Symmetries & Generalisations

Say | generalised symmetries and
geodesics, general fields (with T), linear wave eqn $\nabla^\alpha \nabla_\alpha \phi = 0$.

Symmetries on (M^{1+3}, g)

Def'n A smooth, local, one-parameter group of local ~~isometries~~ diffeomorphism is a function ϕ such that $\forall p \in M$:

1) there's a neighbourhood U of p and an $\varepsilon > 0$ such that
$\phi : (-\varepsilon, \varepsilon) \times U \to M$ is smooth,
2) ~~for each~~ $\forall t \in (-\varepsilon, \varepsilon)$, $\phi_t|_U$ is ~~an isometry~~ diffeomorphism of U onto its range
3) for t, s, $t+s$ all in $(-\varepsilon, \varepsilon)$, $\phi_{t+s} = \phi_t \circ \phi_s$.
(If M is geodesically incomplete, might not be a uniform ε.)

Def'n A ~~Killing~~ vector field $k \in \Gamma(M)$ is Killing if
$\nabla_{(i} k_{j)} = 0$ $\qquad \mathrm{sym}(\nabla k) = 0$ □

k k k
k k k k
k k k k

Lemma If ϕ is a SLOPGOLD, $\frac{\partial}{\partial s}\phi$ generates a vector field.
If $v \in \Gamma(M)$, then flow along v generates a SLOPGOLD.
$\frac{\partial}{\partial s}\phi$ generates ϕ.

Lemma k is a Killing field, if the ϕ_s it generates are local isometries.

proof $(\mathcal{L}_k g)_{ij} = \nabla_{(i} k_{j)}$. □

Theorem [Noether]

Let k be a Killing vector on (M^{1+3}, g), then

1) If γ is an (affinely parameterised geodesic), $k_\alpha \dot{\gamma}^\alpha$ is constant (conserved quantity)
2) If Ψ is some field with stress-energy T, then if Σ_1 and Σ_2 are bounded hypersurfaces with the same boundary
$$\int_{\Sigma_1} T_{\alpha\beta} k^\alpha d\eta^\beta = \int_{\Sigma_2} T_{\alpha\beta} k^\alpha d\eta^\beta . \quad \text{(conserved quantity)}$$

3) $\quad [\nabla^\alpha \nabla_\alpha, k^\beta \nabla_\beta] = 0$ (thus $\Box$ and k simultaneously diagonalise), (quantum number)

pf

(1),(3): direct computation. (2) Stoke's thm and $\nabla^\alpha T_{\alpha\beta} = 0$. $\Box$

Example In $\mathbb{R}^{1+3}$, ∂_t, ∂_{x^i} are Killing.

(1) gives $E = g(\partial_t, \dot\gamma)$, $p_i = g(\partial_{x^i}, \dot\gamma)$ are conserved.

(2) $E = \int_{\{t_0\}\times\mathbb{R}^3} T_{\alpha\beta} (\partial_t)^\alpha \, d\eta^\beta$ and $p_i = \int_{\{t_0\}\times\mathbb{R}^3} T_{\alpha\beta} (\partial_{x^i})^\alpha \, d\eta^\beta$ are conserved.

(3) allows Fourier modes $\psi(t,x) = \int e^{\omega t + k_i x^i} \tilde\psi(\omega, k_i) \, d\omega \, dk_i$.

or $\hat\psi(t, k_i) = e^{it|k|} \hat\psi(0, k_i)$. Improve?

Generalisations (Carter, Penrose-Floyd)

Def'n Given (M^{1+3}, g)

A vector field k is conformal Killing if there's a function $f: M^{1+3} \to \mathbb{R}$ such that $\nabla_{(\alpha} k_{\beta)} = f g_{\alpha\beta}$

A tensor field K is g-Stäckel if K is symmetric $(K_{\alpha\beta} = K_{(\alpha\beta)})$ $\nabla_{(\gamma} K_{\alpha\beta)} = 0$.

It is Killing if, in addition, $\nabla_\beta \left(K_\gamma{}^{[\alpha} R^{\beta]\gamma}\right) = \frac{3}{2} K^{\alpha\beta} \frac{?}{?}\left(\frac{1}{4}\frac{n-2}{n-1}\right)\partial_\beta R$

(I only consider the case when vEeqns, so $R = 0$, & equiv.)

A tensor fied Y is (Killing)-Yano if Y is antisymmetric $(Y_{\alpha\beta} = Y_{[\alpha\beta]})$

and $\nabla_{(\gamma} Y_{\alpha)\beta} = 0$.

conformal (Killing-) Yano $\quad Y_{\lambda[k;\sigma]} - Y_{k[\lambda;\sigma]} + \nabla^\delta g_{\sigma[\lambda} Y_{k]\delta} = 0$

Thm [Conformal Noether]

If k is conformal Killing, then

(1) if γ is a null (aff-par) geodesic, $k_\alpha \dot\gamma^\alpha$ is constant.

(2) if Ψ is a field with $T^\alpha{}_\alpha = 0$, then for Σ_1, Σ_2 as befor

$$\int_{\Sigma_1} T_{\alpha\beta} k^\alpha \, d\eta^\beta = \int_{\Sigma_2} T_{\alpha\beta} k^\alpha \, d\eta^\beta .$$

pf. Direct computation $\Box$

Thm ~~Killing~~

If $K_{\mu\nu}$ is a Killing tensor, then

(1) if γ is a (a p) geodesic, $K_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu$ is constant.

~~(2)~~ (insufficient indices on $\Gamma_{\mu\nu}$)

(3) $[\nabla^\alpha\nabla_\alpha, \nabla_\beta K^{\beta\gamma}\nabla_\gamma]=0$ (simultaneously diagonalise)

~~proof~~ Direct computation. □

(This allows many problems to be investigated)

Furthermore ↗

~~**Thm** If $K^{\mu\nu}$ is a Killing tensor with (locally on $U\subseteq M$) full rank, then (U, K) is a semidefinite metric manifold. (with the (2,0) metric given by $K^{\mu\nu}$)~~

Thm Idea: Derivative of metric is zero.

and $H^{\mu\nu}={}^{(1)}g^{\mu\nu}$

Given $(M^{1+3}, {}^{(1)}g)$. If $K^{\mu\nu}$ a ~~Killing~~ Stackel tensor with (locally on $U\subset M$) full rank, then if ${}^{(2)}g$ is defined by ${}^{(2)}g^{\mu\nu}=K^{\mu\nu}$, then

$(U, {}^{(2)}g)$ is a pseudo-Riemannian manifold and

$H^{\mu\nu}={}^{(1)}g^{\mu\nu}$ is a ${}^{(2)}g$ ~~Killing~~ Stackel tensor

$$\cancel{{}^{(2)}\nabla_{(\gamma}\,{}^{(1)}g_{\alpha\beta)}}\quad {}^{(2)}g\;\; {}^{(2)}\nabla_{(\gamma}\,{}^{(2)}H_{\alpha\beta)}=0$$ □

Thm (Even more hidden)

If Y is Yano and $Ric=0$, then the following are Killing:

$K_{\lambda\mu}=Y_\lambda{}^\rho Y_{\rho\mu}$ ~~is Killing,~~

$k^\lambda=\frac{1}{3!}\varepsilon^{\lambda\mu\rho\sigma}Y_{\mu\rho;\sigma}$ ~~is Killing~~

$h^\lambda=K^{\lambda\rho}k_\rho$ □

Kerr

Metric is

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\Theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\Theta^2 + \frac{\sin^2\Theta}{\rho^2}\left(a\,dt - (r^2+a^2)\,d\phi\right)^2$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\rho^2 = r^2 + a^2\cos^2\Theta.$$

(diagonal form)

Clearly

$$k = \frac{\partial}{\partial t}$$

$$h = \frac{\partial}{\partial \phi} \quad \text{are Killing.}$$

$$K = a^2\cos^2\theta\left(+\frac{\Delta}{\rho^2}\left(dt - a\sin^2\Theta\, d\phi\right)^2 - \frac{\rho^2}{\Delta} dr^2\right) + r^2\left(\rho^2 d\Theta^2 + \frac{\sin^2\Theta}{\rho^2}\left(a\,dt - (r^2+a^2)\,d\phi\right)^2\right)$$

Clear what it reduces to in the limit $a \to 0$. ds^2.

$$Y = r\sin\Theta\, d\Theta \wedge \left((r^2+a^2)\,d\phi - a\,dt\right) + a\cos\Theta\, dr \wedge \left(dt - a\sin^2\Theta\, d\phi\right).$$

Geodesics: $\dot\gamma^\alpha \dot\gamma_\alpha$, $\dot\gamma^\alpha k_\alpha$, $\dot\gamma^\alpha h_\alpha$, $\dot\gamma^\alpha \dot\gamma^\beta K_{\alpha\beta}$

completely integrable.

Wave Equation: [Finster, Kamran, Smoller, Yau]

good ω near $\mathbb{R} \subset \mathbb{C}$
discrete spectrum for $\Theta_{\omega,k}$
indexed by n

Separate $e^{i\omega t} e^{ik\phi}\, \Psi(r,\Theta)$

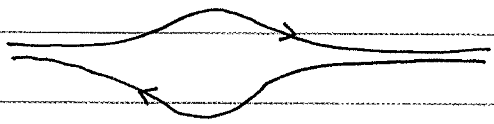
$$= e^{i\omega t} e^{ik\phi} R_{\omega,k}(R)\, \Theta_{\omega,k}(\Theta).$$

represent wave equation as

$$\Psi(t,r,\Theta,\phi) = \frac{-1}{2\pi i}\sum_{k\in\mathbb{Z}} e^{ik\phi}\sum_{n\in\mathbb{N}}$$

$\vec{\Psi}_\phi = \begin{bmatrix} \psi \\ \partial_t \psi \end{bmatrix}$, $i\partial_t \Psi = H \Psi$ initial data in $C^\infty((r_1, \infty) \times S^2)$, $r_1 > r_+$.

$$\psi(t,r,\theta,\phi) = -\frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{ik\phi} \sum_{n \in \mathbb{N}} \lim_{\varepsilon \to 0} \left(\int_{C_\varepsilon} - \int_{\bar{C}_\varepsilon} \right) e^{i\omega t} P_{\omega,k,n} (H-\omega) \vec{\Psi}_0^k (r,\theta)\, d\omega$$



Can use to prove ~~decay~~ pointwise decay. $R > r_1$ $\exists C = C(R, \vec{\Psi}_0)$.

$$\int_{[R,\infty)} e[\Psi](t,x)\, dr\, d\Omega < C.$$

Kerr

Metric is

$$ds^2 = -\frac{\Delta}{\rho}\left(dt - a\sin^2 d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\Theta^2 + \frac{(r^2+a^2)^2}{\rho^2}\sin^2\Theta\left(d\phi - \frac{a}{r^2+a^2}dt\right)^2$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\rho^2 = r^2 + a^2\cos^2\Theta$$

clearly

$$k = \frac{\partial}{\partial t}$$

$$h = \frac{\partial}{\partial \phi} \qquad \text{are Killing}$$

$$K = a^2\cos^2\Theta\left(\frac{\Delta}{\rho}\left(dt - a\sin^2 d\phi\right)^2 - \frac{\rho^2}{\Delta}dr^2\right)$$

$$+ r^2\left(\rho^2 d\Theta^2 + \frac{r^2+a^2}{\rho^2}\sin^2\Theta\left(d\phi - \frac{a}{r^2+a^2}dt\right)^2\right)$$

Clear limit as $a \to 0$. Δ_{S^2}

$$Y = (r^2+a^2) \quad r\,d\Theta \wedge \sin\Theta\left(d\phi - \frac{a}{r^2+a^2}dt\right)$$

$$+ a\cos\Theta \quad dr \wedge \left(dt - a\sin^2\Theta\, d\phi\right).$$