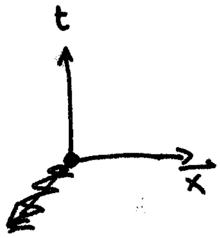
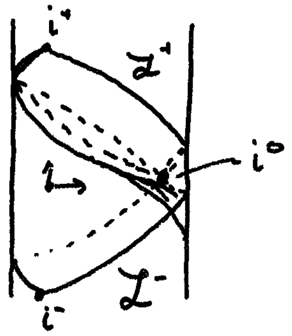
Decay estimates for nonlinear wave equations by the conformal method

Plain, , nice idea, more about eqns already known to be LWP (quickly see comp heavy)
(Not new q linear method for Nikodem)

$$\mathbb{R}^{1+n} \xrightarrow[\text{conformal compactification}]{\phi} \mathbb{R}\times S^n$$





	rescaled metric (conformal factor
u'	$\tilde{u} = u' \circ \phi$
	u rescaled $\tilde{u}$ (conformal factor to dimension dep. power)
some NLW	similar NLW for u
	LWP
GWP for u'	small data $\Rightarrow T_{LWP} > \pi$ for u
decay for u'	boundedness for u

Small data GWP & decay from other methods $\not\longrightarrow$

[Christodoulou, CPAM 1982]

Hormander

<u>Hypotheses</u>

—

Def'n: Weighted Sobolev space norm for $u: \mathbb{R}^n \to \mathbb{R}$

$$\|u'\|^2_{H^{k,\delta}} = \sum_{|\alpha|\le k} \|\langle x\rangle^{\delta+|\alpha|} D^\alpha u\|^2_{L^2}$$

$H^{k,\delta}(\mathbb{R}^n)$ Sobolev space (C^∞ closure, as usual)

[redacted]

Hypotheses on the nonlinearity

Hypothesis A:

$\cancel{f: X_0 \times X_1 \to \mathbb{R}^N}$

- $f: \underset{u}{\mathbb{R}} \times \underset{\nabla u}{\mathbb{R}^{n+1}} \times \underset{\text{sym } \nabla^2 u}{\mathbb{R}^{\frac{(n+1)(n+2)}{2}}} \longrightarrow \mathbb{R}^n$

$$M = 1 + (n+1) + \frac{(n+1)(n+2)}{2}$$

- $f \in C^\infty$
- $f(0,0,0) = 0$

  $Df(0,0,0) = 0$

- $f(u,v,w) = \alpha^{\mu\nu}(u,v)\, w_{\mu\nu} + \beta(u,v)$ (quasilinear)

  $\alpha^{\mu\nu}$ symmetric (linear in $\nabla^2 u$) (arbitrary in u,v)

- $\gamma^{\mu\nu} = \eta^{\mu\nu} - \alpha^{\mu\nu}$ is a hyperbolic matrix. (gives Lorentz metric)

  η: Minkowski metric

Hypothesis B

- $t=0$ is space-like ~~wrt~~ γ

  $\gamma^{00}\big|_{t=0} < 0$,

  $\forall\, \xi_\mu$: if $\gamma^{0\mu}\big|_{t=0} \xi_\mu = 0$ then $\gamma^{\mu\nu}\big|_{t=0} \xi_\mu \xi_\nu > 0$.

Def'n f satisfies the <u>null condition</u> if $\forall\, (u,s,t) \in \mathbb{R}^3$

$$\underbrace{\forall y \in \mathbb{R}^{1+n} :}_{\text{if } \eta^{\alpha\beta} y_\alpha y_\beta = 0 \text{ then}} \quad f(u, sy, t\, y \otimes y) = 0 .$$

Def'n The quadratic part of f is

$$f^{(2)}(z) = \frac{1}{2} \sum_{a_1,a_2=1}^{M} \frac{\partial^2 f}{\partial z^{a_1}\, \partial z^{a_2}}(0)\, z^{a_1} z^{a_2} .$$

Theorem

Consider the IVP on $\mathbb{R}^{1+n}$:

$$\left.\begin{aligned} \Box u' &= f(u', \partial u', \partial^2 u') \\ u'(0,x) &= Q' \\ \partial_t u(0,x) &= P' \end{aligned}\right\} \quad (NLW')$$

satisfying hypotheses A and B

n odd

$n > 3$ or $n=3$ and $f^{(2)}$ satisfies the null condition.

$k = \frac{1}{2}(n+1)+2$

$Q' \in H^{k,k-1}(\mathbb{R}^n)$, $P' \in H^{k-1,k}(\mathbb{R}^n)$

then $\exists \varepsilon > 0$:

$$\| Q' \|_{H^{k,k-1}} + \| P' \|_{H^{k-1,k}} < \varepsilon$$

implies there's a unique global solution to (1) and for which

$\exists C$:

$$|u'(t', \vec{x}')| \leq C \left[\left(1 + (t' + |\vec{x}'|)^2\right)\left(1 + (t' - |\vec{x}'|)^2\right)\right]^{-\frac{n-1}{4}}.$$

For $n=3$, this is $\displaystyle |u'| \leq \frac{C}{(1 + t' + |\vec{x}'|)(1 + (t' - |\vec{x}'|))}$

proof

Step 1: Conformal compactification

Use $T, \vec{X}$ on $\mathbb{R} \times S^n$

with $\vec{X}^a$ being south-pole stereographic projection.

Standard metric on $\mathbb{R} \times S^n$ is

$$-dT^2 + h$$

with h the standard S^n metric: $\displaystyle h = \frac{1}{1 + \frac{1}{4}|\vec{X}|^2} \sum_{a=1}^{n} (dX^a)^2.$

Let $\phi : \mathbb{R}^{1+n} \to \mathbb{R} \times S^n$ given by

use primes — use capitals (some ~ some not)

$$\tilde{T} = \arctan(t' + |\vec{x}'|) + \arctan(t' + |\vec{x}'|)$$

$$\tilde{X}^a = \tilde{R}(t', |\vec{x}'|) \frac{x'^a}{|\vec{x}'|}$$

$$\tilde{R}(t', r') = \frac{1}{r'}\left[\sqrt{(1+(t'+r')^2)(1+(t'-r')^2)} - (1+t'^2-r'^2)\right]$$

like a radial coordinate
or like Θ on S^2

$\phi(\mathbb{R}^{1+n}) \subset \{2\arctan(\tfrac{1}{2}\tilde{R}) - \pi < \tilde{T} < 2\arctan(\tfrac{1}{2}\tilde{R})\}$ More precisely: the stereographic projection radius.

On $\phi(\mathbb{R}^{n+1})$ let

$$\tilde{g} = \phi_*^{-1}\eta.$$

By direct computation

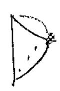
$$g = \Omega^2(\tilde{g})$$

$$\Omega = \cos\tilde{T} - R$$

$$R = \frac{1 - \frac{1}{4}|\tilde{R}|^2}{1 + \frac{1}{4}|\tilde{R}|^2} \in (-1, 1]$$

relate $\cos\tilde{T}$ and R
to coordinates in conf diag

$\cos T$ ↑



← R

$$\Omega(\phi(t', x')) = \frac{2}{\sqrt{(1+(t'+|\vec{x}'|^2))(1+(t'-|\vec{x}'|)^2)}}$$

This vanishes exactly on the boundary (ie conformal infinity) and is positive in $\phi(\mathbb{R}^{n+1})$

Step 2: <u>Derive Conformal equation</u>

Construct $E : T^*\phi(\mathbb{R}^{1+n}) \to T^*\phi(\mathbb{R}^{1+n})$

analytic in p
linear in vectors, essentially transition function.

Let $\tilde{u} = u' \circ \phi^{-1}$

then $\partial' u' = E(\tilde{\nabla} \tilde{u})$

$$\partial'^2 u' = (E \otimes E)(\tilde{\nabla}^2 \tilde{u})$$

and

$$(\Box' u')(t', x') = (\Box_{\tilde{g}} \tilde{u})(\phi(t', x')) .$$

$\tilde{F} : \phi(\mathbb{R}^{1+n}) \times \mathbb{R}^{M} \to \mathbb{R}$

Let $\tilde{F}(p, \tilde{u}, \tilde{v}, \tilde{w}) = f(\tilde{u}, E(\tilde{v}), (E \otimes E)\tilde{w})$

(depends on $p \in \phi(\mathbb{R}^{1+n})$ through E)

E: Analytic map in $p \in \mathbb{R} \times S^n$ linear in tangent vector relating tangent vectors.

(NLW$'$) is equivalent to

$$(\widetilde{NLW}) \quad (\Box_{\tilde{g}} \tilde{u})(p) = \tilde{F}(p, \tilde{u}, E(\tilde{\nabla} \tilde{u}), E \otimes E(\tilde{\nabla}^2 u)) .$$

Let $u = \Omega^{-\frac{(n-1)}{2}} \tilde{u}$.

$\Box_g u - \frac{n-1}{4n} R[g] u$ is conformally ~~invariant~~ covariant

metric Ω^2 $\quad u \mapsto \tilde{u}$

$$\Box_g u - \frac{n-1}{4n} R[g] u = \Omega^{-2} \Omega^{-\frac{(n-1)}{2}} \left(\Box_{\tilde{g}} \tilde{u} - \frac{n-1}{4n} R[\tilde{g}] \tilde{u}\right)$$

$$R[\tilde{g}] = R[\eta] = 0$$

$$R[g] = R[h] = n(n-1)$$

There's an explicit relation between

$\tilde{\nabla} u$ and ∇u

$\tilde{\nabla}^2 u$ and $\nabla^2 u$

involving only derivatives of lower order terms (fewer derivatives)

This can be used to define

$$F(p, u, v, w) = \{\Omega(p)\}^{-\frac{3+n}{2}} \tilde{F}(p, \tilde{u}, \tilde{v}, \tilde{w})$$

$$F(p, u, \nabla u, \nabla^2 u) = \Omega^{-\frac{3+n}{2}} \tilde{F}(p, \tilde{u}, \tilde{\nabla} \tilde{u}, \tilde{\nabla}^2 \tilde{u})$$

Thus (NLW$'$) and ($\widetilde{\text{NLW}}$) are equivalent to

$$\text{(NLW)} \quad \Box_g u - \tfrac{1}{4}(n-1)^2 u = F(p, u, \nabla u, \nabla^2 u).$$

Step 3: <u>Analyse F</u>

By (A) $\forall z \in \mathbb{R}^M, \lambda \in \mathbb{R}$

$$f(\lambda z) = \lambda^2 h(\lambda, z)$$

with $h \in C^\infty$

For $\lambda = \Omega^{(n-1)/2}$

$$F = \Omega^{\frac{n-5}{2}} h\left(\Omega^{\frac{n-1}{2}}, u, \zeta(u, \nabla u, \Omega, \nabla\Omega), \xi(u, \nabla u, \nabla^2 u, \Omega, \nabla\Omega, \nabla^2\Omega)\right)$$

~~where~~

$\therefore$ for ~~n odd,~~ $n > 5$, F extends smoothly to $\mathbb{R} \times S^n$ ($\times$ appropriate tangents, etc)

for $n = 3$, if f has no quadratic part

$$f(\lambda z) = \lambda^3 h(\lambda, z)$$

$$F = \Omega^{\frac{2n-6}{2}} h$$

Careful analysis of null condition also gives a smooth continuation.

~~Step 4 Compare Initial Data~~

~~The highest order term~~

The second order term in F occurs only when all derivatives land on u, so that the quasilinear term is a multiple of $\gamma^{\mu\nu}$, so ~~F satisfies hypotheses (A-B) if f does~~ the quasilinear part of F is hyperbolic if it is for f. Similarly for hyp B.

Step 4 Compare initial data

Note $\phi(\{t=0\}) \longrightarrow \{\tilde{T}=0\}$

$\phi\big|_{t=0}$ is a diffeomorphism $\mathbb{R}^n \to S^n / \{\text{south pole}\}$

Thus (Q', P') on $\{t=0\}$ induces data (Q, P) on $\{\tilde{T}=0\}$.

Since $\dfrac{\partial \tilde{T}}{\partial t} = \langle |\vec{x}'| \rangle^{-2}$

$$\Omega \circ \phi = \langle |\vec{x}'| \rangle^{-2}$$

$$Q \circ \phi = \langle |\vec{x}'| \rangle^{n-1} Q' \qquad \left(\text{from } \Omega^{-\frac{n-1}{2}} \text{ factor}\right)$$

$$P \circ \phi = \langle |\vec{x}'| \rangle^{n+1} P' \qquad \left(\text{extra factor from } \tfrac{\partial \tilde{T}}{\partial t}\right)$$

$$h \circ \phi = \langle |\vec{x}'| \rangle^{-4} \eta \qquad (\text{as metrics})$$

Lemma

$$\| Q \|_{H^k(S^n)} \leq C_{k,n} \| Q' \|_{H^{k,k-1}(\mathbb{R}^n)}$$

$$\| P \|_{H^{k-1}(S^n)} \leq C_{k,n} \| P' \|_{H^{k-1,k}(\mathbb{R}^n)} \qquad \square$$

(extra factor in weight from P having extra weight).

Step 5 Apply LWP

By standard LWP arguments

Uncited in Christodoulou

Citation in Hormander doesnot apply b/c on S^n not $\mathbb{R}^n$.

Use several charts in space, patch together & repeat to get to time π.

$$\text{(NLW)} \qquad \square_g u - \tfrac{1}{4}(n-1)^2 u = F(p, u, \zeta, \xi)$$

$$u(0, \tilde{X}) = Q(\tilde{X})$$

$$\frac{\partial u}{\partial \tilde{T}}(0, \tilde{X}) = P(\tilde{X})$$

has a local solution and if $\| Q \|_{H^k} + \| P \|_{H^{k-1}} < \varepsilon$,

then the time of existence $\tilde{T}_{LWP} > \pi$.

ie $\forall \tilde{T} > T_{LWP} : \| u(\tilde{T}) \|_{H^k} < \varepsilon'$

By Sobolev, on

$[-\pi, \pi] \times S^n$

$$|u(p)| \leq C$$

$$|u'(t', \vec{x}')| = |\tilde{u}(\phi(t', x'))|$$

$$\leq \Omega^{\frac{n-1}{2}} |u|$$

$$\leq \frac{C}{\left((1+(t'+r')^2)(1+(t'-r')^2) \right)^{+\frac{n-1}{4}}}$$

QED