FRW Cosmologies

(Nonlinear theory, $\mathbb{R}^{1+3}$ trivial solutions, need to understand large data solutions (like solitons), ~~other techniques for analyzing geometry~~) illustrate ideas: pert, conformal trans, Killing vectors)

Einstein's equation

$$G = 8\pi T$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{c^4} T_{\mu\nu} \qquad \text{(with } + - - - \text{ signature)}$$

Energy-Momentum

$(T) = 0$ is symmetric $\qquad T_{[\alpha\beta]} = 0$ Ricci properties, matter

$\mathrm{div}(T) = 0$ $\qquad \nabla^\alpha T_{\alpha\beta} = 0$. Bianchi, matter

Particle $\quad p_\mu \quad$ 4-momentum

Field $\quad T_{\mu\nu}$

In $(\mathbb{R}^{1+3}, -dt^2 + d\vec{x}^{\,2})$,

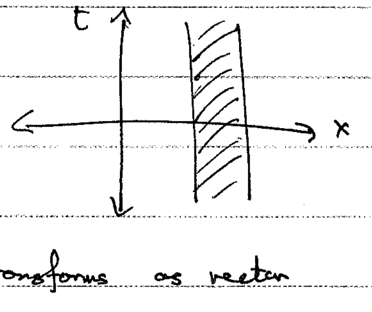
$p_t \quad T_{tt}$ energy (mass density)

$p_i \quad T_{ti}$ momentum

T_{ij} stress

(any combination of these words).

For $T(t, x_i) = \chi(x_i)\, T_0$



$$p_\mu = \int_{t=c} T_{\mu\nu}\, d\eta^\nu \qquad \text{transforms as vector}$$

$$= \int_{\mathbb{R}^3} T_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\nu d^3x \,.$$

Homogeneous & Isotropic spaces (assume connected)

Def: A Riemannian manifold $(\tilde{M}^n, \tilde{g})$ is homogeneous if there is a Lie group A (of translations) acting transitively on $\tilde{M}$ such that $\forall a \in A$, $\phi_a^* \tilde{g} = \tilde{g}$.

It is isotropic if, at each point $\tilde{p}$, the sectional curvature is the same for all planes in $T\tilde{M}_{\tilde{p}}$. (Isotropic at $\tilde{p}$).

__Schur's thm__ [Kobiyashi-Nomizu] ($\tilde{M}$ understood to be Riemannian)

If $(\tilde{M}^n, \tilde{g})$ is an ~~Riemannian~~ isotropic manifold and $n \geq 3$, then $\tilde{M}$ is a homogeneous space.

__Corollary__

$\tilde{R}_{ijkl} = k(\tilde{g}_{il}\tilde{g}_{jk} - \tilde{g}_{ik}\tilde{g}_{jl})$

If $\tilde{M}^3$ is simply connected then $\tilde{g}$ has the form

$$\tilde{g} = \begin{cases} C(d\psi^2 + \sin^2\psi\,(d\Theta^2 + \sin^2\Theta\, d\phi^2)) & k>0 \\ d\psi^2 + \psi^2(d\Theta^2 + \sin^2\Theta\, d\phi^2) & k=0 \\ C(d\psi^2 + \sinh^2\psi\,(d\Theta^2 + \sin^2\Theta\, d\phi^2)) & k<0 \end{cases}$$

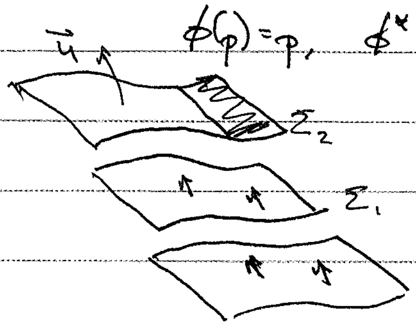
__Physics concept__ (throw in hypotheses as needed to get results) (No tilde is Lorentz)

A ~~Lorentz~~ manifold (M^{1+n}, g) is __spatially homogeneous__ if there is a foliation $M^{1+n} = \bigcup_{t\in I} \Sigma_t$ with each Σ_t a homogeneous Riemannian manifold (+smoothness)

It is __spatially isotropic__ if there is a time-like vector field u^a such that (defines simultaneous)

$\forall p \in M$, $s_1, s_2 \in T_pM$ if $g(s_1,s_1) = g(s_2,s_2)$, $g(s_1,u) = g(s_2,u) = 0$ then there is a diffeomorphism ϕ with

$\phi(p) = p$, $\phi^*(u) = u$, $\phi^*(s_1) = s_2$. (direction of time) (family of observers that see isotropy)



(Isotropy means $P_\Sigma u = 0$, so $u \perp \Sigma$ gives good coordinates or such massive degeneracy that can anyway)

(simple connectedness held by other things)

Both $+ \ldots \Rightarrow$ $M = I \times \tilde{M}$ with $\tilde{M}$ homogeneous

$$g = -dt^2 + a(t)^2\tilde{g}$$

Can take $k \in \{-1, 0, 1\}$ by absorbing in a.

Check

~~Friedman Robertson Walker~~ FRW

fluids

$$T = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} = (\rho \cancel{\ } + P)\, dt^2 + P\tilde{g}.$$ In Minkowski:

Natural extension when good time-space decomposition
$\Rightarrow$ Homo & Isotropic

$\cancel{\ }P = 0$ dust

$= P/3$ radiation

~~$= -\rho = \Lambda$ cosmological constant (here add $-\Lambda g_{ab}$ to rhs of eqn)~~

From Einstein equation:

$$3\dot{a}^2/a^2 = 8\pi\rho - 3k/a^2,$$

$$3\ddot{a}/a = -4\pi(\rho + 3P)$$ (with matter model 2 unknowns & 2 eqns)

Remarks:

① $\rho > 0$, $P \geq 0 \Rightarrow \ddot{a} < 0$, no static solutions

skip

~~Ex~~ For $G_{ab} = -\Lambda g_{ab}$ (cosmological constant)

effectively, $\rho = \Lambda$, $P = -\Lambda a^2$

$k = 1$, $a^2 = +\frac{1}{3}$, $\rho = \frac{1}{8\pi}$ gives Einstein static universe.

② For $\tilde{p}, \tilde{q} \in \tilde{M}$, the curves $(t, \tilde{p})$, $(t, \tilde{q})$ are geodesics.

skip

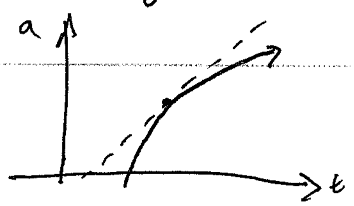
If $\tilde{d}(\tilde{p}, \tilde{q}) = L'$, (distance in $\tilde{M}$ wrt $\tilde{g}$)

$L = d((\lambda, \tilde{p}), (\lambda, \tilde{q})) = aL'$ (in $\{\lambda\} \times \tilde{M}$)

$$\frac{d}{dt}L = \dot{a}L' = \left(\frac{\dot{a}}{a}\right)L$$

$H = \frac{\dot{a}}{a}$ Hubble constant (constant in space, not time. Observe speed $\propto$ distance)

③ Since $\dot{a}$ is increasing and $\dot{a}$ is observed to be positive



~~at some point in the previously~~, it was zero at most $H^{-1} = \frac{a}{\dot{a}}$ time ago.

(r=0 in spherical coordinates)

SKIP As ~~R~~ $a \to 0$, $R \to \infty$: Big bang singularity (Singularity thm, expect singularity to be real)

④ SKIP Recent observation $\ddot{a} > 0$: strange matter with $\rho + 3P < 0$ or incorrect model.

$$3\dot{a}^2 - 8\pi\rho a^2 = -3k$$

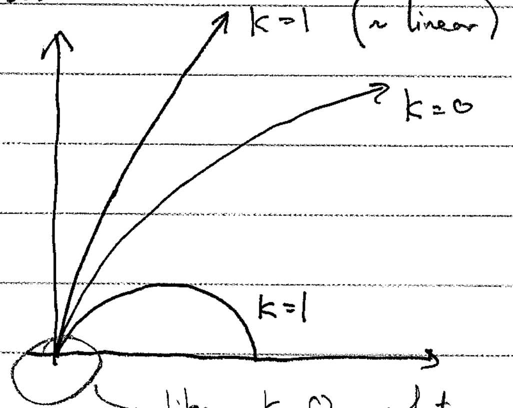
$$6\ddot{a}\dot{a} - 16\pi\rho a\dot{a} - 8\pi\dot{\rho}a^2 = 0$$

$$-8\pi(3\rho + 3P)\, a\dot{a} - 8\pi\dot{\rho}a^2 = 0$$

Dust $\rho a^3 = C$ (Matter per unit volume decreases as volume increases)

Radiation $\rho a^4 = C$

Exact solutions



like k=0 solution.

(Note singularity)

$k=0$

dust $Ct^{2/3}$

radiation $Ct^{1/2}$

Physicists imagine $t^{2/3}$ recently, $t^{1/2}$ before, quantum before that.

Particle Horizon

Defn ~~A Lorentz manifold~~ (M^{1+n}, g) is time-orientable if there is a non-vanishing time-like vector field, $\vec{T}$.

A ~~...~~ C^1 ~~curve~~ time-like curve $\gamma : I \to M$ ~~on~~ of such a manifold is future-directed if $\forall t \in I : g(\dot{\gamma}(t), \vec{T}) > 0$

On such a manifold, the causal past of a point is

$$J^-(p) = \{q \in M \mid \exists \gamma \text{ future-directed from } q \text{ to } p\}.$$

Def'n

If $\Omega : M \to \mathbb{R}^+$, then $(M, \Omega^2 g)$ is a ~~Lorentz manifold~~ Conformal transform of (M,g).

(Conformal transform)

Lemma

If (M,g) is a Lorentz so is $(M, \Omega^2 g)$.

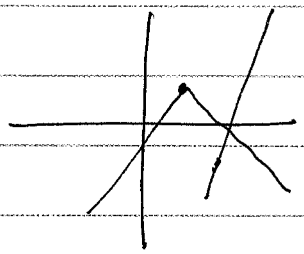
Conformal transforms preserve time-orientability, future-directedness, the causal past, and null-geodesics (upto parameterisation).

(all obvious, but last. Brief computation)

Defn (M,g) has property P if $\exists p \in M$, and a ~~complete~~ maximal time-like ~~geodesic~~ $\gamma \subset M$:

$$J^-(p) \cap \gamma = \emptyset.$$

Ex: Minkowski does not have property P



Ex: For FRW: (take $k=0$) radiation (but the same in other cases.

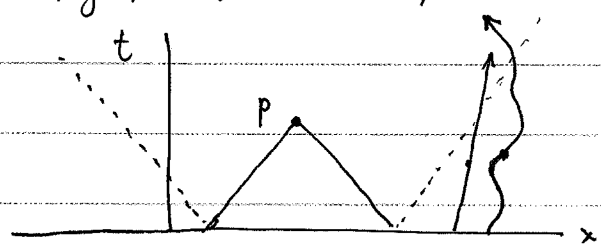
$$(M_1, g_1) = (\mathbb{R}^+ \times \mathbb{R}^3, -dt^2 + t\, dx^2)$$

conformal to $(\mathbb{R}^+ \times \mathbb{R}^3, -t^{-1} dt^2 + dx^2) = (M_2, g_2)$

let $d\tau = t^{-1/2} dt$

$$\tau = t^{1/2}$$

$$(M_2, g_2) = (\mathbb{R}^+ \times \mathbb{R}^3, -d\tau^2 + dx^2)$$ Easier structure



pick p, c

draw light-cone back

draw light-cone forward

any future-directed t-like starting outside can't get in inner

(M_2, g_2) has property P.

~~(M_1, g_1) has property P~~

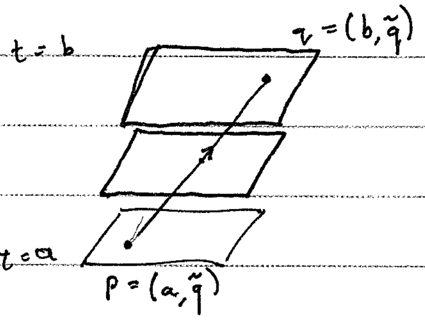
Since curve-type & causal past are preserved & (an original geodesic is a new t-like curve, not geo)
(M_1, g_1) has property P.
(Near the singularity, particles were not close enough to talk to each other - they couldn't send light rays)

P $\Leftrightarrow$ Particle horizon (particle horizon of a point is defined)
(~~???~~ Black hole event horizon is submanifold).

Redshift

In homogeneous $\tilde{M}$, $\forall \tilde{p}, \tilde{q} \in \tilde{M} : \exists \tilde{\gamma} : (a,b) \to \tilde{M}$ with $\tilde{\gamma}$ geodesic ~~geodesic~~ and a symmetry generating vector field $\tilde{K}$ with $\tilde{K}|_{\gamma} = \dot{\gamma}$. (Assume arclength para, so $\tilde{g}(\tilde{K}, \tilde{K}) = 1$)

In FRW, $\tilde{p}, \tilde{q} \in \tilde{M}$, $\exists f_1, f_2 : (a,b) \to \mathbb{R}$: a parameterisation such that $\gamma(\lambda) = (f_1(\lambda), \tilde{\gamma}(f_2(\lambda))) : (a,b) \to I \times \tilde{M}$ is a null geodesic.



$I \times \tilde{M}$.

Extend $\tilde{K}$ to $\Gamma(I \times \tilde{M})$ as symmetry generator $K = \tilde{K}$
$\Rightarrow K$ is a <u>Killing vector field</u>

sym $(\nabla K) = 0$ $\qquad \nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0$.

Along γ:

$$\dot{\gamma}\left(g(\dot{\gamma}, K)\right) = \dot{\gamma}^\alpha \nabla_\alpha (\dot{\gamma}^\beta K_\beta)$$
$$= \underbrace{(\dot{\gamma}^\alpha \nabla_\alpha \dot{\gamma}^\beta)}_{\text{geodesic} \to 0} K_\beta + \underbrace{\dot{\gamma}^\alpha \dot{\gamma}^\beta}_{\text{sym}} \underbrace{\nabla_\alpha K_\beta}_{\text{sym part zero}}$$
$$= 0$$

$g(\dot{\gamma}, K)$ is conserved.

$$E(q) = g(\dot{\gamma}(q), \tfrac{\partial}{\partial t})$$ ← Recall when good $\frac{\partial}{\partial t}$, energy is $g(\frac{\partial}{\partial t}, p)$.

$$= g(\dot{\gamma}(q), \tfrac{1}{a} K)$$ since $\dot{\gamma} \in \operatorname{span}(\cancel{\frac{\partial}{\partial t}}, K)$

$$= \frac{1}{a(q)} g(\dot{\gamma}, K)$$ Need $\frac{1}{a}$ to get null, recall, $\dot{\gamma}$ in span

$$E(q) = \frac{a(p)}{a(q)} E(p).$$

Wave length gets stretched ~~with~~ with space.

Can attempt to use similar calculations with ~~the~~ wave equations.

Killing vectors & symmetries give conserved quantities.

Key to nonlinear stability of $\mathbb{R}^{1+3}$, which ~~is~~ preeminant result in analysis of mathematical relativity.