_3, \lambda_4) \cdot \prod_{j=1,2,5} \int_0^1 1\, dW_{s_j}^{\varphi_j}(n_j) \Big\|_{n_3 n_4 \to n_0} \Big\|_{L^2_\lambda} \Big]$$

We bound the $\|\cdot\|_{n_3 n_4 \to n_0}$-norm by the Hilbert-Schmidt norm $\|\cdot\|_{n_0 n_3 n_4}$

Using the p-moment estimate reduction in Subsection 5.7

and Lemma 5.11 (5.52) (note 11) ,

$$\mathbb{E}\big[(10.33)^p \big]^{1/p} \lesssim p^{\frac{3}{2}} N_{\max}^{\varepsilon} N_0^{-\frac{1}{2}+\delta_2} N_3^2 N_4^2$$

$$\times \min(N_0, N_1, N_2, N_5)^{\frac{1}{2}} \max(N_0, N_1, N_2, N_5)^{-\frac{1}{2}} N_0 N_2^{-1} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}$$

$$\lesssim p^{\frac{3}{2}} N_{\max}^{\varepsilon+\delta_2} N_2^{-\frac{1}{2}} \underbrace{N_3^{\frac{5}{2}} N_4^{\frac{5}{2}}}_{\le N_2^{5\eta}} \qquad N_2 \sim N_{234} > \max(N_1, N_5)^{\eta} \qquad \checkmark \qquad 0 < \varepsilon \ll \delta_2 \ll \eta \ll 1$$

For the resonant part (10.29) :

We use the sine-cancellation tensor from Lemma 5.18 (note 13) :

$$h^{\text{sine}}_{n_0 n_3 n_4 n_5}(t, \lambda_3, \lambda_4) = h^{\text{sine}}_{n_0 n_3 n_4 n_5}[N_0, N_1, \dots, N_5, N_{234}](t, \lambda_3, \lambda_4)$$

where we set $\lambda_5 = 0$.

Thus, we can write

$$(10.29) = \sum_{\pm j} \int_{\mathbb{R}^2} \sum_{n_0, n_3, n_4, n_5} \Big[e^{i\langle n_0, x\rangle} h^{\text{sine}}_{n_0 n_3 n_4 n_5}(t, \lambda_3, \lambda_4) \\ \times \widetilde{\langle \nabla \rangle w_3^{\pm_3}}(n_3, \lambda_3) \; \widetilde{\langle \nabla \rangle w_4^{\pm_4}}(n_4, \lambda_4) \cdot \int_0^1 1 \, dW^{\varphi_5}_{s_5}(n_5) \Big] d\lambda_3 \, d\lambda_4 \qquad (10.35)$$

Since $N_3, N_4 \leq N_2^\eta$, the $\langle \nabla \rangle$-multipliers are essentially irrelevant

Using the reduction argument in Subsection 5.7, we have

$$\| (10.29) \|_{X^{-1,b} \times X^{-1,b} \to X^{-\frac{1}{2}+\delta_2, b_+-1}} \\ \lesssim N_0^{-\frac{1}{2}+\delta_2} N_3^2 N_4^2 \max_{\pm j} \sup_{\lambda_3, \lambda_4 \in \mathbb{R}} \Big[\langle \lambda_3 \rangle^{-(b_- - \frac{1}{2})} \langle \lambda_4 \rangle^{-(b_- - \frac{1}{2})} \\ \times \Big\| \langle \lambda \rangle^{b_+-1} \Big\| \sum_{n_5} \tilde{h}^{\text{sine}}_{n_0 n_3 n_4 n_5}(\lambda, \lambda_3, \lambda_4) \cdot \int_0^1 1 \, dW^{\varphi_5}_{s_5}(n_5) \Big\|_{n_3 n_4 \to n_0} \Big\|_{L^2_\lambda} \Big] \qquad (10.36)$$

We bound the $\| \cdot \|_{n_3 n_4 \to n_0}$-norm by the Hilbert-Schmidt norm $\| \cdot \|_{n_0 n_3 n_4}$

Using the p-moment estimate reduction in Subsection 5.7

and Lemma 5.18 (5.89) (note 13) ,

$$\mathbb{E}\big[(10.36)^p \big]^{1/p} \lesssim p^{\frac{1}{2}} N_{\max}^{\varepsilon} N_0^{-\frac{1}{2}+\delta_2} N_3^2 N_4^2 \cdot \min(N_0, N_5)^{\frac{1}{2}} N_2^{-1} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}$$

$$\lesssim p^{\frac{1}{2}} N_{\max}^{\varepsilon+\delta_2} N_2^{-1} \underbrace{N_3^{\frac{5}{2}} N_4^{\frac{5}{2}}}_{\leq N_2^{5\eta}} \qquad N_2 \sim N_{234} > \max(N_1, N_5)^\eta \qquad \checkmark \quad 0 < \varepsilon \ll \delta_2 \ll \eta \ll 1$$

(ii) Orthogonality argument : (see note 17)

- localize w_3 and w_4 to balls of radius N_{34}
- repeat part (i)
- sum up balls using the Cauchy-Schwarz inequality

(iii) Consider following cases for $\mathbb{X}^{(1),\mathrm{expl}}_{\le N}$

① $\mathcal{C}_{\le N}\, \rho_{\le N}$:

Lemma 6.23 (note 6) $\Rightarrow \|\chi(t)\mathcal{C}_{\le N}(t)\|_{H_t^{\frac12+}} \lesssim_\chi 1$ $\quad \chi \equiv 1$ on J

$\Rightarrow$ similar (and easier) to part (i)

② $\displaystyle\sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N\\ \max(N_2,N_3)\le N_1^\eta}} P_{N_0}\left(\rho_{N_1}\mathcal{C}^{(1,5)}_{\le N}[N_2,N_3](t)\right)$ and $\displaystyle\sum_{\substack{N_0,N_1,N_2,N_3:\\ N_{max}\le N\\ \max(N_2,N_3)\le N_1^\eta}} P_{N_0}\left(\rho_{N_1}\mathcal{C}^{(3,3)}_{\le N}[N_2,N_3](t)\right)$;

Lemma 7.17 (note 14) $\Rightarrow$ $\|\chi^2(\frac{t}{T})\mathcal{C}^{(1,5)}_{\le N}[N_2,N_3](t)\|_{H_t^{\frac12+}} + \|\chi^2(\frac{t}{T})\mathcal{C}^{(3,3)}_{\le N}[N_2,N_3](t)\|_{H_t^{\frac12+}}$

$$\lesssim \max(N_2,N_3)^2\, T^\alpha \le N_1^{2\eta}\, T^\alpha$$

$\Rightarrow$ similar (and easier) to part (i)

③ $(\gamma_{\le N} - \Gamma_{\le N})\,\rho_{\le N}$:

We set $N_j := K_j$, $j = 0,1,2,3$

We decompose

$$P_{N_0}\left[:\rho_{N_1}\rho_{N_2}:\ (\gamma_{\le N} - \Gamma_{\le N})\, I[\rho_{N_3}]\right]$$

$$= \sum_{\substack{\varphi_1,\varphi_2,\varphi_3\in\\ \{\cos,\sin\}}}\ \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3\\ n_0=n_{123}}} \Big[\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-2}\left(\gamma_{\le N} - \Gamma_{\le N}(n_3)\right)e^{i\langle n_0,x\rangle}$$
$$\times\ \varphi_1(t\langle n_1\rangle)\,\varphi_2(t\langle n_2\rangle)\int_0^t \sin((t-t')\langle n_3\rangle)\,\varphi_3(t'\langle n_3\rangle)\,dt' \cdot \prod_{j=1}^{3}\int_0^1 1\, dW^{\varphi_j}_{s_j}(n_j)\Big] \qquad (10.37)$$

$$+\ \mathbb{1}_{\{N_2=N_3\}}\sum_{\varphi_1\in\{\cos,\sin\}}\ \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3\\ n_0=n_1}}\Big[\mathbb{1}_{\{n_{23}=0\}}\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-2}$$
$$\times\ \left(\gamma_{\le N} - \Gamma_{\le N}(n_3)\right)e^{i\langle n_0,x\rangle}\varphi_1(t\langle n_1\rangle)\int_0^t \sin((t-t')\langle n_3\rangle)\cos((t-t')\langle n_3\rangle)\,dt' \cdot \int_0^1 1\, dW^{\varphi_1}_{s_1}(n_1)\Big] \qquad (10.38)$$

$$+\ \mathbb{1}_{\{N_1=N_3\}}\sum_{\varphi_2\in\{\cos,\sin\}}\ \sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^3\\ n_0=n_2}}\Big[\mathbb{1}_{\{n_{13}=0\}}\Big(\prod_{j=0}^{3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-2}$$
$$\times\ \left(\gamma_{\le N} - \Gamma_{\le N}(n_3)\right)e^{i\langle n_0,x\rangle}\varphi_2(t\langle n_2\rangle)\int_0^t \sin((t-t')\langle n_3\rangle)\cos((t-t')\langle n_3\rangle)\,dt' \cdot \int_0^1 1\, dW^{\varphi_2}_{s_2}(n_2)\Big] \qquad (10.39)$$

By symmetry in n_1 and n_2, it suffices to consider (10.37) and (10.38)

Using Lemma 7.1 (note 9), the non-resonant part (10.37) follows from the same way as part (i)

For the resonant part (10.38), we have

$$(10.38) = -\frac14\cdot\mathbb{1}_{\{N_0=N_1\}}\mathbb{1}_{\{N_2=N_3\}}\sum_{\varphi_1\in\{\cos,\sin\}}\ \sum_{n_1,n_3\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1,3}\mathbb{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_3\rangle^{-4}$$
$$\times\ \left(\gamma_{\le N} - \Gamma_{\le N}(n_3)\right)e^{i\langle n_0,x\rangle}\varphi_1(t\langle n_1\rangle)\left(\cos(2t\langle n_3\rangle) - 1\right)\cdot\int_0^1 1\, dW^{\varphi_1}_{s_1}(n_1)\Big]$$

By Gaussian hypercontractivity and Lemma 7.1 (note 9), we have

$$\mathbb{E}\Big[\big\| (10.38) \big\|^{p}_{X^{-\frac{1}{2}+\delta_2, b_+-1}} \Big]^{2/p}$$

$$\lesssim \mathbb{1}_{\{N_0=N_1\}} \mathbb{1}_{\{N_2=N_3\}} \sum_{n_1\in\mathbb{Z}^3} \mathbb{1}_{N_1}(n_1) \langle n_1\rangle^{-3+2\delta_2} \Big(\sum_{n_3\in\mathbb{Z}^2} \mathbb{1}_{N_3}(n_3) \langle n_3\rangle^{-4} \big| \gamma_{\leq N} - \Gamma_{\leq N}(n_3) \big| \Big)^2$$

$$\lesssim \mathbb{1}_{\{N_0=N_1\}} \mathbb{1}_{\{N_2=N_3\}} N_1^{2\delta_2} N_3^{-2+\varepsilon} \qquad N_3 > \max(N_1,N_2)^\eta \quad \checkmark \quad 0<\delta_2 \ll \eta \ll 1 \quad \square$$

To treat the term involving $\vee$ and $\mathbb{X}^{(2)}$, it is convenient to make the following definition.

Definition 10.12 (Frequency-localized operator version of $\Gamma_{\leq N}$)

For all frequency-scales K_0, K_1, K_2, K_3 and $w : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we define

$$\Gamma^{op}[K_*](w) := 18 \sum_{\substack{k_0,k_1,k_2,k_3\in\mathbb{Z}^3\\ k_0=k_{123}}} \Big[\Big(\prod_{j=0}^{3} \mathbb{1}_{K_j}(k_j) \Big) e^{i\langle k_0, x\rangle} \int_0^t \frac{\sin((t-t')\langle k_3\rangle)}{\langle k_3\rangle} \Big(\prod_{j=1}^{2} \frac{\cos((t-t')\langle k_j\rangle)}{\langle k_j\rangle^2} \Big) \widehat{w}(t',k_0)\, dt' \Big]$$

Lemma 10.14 (Decomposition of the $\vee_{\leq N} \mathbb{X}^{(2)}_{\leq N}$- term)

For all $N \geq 1$, we have

$$3 \vee_{\leq N} \mathbb{X}^{(2)}_{\leq N} - \big(6\, \Pi^{hi,lo,lo}_{\leq N} + 3\, \Pi^{hi,hi,lo}_{\leq N} \big) \big(\varphi_{\leq N}, \varphi_{\leq N}, \mathbb{X}^{(2)}_{\leq N} \big) + \Gamma_{\leq N} \big(3 \Psi_{\leq N} + v_{\leq N} \big)$$

$$= \sum_{w_4} \sum_{\substack{N_0,\dots,N_5,N_{234}\in N\\ N_{234}>\max(N_1,N_5)^\eta \\ N_4 \in \max(N_2,N_3)^\eta < \min(N_2,N_3)}} P_{N_0}\Big[3 : \varphi_{N_1} \varphi_{N_5} : \mathbb{X}^{(2),op}[N_*, w_4] + \mathbb{1}_{\{N_0=N_4,\, N_1=N_2,\, N_3=N_5\}} \Gamma^{op}[N_4,N_2,N_3,N_{234}](w_4) \Big] \tag{10.42}$$

$$+ \sum_{w_4} \Big[\Gamma_{\leq N} w_4 - \sum_{\substack{N_2,N_3,N_4,N_{234}\in N\\ N_{234}>\max(N_2,N_3)^\eta \\ \min(N_2,N_3)>\max(N_2,N_3)^\eta \\ N_4 \leq \max(N_2,N_3)^\eta}} \Gamma^{op}[N_4,N_2,N_3,N_{234}](w_4) \Big], \tag{10.43}$$

where the sum in w_4 is taken over $3\Psi_{\leq N}$ and $v_{\leq N}$

Proof: It follows directly from Lemma 10.3 (note 17) and the definitions of $\Pi^{hi,lo,lo}_{\leq N}$ and $\Pi^{hi,hi,lo}_{\leq N}$ in Definition 3.14 :

$$\{\max(N_5,N_{234}) \leq N_1^\eta\} \cup \{\max(N_1,N_{234}) \leq N_5^\eta\} \cup \{\min(N_1,N_5) > \max(N_1,N_5)^\eta \geq N_{234}\}$$

$$= \underbrace{\{N_1, N_{234} \leq N_5^\eta\} \cup \{N_1 > N_5^\eta \geq N_{234}\}}_{N_1 \in N_5} \cup \underbrace{\{N_5, N_{234} \leq N_1^\eta\} \cup \{N_5 > N_1^\eta \geq N_{234}\}}_{N_1 > N_5}$$

$$= \{N_1 \leq N_5,\ N_{234} \leq N_5^\eta\} \cup \{N_1 > N_5,\ N_{234} \leq N_1^\eta\}$$

$$= \{N_{234} \leq \max(N_1,N_5)^\eta\} \qquad \square$$

We now separately treat (10.42) and (10.43)

Lemma 10.15 (Estimate of the zero and one-pairing parts of $\mathcal{V}_{\leq N} \mathbb{X}^{(2)}_{\leq N}$)

For all $T \geq 1$ and $p \geq 2$, we have ($0 \in J \subseteq [-T, T]$ closed interval)

$$\mathbb{E}\Bigg[\sup_N \sup_J \Big\| w_4 \mapsto \sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} P_{N_0}\big[3 : \ell_{N_1} \ell_{N_5} : \mathbb{X}^{(2), op}[N_*, w_4]$$

$$+ \mathbb{1}\{N_0 = N_4, N_1 = N_2, N_3 = N_5\} \Gamma^{op}[N_4, N_2, N_3, N_{234}](w_4)\big] \Big\|^p_{X^{-1,b}(J) \to X^{-\frac{1}{2}+\delta_2, b_+ - 1}(J)}\Bigg]^{1/p} \lesssim p^2 T^\alpha$$

Proof: As before, we only consider $T = 1$ and $J = [-1, 1]$.

By the definition of the Sine-kernel in Definition 5.13 (note 12),

$$\sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} P_{N_0}\big[3 : \ell_{N_1} \ell_{N_5} : \mathbb{X}^{(2), op}[N_*, w_4]\big]$$

$$= -9 \sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} P_{N_0}\Big[: \ell_{N_1} \ell_{N_5} : P_{N_{234}}\big[: \ell_{N_2} \ell_{N_3} : P_{N_4} w_4 \big]\Big]$$

$$= -9 \sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} \sum_{\substack{\varphi_1, \varphi_2, \varphi_3, \varphi_5 \\ \in \{\cos, \sin\}}} \sum_{\substack{n_0, n_1, \dots, n_5 \in \mathbb{Z}^3 \\ n_0 = n_{12345}}} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=0}^{5} \mathbb{1}_{N_j}(n_j) \Big) \langle n_{234} \rangle^{-1} \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \langle n_3 \rangle^{-1} \langle n_5 \rangle^{-1}$$

$$\times e^{i\langle n_0, x \rangle} \varphi_1(t\langle n_1 \rangle) \varphi_5(t\langle n_5 \rangle) \int_0^t \sin((t-t')\langle n_{234} \rangle) \varphi_2(t'\langle n_2 \rangle) \varphi_3(t'\langle n_3 \rangle) \widehat{w}_4(t', n_4) \, dt' \cdot \prod_{j=1,2,3,5} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) \Big] \quad (10.44)$$

$$-36 \sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} \sum_{\substack{\varphi_3, \varphi_5 \in \\ \{\cos, \sin\}}} \sum_{\substack{n_0, n_3, n_4, n_5 \in \mathbb{Z}^3 \\ n_0 = n_{345}}} \Big[\mathbb{1}\{N_1 = N_2\} \mathbb{1}_{N_0}(n_0) \frac{\mathbb{1}_{N_3}(n_3)}{\langle n_3 \rangle} \mathbb{1}_{N_4}(n_4) \frac{\mathbb{1}_{N_5}(n_5)}{\langle n_5 \rangle}$$

$$\times e^{i\langle n_0, x \rangle} \varphi_5(t\langle n_5 \rangle) \int_0^t \mathrm{Sine}[N_{234}, N_2](t - t', n_{34}) \varphi_3(t'\langle n_3 \rangle) \widehat{w}_4(t', n_4) \, dt' \cdot \prod_{j=3,5} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) \Big] \quad (10.45)$$

$$-18 \sum_{\substack{N_0, \dots, N_5, N_{234} \in N \\ N_{234} > \max(N_1, N_5)^\eta \\ N_4 \in \max(N_2, N_3)^\eta < \min(N_2, N_3)}} \sum_{\substack{n_0, n_1, \dots, n_5 \in \mathbb{Z}^3 \\ n_0 = n_{12345}}} \Big[\mathbb{1}\{n_{12} = n_{35} = 0\} \mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=0}^{5} \mathbb{1}_{N_j}(n_j) \Big) e^{i\langle n_0, x \rangle}$$

$$\times \int_0^t \frac{\sin((t-t')\langle n_{234} \rangle)}{\langle n_{234} \rangle} \frac{\cos((t-t')\langle n_2 \rangle)}{\langle n_2 \rangle^2} \frac{\cos((t-t')\langle n_3 \rangle)}{\langle n_3 \rangle^2} \widehat{w}_4(t', n_4) \, dt' \Big] \quad (10.46)$$

In (10.45) and (10.46), we already used symmetry between n_1, n_5 and between n_2, n_3

Since the two-pairing term (10.46) is exactly cancelled by the $\Gamma^{op}[N_4, N_2, N_3, N_{234}](w_4)$ term, we only need to consider (10.44) and (10.45).

For the zero-pairing term (10.44) : (similar to Lm 10.11)

We use the quintic tensor from Lemma 5.11 (note 11) :

$$h_{n_0 n_1 \cdots n_5}(t, \lambda_4) = h_{n_0 n_1 \cdots n_5}[N_0, \dots, N_5, N_{234}, \pm_1, \dots, \pm_5](t, \lambda_4)$$

where we set $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = 0$.

Thus, we can write

$$(10.44) = \sum_{\pm_j} \int_{\mathbb{R}^2} \sum_{n_0, \dots, n_5} \Big[e^{i\langle n_0, x \rangle} h_{n_0 n_1 \dots n_5}(t, \lambda_4) \\ \times \widetilde{\langle \nabla \rangle w_4^{\pm_4}}(n_4, \lambda_4) \cdot \prod_{j=1,2,3,5} \int_0^1 1 \, dW_{s_j}^{\varphi_j}(n_j) \Big] d\lambda_4$$

Since $N_4 \leq \max(N_2, N_3)^\eta$, the $\langle \nabla \rangle$-multipliers are essentially irrelevant

Using the reduction argument in Subsection 5.7, we have

$$\| (10.44) \|_{X^{-1,b} \to X^{-\frac{1}{2}+\delta_2, b_+ - 1}} \\ \lesssim N_0^{-\frac{1}{2}+\delta_2} N_4^2 \max_{\pm_j} \sup_{\lambda_4 \in \mathbb{R}} \Big[\langle \lambda_4 \rangle^{-(b_- - \frac{1}{2})} \qquad (10.49) \\ \times \Big\| \langle \lambda \rangle^{b_+ - 1} \Big\| \sum_{n_1, n_2, n_3, n_5} \tilde{h}_{n_0 n_1 \dots n_5}(\lambda, \lambda_4) \cdot \prod_{j=1,2,3,5} \int_0^1 1 \, dW_{s_j}^{\varphi_j}(n_j) \Big\|_{n_4 \to n_0} \Big\|_{L^2_\lambda} \Big]$$

We bound the $\| \cdot \|_{n_4 \to n_0}$-norm by the Hilbert-Schmidt norm $\| \cdot \|_{n_0 n_4}$

Using the p-moment estimate reduction in Subsection 5.7

and Lemma 5.11 (5.51) (note 11) ,

$$\mathbb{E}\big[(10.49)^p \big]^{1/p} \lesssim p^2 N_{max}^{\varepsilon} N_0^{-\frac{1}{2}+\delta_2} N_4^2 \\ \times N_0 \min(N_2, N_3, N_4)^{\frac{1}{2}} \max(N_2, N_3, N_4)^{-\frac{1}{2}} \max(N_0, N_1, N_5)^{-\frac{1}{2}} \\ \lesssim p^2 N_{max}^{\delta_2+\varepsilon} N_4^{\frac{5}{2}} \max(N_2, N_3, N_4)^{-\frac{1}{2}} \\ \overset{N_4 \leq \max(N_2,N_3)^\eta}{\leq} p^2 N_{max}^{\delta_2+\varepsilon} \max(N_2, N_3)^{-\frac{1}{2}+\frac{5}{2}\eta} \qquad \checkmark \quad 0 < \varepsilon \ll \delta_2 \ll \eta \ll 1 \\ \max(N_2, N_3) \gtrsim N_{234} > \max(N_1, N_5)^\eta$$

For the one-pairing term (10.45) : (similar to Lm 10.11)

We use the sine-cancellation tensor from Lemma 5.18 (note 13) :

$$h^{sine}_{n_0 n_3 n_4 n_5}(t, \lambda_4) = h^{sine}_{n_0 n_3 n_4 n_5}[N_0, N_1, \dots, N_5, N_{234}](t, \lambda_4)$$

where we set $\lambda_3 = \lambda_5 = 0$.

Thus, we can write

$$(10.45) = \sum_{\pm j} \int_{\mathbb{R}^2} \sum_{n_0,n_3,n_4,n_5} \Big[e^{i\langle n_0, x\rangle} h^{\text{sine}}_{n_0 n_3 n_4 n_5}(t, \lambda_4)$$

$$\times \widetilde{\langle \nabla \rangle w_4^{\pm_4}}(n_4, \lambda_4) \cdot \prod_{j=3,5} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) \Big] d\lambda_4$$

Since $N_4 \le \max(N_2, N_3)^\eta$, the $\langle \nabla \rangle$-multipliers are essentially irrelevant

Using the reduction argument in Subsection 5.7, we have

$$\| (10.45) \|_{X^{-1,b} \to X^{-\frac{1}{2}+\delta_2, b_+-1}}$$

$$\lesssim N_0^{-\frac{1}{2}+\delta_2} N_4^2 \max_{\pm j} \sup_{\lambda_4 \in \mathbb{R}} \Big[\langle \lambda_4 \rangle^{-(b_- - \frac{1}{2})}$$

$$\times \Big\| \langle \lambda \rangle^{b_+-1} \Big\| \sum_{n_5} \tilde{h}^{\text{sine}}_{n_0 n_3 n_4 n_5}(\lambda, \lambda_4) \cdot \prod_{j=3,5} \int_0^1 1 \, dW^{\varphi_j}_{s_j}(n_j) \Big\|_{n_4 \to n_0} \Big\|_{L^2_\lambda} \Big] \qquad (10.51)$$

We bound the $\| \cdot \|_{n_4 \to n_0}$-norm by the Hilbert-Schmidt norm $\| \cdot \|_{n_0 n_4}$

Using the p-moment estimate reduction in Subsection 5.7

and Lemma 5.18 (5.89) (note 13),

$$\mathbb{E}\big[(10.51)^p \big]^{1/p} \lesssim p N_{\max}^{\varepsilon} N_0^{-\frac{1}{2}+\delta_2} N_4^2$$

$$\times \min(N_0, N_5)^{\frac{3}{2}} \max(N_2, N_{234})^{-1} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}} N_5^{-1}$$

$$\lesssim p N_{\max}^{\varepsilon+\delta_2} \underbrace{\max(N_2, N_{234})^{-1}}_{\sim \max(N_2, N_3, N_4)^{-1}} N_3^{\frac{1}{2}} N_4^{\frac{5}{2}} \qquad \text{since } N_4 \le \max(N_2, N_3)^\eta$$

$$\lesssim p N_{\max}^{\varepsilon+\delta_2} \max(N_2, N_3, N_4)^{-\frac{1}{2}} N_4^{\frac{5}{2}} \qquad ✓ \quad 0 < \varepsilon \ll \delta_2 \ll \eta \ll 1$$

$$\max(N_2, N_3, N_4) \gtrsim N_{234} > \max(N_1, N_5)^\eta$$

□

Lemma 10.15 (Estimate of the renormalized two-pairing term in $\mathcal{V}_{\le N} \mathbb{X}^{(2)}_{\le N}$)

For all $N \ge 1$, $T \ge 1$, and closed intervals $0 \in J \subseteq [-T, T]$, we have

$$\Big\| \sum_{\substack{N_2, N_3, N_4, N_{234} \le N \\ N_{234} > \max(N_2, N_3)^\eta \\ \min(N_2, N_3) > \max(N_2, N_3)^\eta \\ N_4 \le \max(N_2, N_3)^\eta}} \Gamma^{op}[N_4, N_2, N_3, N_{234}](w_4) - \Gamma_{\le N} w_4 \Big\|_{X^{-\frac{1}{2}+\delta_2, b_+-1}(J)} \lesssim T^\alpha \| w_4 \|_{X^{-\varepsilon, b}}$$

Proof: As before, we only consider $T = 1$ and $J = [-1, 1]$.

From Definition 10.12, we can write

$$\sum_{\substack{N_2,N_1,N_4,N_{234}\in N \\ N_{234}>\max((N_2,N_3)^\eta \\ \min(N_2,N_3)>\max(N_2,N_3)^\eta \\ N_4\leq\max(N_2,N_3)^\eta}} \Gamma^{op}[N_4,N_2,N_3,N_{234}](w_4) - \Gamma_{\leq N} w_4$$

$$= \sum_{N_2,N_3,N_4,N_{234}\in N} \Gamma^{op}[N_4,N_2,N_3,N_{234}](w_4) - \Gamma_{\leq N} w_4 \tag{10.55}$$

$$+ \sum_{N_2,N_3,N_4,N_{234}\in N} \Big[\Big(\mathbb{1}_{\{N_{234}>\max(N_2,N_3)^\eta,\ N_4\leq\max(N_2,N_3)^\eta<\min(N_2,N_3)\}} - 1 \Big) \Gamma^{op}[N_4,N_2,N_3,N_{234}](w_4) \Big] \tag{10.56}$$

The contribution of (10.55) :

By Definition 10.12 and the symmetry in $-n_{234}$, n_2, and n_3,

$$(10.55) = 18 \sum_{\substack{n_2,n_3,n_4,n_{234}\in\mathbb{Z}^3 \\ n_{234}=n_2+n_3+n_4}} \Big[\mathbb{1}_{\leq N}(n_{234}) \Big(\prod_{j=2}^{4} \mathbb{1}_{\leq N}(n_j)\Big) e^{i\langle n_4,x\rangle}$$

$$\times \int_0^t \partial_{t'} \Big(\frac{\cos((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle^2} \Big) \Big(\prod_{j=2}^{3} \frac{\cos((t-t')\langle n_j\rangle)}{\langle n_j\rangle^2} \Big) \widehat{w}_4(t',n_4)\, dt' \Big] - \Gamma_{\leq N} w_4$$

$$= 6 \sum_{\substack{n_2,n_3,n_4,n_{234}\in\mathbb{Z}^3 \\ n_{234}=n_2+n_3+n_4}} \Big[\mathbb{1}_{\leq N}(n_{234}) \Big(\prod_{j=2}^{4} \mathbb{1}_{\leq N}(n_j)\Big) e^{i\langle n_4,x\rangle}$$

$$\times \int_0^t \partial_{t'} \Big(\frac{\cos((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle^2} \Big(\prod_{j=2}^{3} \frac{\cos((t-t')\langle n_j\rangle)}{\langle n_j\rangle^2} \Big) \Big) \widehat{w}_4(t',n_4)\, dt' \Big] - \Gamma_{\leq N} w_4$$

$$\overset{\text{Def 6.11 (note 4)}}{=} \sum_{n_4\in\mathbb{Z}^3} \Big[e^{i\langle n_4,x\rangle} \Big(\int_0^t \partial_{t'} \big(\Gamma_{\leq N}(n_4, t-t') \big) \widehat{w}_4(t',n_4)\, dt' - \Gamma_{\leq N}(n_4)\, \widehat{w}_4(t,n_4) \Big) \Big] \tag{10.57}$$

By integration by parts, we have

$$(10.57) = -\sum_{n_4\in\mathbb{Z}^3} e^{i\langle n_4,x\rangle}\, \Gamma_{\leq N}(n_4,t)\, \widehat{w}_4(0,n_4) \tag{10.58}$$

$$- \sum_{n_4\in\mathbb{Z}^3} e^{i\langle n_4,x\rangle} \int_0^t \Gamma_{\leq N}(n_4,t-t')\, \partial_{t'} \widehat{w}_4(t',n_4)\, dt' \tag{10.59}$$

For (10.58), by (7.5) in Lemma 7.3 (note 9),

$$\|(10.58)\|_{X^{-\frac{1}{2}+\delta_2,b_+-1}(J)} \leq \sum_{N_2,N_3,N_4,N_{234}\leq N} \Big\| \langle\lambda\rangle^{b_+-1} \int_{-1}^{1} \Gamma_{\leq N}[N_*](n_4,t)\, e^{-i\lambda t}\, dt \cdot \langle n_4\rangle^{-\frac{1}{2}+\delta_2} \widehat{w}_4(0,n_4) \Big\|_{\ell^2_{n_4} L^2_\lambda}$$

$$\lesssim \sum_{N_2,N_3,N_4,N_{234}\leq N} \log(N_{max})\, \Big\| \langle\lambda\rangle^{b_+-1} \max(N_{max}, \langle\lambda\rangle)^{-1} \cdot \langle n_4\rangle^{-\frac{1}{2}+\delta_2} \widehat{w}_4(0,n_4) \Big\|_{\ell^2_{n_4} L^2_\lambda}$$

$$\lesssim \sum_{N_2,N_3,N_4,N_{234}\leq N} \log(N_{max})\, N_{max}^{b_+-\frac{3}{2}} \Big\| \langle n_4\rangle^{-\frac{1}{2}+\delta_2} \widehat{w}_4(0,n_4) \Big\|_{\ell^2_{n_4}}$$

$$\lesssim \|w_4\|_{X^{-\varepsilon,b}}$$

For (10.59), as in Subsection 5.7 (note 7), we write

$$w_4(t,x) = \sum_{\pm_4} \sum_{n_4\in\mathbb{Z}^3} \int_{\mathbb{R}} e^{i(\pm_4\langle n_4\rangle+\lambda_4)t} e^{i\langle n_4,x\rangle}\, \widetilde{w}_4^{\pm_4}(\lambda_4,n_4)\, d\lambda_4 ,$$

so that

$$(10.59) = i \sum_{\pm_4} \sum_{n_4 \in \mathbb{Z}^3} \Big[e^{i\langle n_4, x\rangle} \Big(\int_{\mathbb{R}} (\pm_4 \langle n_4\rangle + \lambda_4) \Big(\int_0^t \Gamma_{\leq N}(n_4, t-t') e^{i(\pm_4\langle n_4\rangle + \lambda_4)t} dt' \Big) \widetilde{w}_4^{\pm_4}(\lambda_4, n_4)\, d\lambda_4 \Big) \Big]$$

By (7.5) in Lemma 7.3 (note 9), we have

$$\Big| (\pm_4\langle n_4\rangle + \lambda_4) \Big(\int_0^t \Gamma_{\leq N}(n_4, t-t') e^{i(\pm_4\langle n_4\rangle + \lambda_4)t} dt' \Big) \Big|$$

$$\lesssim \sum_{N_2, N_3, N_4, N_{234} \leq N} \log(N_{max}) \; |\pm_4\langle n_4\rangle + \lambda_4| \; \max\big(N_{max}, \langle \pm_4\langle n_4\rangle + \lambda_4\rangle\big)^{-1}$$

$$\lesssim \sum_{N_4 \leq N} (N_4 + \langle\lambda_4\rangle)^{0+}$$

Thus, by Cauchy-Schwarz in λ_4, we have

$$\| (10.59) \|_{L_t^\infty H_x^{-2\varepsilon}} \lesssim \| w_4 \|_{X^{-\varepsilon, b}} .$$

The contribution of (10.56):

By Definition 10.12, the dyadic components in (10.56) are given by

$$\sum_{n_2, n_3, n_4 \in \mathbb{Z}^3} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=2}^{4} \mathbb{1}_{N_j}(n_j) \Big) e^{i\langle n_4, x\rangle} \times \int_0^t \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \Big(\prod_{j=2}^{3} \frac{\cos((t-t')\langle n_j\rangle)}{\langle n_j\rangle^2} \Big) \widehat{w}_4(t', n_4)\, dt' \Big] \tag{10.60}$$

under at least one of the conditions:

$$\min(N_2, N_3) \leq \max(N_2, N_3)^\eta, \quad N_4 > \max(N_2, N_3)^\eta, \quad \text{or } N_{234} \leq \max(N_2, N_3)^\eta$$

By symmetry, we also assume $N_2 \geq N_3$

Case 1: $\min(N_2, N_3) \leq \max(N_2, N_3)^\eta$ or $N_4 > \max(N_2, N_3)^\eta$

By the definition of the Sine-kernel in Definition 5.13 (note 12),

$$(10.60) = \sum_{n_3, n_4 \in \mathbb{Z}^3} \Big[\langle n_3\rangle^{-2} \mathbb{1}_{N_3}(n_3) \mathbb{1}_{N_4}(n_4) e^{i\langle n_4, x\rangle} \times \int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34}) \cos((t-t')\langle n_3\rangle) \widehat{w}_4(t', n_4)\, dt' \Big]$$

so that

$$\| (10.60) \|^2_{X^{-\frac{1}{2}+\delta_2, b_+-1}} \lesssim \| (10.60) \|^2_{L_t^2 H^{-\frac{1}{2}+\delta_2}}$$

$$\lesssim \sup_{t \in [-1,1]} \sum_{n_4 \in \mathbb{Z}^3} \mathbb{1}_{N_4}(n_4) \langle n_4\rangle^{-1+2\delta_2} \Big| \sum_{n_3 \in \mathbb{Z}^3} \Big[\mathbb{1}_{N_3}(n_3) \langle n_3\rangle^{-2} \times \int_0^t \mathrm{Sine}[N_{234}, N_2](t-t', n_{34}) \cos((t-t')\langle n_3\rangle) \widehat{w}_4(t', n_4)\, dt' \Big] \Big|^2$$

As in Subsection 5.7 (note 7), we write

$$\widehat{w}_4(t, n_4) = \sum_{\pm_4} \int_{\mathbb{R}} e^{i(\pm_4\langle n_4\rangle + \lambda_4)t} \, \widetilde{w}_4^{\pm_4}(\lambda_4, n_4)\, d\lambda_4$$

Thus, by Lemma 5.17, we have

$$\begin{aligned}\|(10.60)\|^2_{X^{-\frac{1}{2}+\delta_2,b_+-1}} &\lesssim \max(N_{234},N_2)^{-2+2\varepsilon} N_3^2 \|P_{N_4} w_4\|^2_{X^{-\frac{1}{2}+\delta_2,b}} \\ &\lesssim \max(N_{234},N_2)^{-2+2\varepsilon} N_3^2 N_4^{-1+2\delta_2+2\varepsilon} \|P_{N_4} w_4\|^2_{X^{-\varepsilon,b}}\end{aligned} \tag{10.63}$$

If $\min(N_2,N_3) \le \max(N_2,N_3)^\eta$, we have $(N_3 \le N_2^\eta)$

$$(10.63) \lesssim N_2^{-2+2\varepsilon+2\eta} N_4^{-1+2\delta_2+2\varepsilon} \|P_{N_4} w_4\|^2_{X^{-\varepsilon,b}} \quad \checkmark$$

If $N_4 > \max(N_2,N_3)^\eta$, we have $(N_3^2 \le N_2^2 \le \max(N_{234},N_2))$

$$\begin{aligned}(10.63) &\lesssim \max(N_2,N_3,N_4)^{2\varepsilon} N_4^{-1+2\delta_2+2\varepsilon} \|P_{N_4} w_4\|^2_{X^{-\varepsilon,b}} \\ &\lesssim \max(N_2,N_3,N_4)^{-\eta+2\delta_2+4\varepsilon} \|P_{N_4} w_4\|^2_{X^{-\varepsilon,b}} \quad \checkmark\end{aligned}$$

Case 2: $N_{234} \le \max(N_2,N_3)^\eta$

We write the dyadic component of (10.56) as

$$\begin{aligned}&\sum_{n_4\in\mathbb{Z}^3} \int_0^t \mathbb{1}_{N_4}(n_4)\, \widehat{w}_4(t',n_4)\, e^{i\langle n_4,x\rangle} \sum_{n_2,n_3\in\mathbb{Z}^3} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=2}^3 \mathbb{1}_{N_j}(n_j') \Big) \\ &\quad \times \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \Big(\prod_{j=2}^3 \frac{\cos((t-t')\langle n_j'\rangle)}{\langle n_j'\rangle^2} \Big) \Big]\, dt'\end{aligned} \tag{10.64}$$

Since $N_{234} \le \max(N_2,N_3)^\eta$ and $N_2 \ge N_3$, we have

$$\begin{aligned}&\Big| \sum_{n_2,n_3\in\mathbb{Z}^3} \Big[\mathbb{1}_{N_{234}}(n_{234}) \Big(\prod_{j=2}^3 \mathbb{1}_{N_j}(n_j') \Big) \frac{\sin((t-t')\langle n_{234}\rangle)}{\langle n_{234}\rangle} \Big(\prod_{j=2}^3 \frac{\cos((t-t')\langle n_j'\rangle)}{\langle n_j'\rangle^2} \Big) \Big] \Big| \\ &\lesssim N_{234}^{-1} N_2^{-2} N_3^{-2} \cdot N_{234}^3 N_3^3 \lesssim N_2^{-1+2\eta}\end{aligned}$$

Thus,

$$\begin{aligned}\|(10.64)\|^2_{X^{-\frac{1}{2}+\delta_2,b_+-1}} &\lesssim \|(10.64)\|^2_{L_t^2 H_x^{-\frac{1}{2}+\delta_2}} \\ &\lesssim N_2^{-2+4\eta} N_4^{-1+2\delta_2+2\varepsilon} \|P_{N_4} w_4\|^2_{L_t^2 H_x^{-\varepsilon}} \\ &\lesssim N_2^{-2+4\eta} N_4^{-1+2\delta_2+2\varepsilon} \|P_{N_4} w_4\|^2_{X^{-\varepsilon,b}} \quad \checkmark\end{aligned}$$

$\square$