Skein Module Dimensions of Twisted Tori

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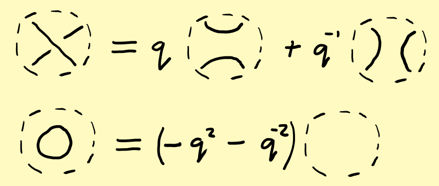
Skein Modules

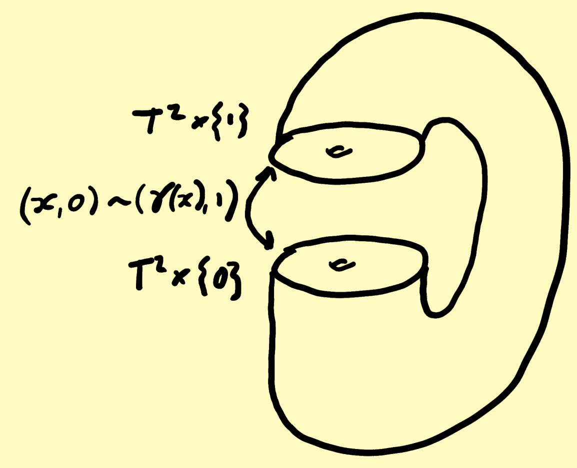
The **skein module** $\mathrm{Sk}_q(M)$ of a 3-manifold M is the $\mathbb{C}(q)$-module spanned by isotopy classes of framed links embedded in M, modulo the skein relations:



It was conjectured by Witten that, for q generic, then $\mathrm{Sk}_q(M)$ should be finite-dimensional. This question was only recently settled by Gunningham-Jordan-Safronov [2], and still the actual dimension of the skein module has only been computed in a few cases.

Twisted Tori

The **twisted 3-torus** M_f, with respect to a diffeomorphism $f : T^2 \to T^2$, is the mapping torus of f. Since we only consider skein modules up to isotopy, it is enough to consider $f = \gamma \in \mathrm{Mod}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$.



Skein Categories

The **skein category** of a surface Σ has

- Objects: unions of marked points on Σ up to isotopy.
- Hom-spaces: $\mathbb{C}(q)$-modules spanned by isotopy classes of framed tangles in $\Sigma \times I$ beginning at one set of points and ending at another, modulo the skein relations.

Given an object x, then $\mathrm{End}_{\mathrm{SkCat}(\Sigma)}(x)$ is an algebra with the product given by stacking.
The **skein algebra** of Σ is the endomorphism algebra of $\emptyset \in \mathrm{SkCat}(\Sigma)$, denoted $\mathrm{SkAlg}_q(\Sigma)$.

Algebraic Setting

We can obtain the skein module of M_γ from the skein category of T^2. We should consider Hom-spaces quotiented by the action of γ, and clearly we need only consider endomorphisms. Therefore, the skein module should be computed as **twisted Hochschild homology:**

$$\mathrm{Sk}_q(M_\gamma) \cong \bigoplus_{x\in\mathrm{SkCat}(T^2)} \mathrm{End}_{\mathrm{SkCat}(T^2)}(x)/(ab-\gamma(b)a) \cong \bigoplus_{x\in\mathrm{SkCat}(T^2)} HH_0^\gamma(\mathrm{End}_{\mathrm{SkCat}(T^2)}(x)).$$

It is clear, from parity considerations, that the skein category breaks into even and odd components. Moreover, in forthcoming work of Gunninham-Jordan-Vazirani [3], it is shown that the skein category is generated by just the empty object $\emptyset$ and the singleton object $\boxdot$. So we have that

$$\mathrm{Sk}_q(M_\gamma) \cong HH_0^\gamma(\mathrm{SkAlg}_q(T^2)) \oplus HH_0^\gamma(\mathrm{End}_{\mathrm{SkCat}(T^2)}(\boxdot)).$$

In our computations, we make use of the following isomorphism of Frohman-Gelca [1]:

$$\mathrm{SkAlg}_q(\Sigma) \cong \mathbb{C}_q[X, Y, X^{-1}, Y^{-1}]^{\mathbb{Z}/2\mathbb{Z}}$$

where the $\mathbb{Z}/2\mathbb{Z}$-action is $X \mapsto X^{-1}, Y \mapsto Y^{-1}$. Then a generating set of $HH_0^\gamma(\mathrm{SkAlg}(T^2))$ is $\{m_{r,s} = X^rY^s + X^{-r}Y^{-s}\}$ for (r,s) in a fundamental domain for the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{Z}^2$. The $\mathrm{SL}_2(\mathbb{Z})$-action is via its action on $\mathbb{Z}^2$.
Moreover, it can be shown that $\mathrm{End}_{\mathrm{SkCat}(T^2)}(\boxdot) \cong \mathbb{C}(q)[X,Y]/(X^2-1, Y^2-1)$, and the $\mathrm{SL}_2(\mathbb{Z})$ action is similar to the above, working modulo 2.

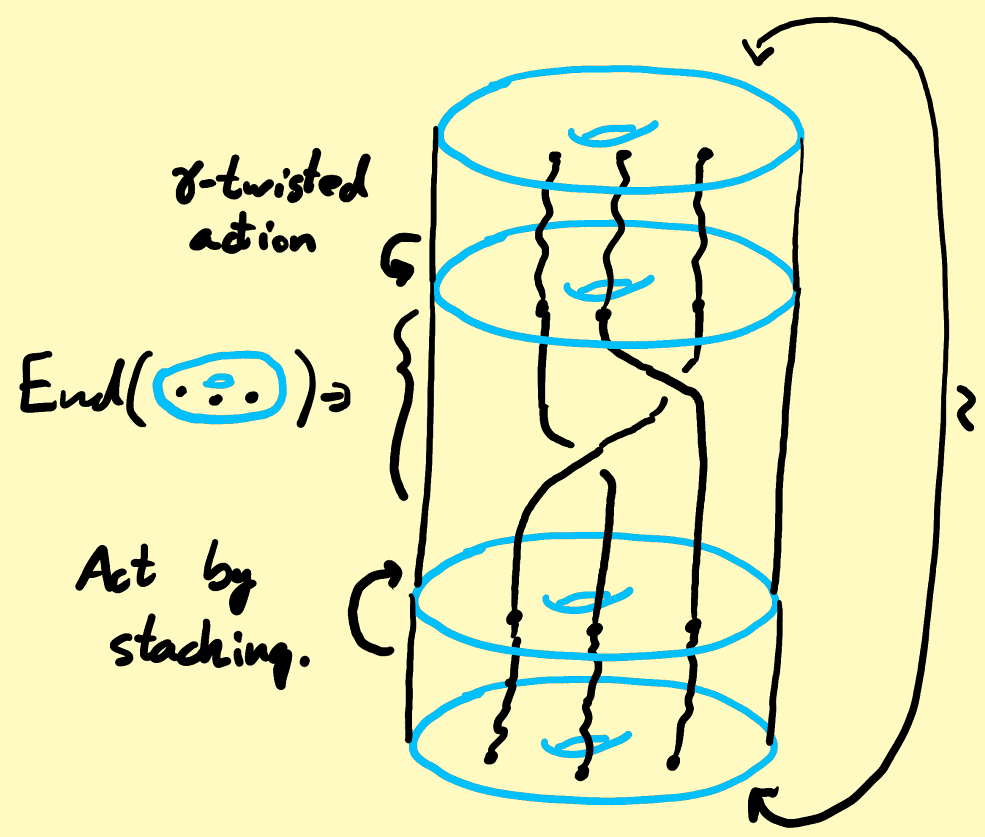


Figure 1: The endomorphism algebra acts on itself by stacking; this can be twisted on one side by γ.

Estimating Dimension by Computer

The computation of $\dim HH_0^\gamma(\mathrm{End}_{\mathrm{SkCat}(T^2)}(\boxdot))$ is straightforward, since this simply is the quotient of a 4-dimensional vector space by 16 relations.
For $HH_0^\gamma(\mathrm{SkAlg}_q(T^2))$, we consider the generators $m_{r,s}$ as lying on a half-lattice. We notice that the relation

$$m_{r,s}m_{t,u} - \gamma(m_{t,u})m_{r,s}$$

relates the generators $m_{(r,s)\pm(t,u)}, m_{(r,s)\pm\gamma(t,u)}$. This allows us to express, say, $m_{(r,s)+\gamma(t,u)}$ in terms of three other generators nearer to the origin: see Figure 2.
Our hope is that there will exist a shell of lattice points around the origin to which, applying such relations, we can always return. This will be a generating set for $HH_0^\gamma(\mathrm{SkAlg}_q(T^2))$, and hence an **upper bound** on the dimension.
We can use a computer to verify this. Taking successively larger shells, we compute the dimension of the space with generators and relations in the shell. If this dimension stabilises as the shell size grows, this reflects that outside of a certain shell, the generators further from the origin are all obtained from those in the shell by the commutator relations. Therefore **we can conjecture the dimension** of $HH_0^\gamma(\mathrm{SkAlg}_q(T^2))$, and hence $\mathrm{Sk}_q(M_\gamma)$.

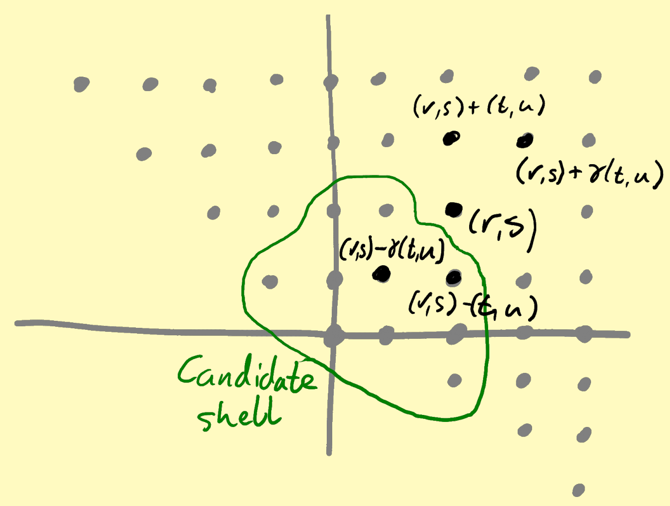


Figure 2: The point $(3,3)$ is brought nearer 0 using the $(2,2),(0,1)$ commutator ($\gamma = T$).

Table of Computations

Matrix	Single	Empty	Total
I	4	5	9
S	2	0	2
T	2	1	3
T^2	4	2	6
T^3	2	3	5
T^4	4	4	8
T^5	2	5	7

Conjectured Dimensions

Our method recovers the known fact that $\dim \mathrm{Sk}_q(M_{\mathrm{Id}}) = 9$. We also conjecture that

- $\dim \mathrm{Sk}_q(M_S) = 2$
- $\dim \mathrm{Sk}_q(M_T) = 3$
- $\dim \mathrm{Sk}_q(M_{T^n}) = 4 - 2(n \mod 2) + n$

Where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ as in the familiar presentation of $\mathrm{SL}_2(\mathbb{Z})$.

References

[1] Frohman, C., and Gelca, R. Skein modules and the noncommutative torus. *Trans. Amer. Math. Soc. 352*, 10 (June 2000), 4877–4888.

[2] Gunningham, S., Jordan, D., and Safronov, P. The finiteness conjecture for skein modules. *arXiv:1908.05233 [math]* (Sept. 2019).

[3] Gunningham, S., Jordan, D., and Vazirani, M. Quantum Springer Theory. *In preparation.*