Periods, tropical geometry and mirror symmetry
(joint w/ Abouzaid, Ganatra, Iritani)

I. Tropical asymptotic expansions

Maslov dequantisation:

$$\lim_{T\to\infty} \log_T \left(T^x + T^y\right) = \max(x,y) + O\left(\frac{T^{-|x-y|}}{\log T}\right)$$

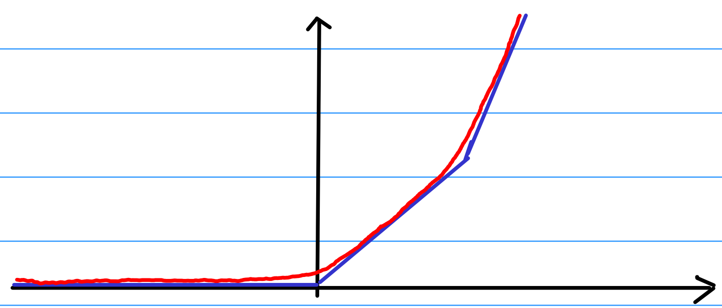
$$\cap$$

$$O(T^{-\varepsilon}) \text{ if } |x-y| < \varepsilon$$

alg. geom. $\longleftrightarrow$ trop. geom.

E.g. $p_T(a) = 1 + a + T^{-1}a^2$

$$\lim_{T\to\infty} \log_T \left(p_T(T^x)\right) = \max(0, x, 2x-1)$$



Why?

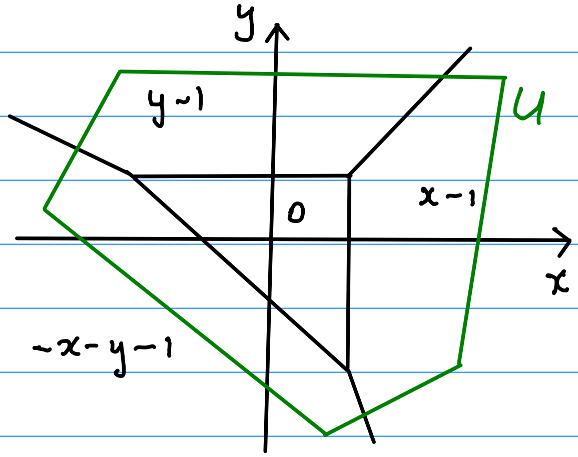
$$\int_{-B}^{B} \text{LHS}\, dx = \int_{-B}^{B} \text{RHS}\, dx + \frac{2\zeta(2)}{\log T} + O(T^{-\varepsilon}),\ T\to\infty.$$

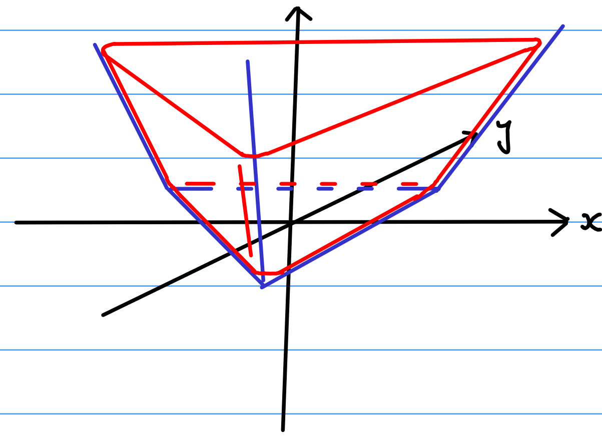
$\Rightarrow$ error term comes from 'bends'; local contribution is

$$\int_{-\infty}^{\infty} \log(1+e^x) - \max(0,x)\, dx = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}.$$

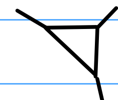
E.g. $p_T(a,b) = -1 + T^{-1}\left(a + b + \frac{1}{ab}\right)$

$$\lim_{T\to\infty} \log_T\left(p_T(T^x, T^y)\right) = \max\left(0, x-1, y-1, -x-y-1\right)$$

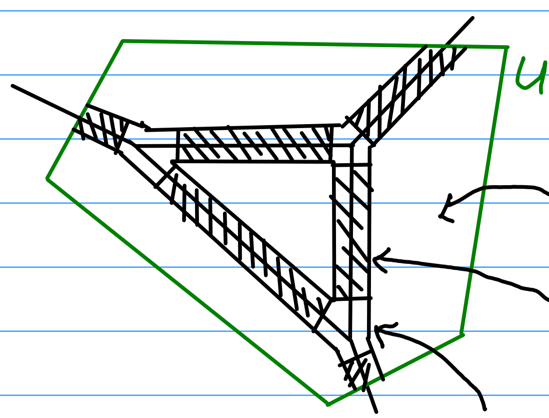




$$\iint_U \text{LHS}\, dA = \iint_U \text{RHS}\, dA + L \cdot \frac{\zeta(2)}{\log T} + 3\frac{\zeta(3)}{(\log T)^2} + O(T^{-\varepsilon})$$

length of  contained inside U.

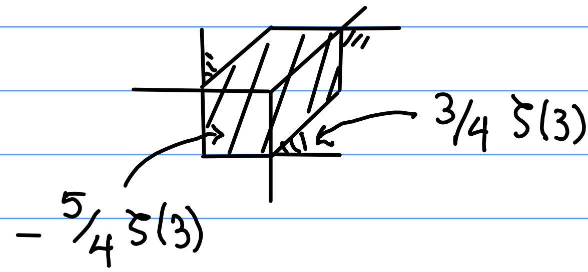
Why?



LHS = RHS + $O(T^{-\varepsilon})$

reduces to $\int_{\text{edge}} \frac{\zeta(2)}{\log T}$

new integral: $\zeta(3)$.

Cautions : (1) 

'kinks' along the edge contribute : $\mathcal{U}$ must cross edges <u>transversely</u>.

(2) In higher dims, integral depends on <u>angle</u> at which $\mathcal{U}$ crosses (not here).

II. Periods, mirror symmetry

$$Y_T \xleftrightarrow{\text{mirror}} (X, \log T \cdot \omega)$$

fam. of alg. var. $\qquad$ fam. of symp. mfd $\qquad T \rightsquigarrow \infty$

$T^{-1} \in \Delta^*$
punctured disc

$$\left\{ \begin{matrix} H^*(Y_T;\mathbb{C}) \\ \downarrow \\ \Delta^* \end{matrix} \quad \begin{matrix} \text{Hodge filtration} \\ \text{Gauss-Manin} \\ \text{connection} \end{matrix} \right\} \overset{\text{VHS}}{\simeq} \left\{ \begin{matrix} H^*(X;\mathbb{C}) \\ \downarrow \\ \Delta^* \end{matrix} \quad \begin{matrix} \text{deg. filt.} \\ \text{"quantum"} \\ \text{connection} \end{matrix} \right\}$$

$\rightsquigarrow$ enumerative predictions.

Γ Conjecture (Iritani, KKP, Hosono, Horja, Libgober, GKZ, Golyshev ...)

Lattice $H^*(Y_T;\mathbb{Z}) \subset H^*(Y_T;\mathbb{C})$ matches with lattice of flat sections asymptotic to

$$T^\omega \cdot \hat{\Gamma}_X \cdot H^*(X;\mathbb{Z}) + O(T^{-\varepsilon}) \qquad T \to \infty$$

GW invariants here

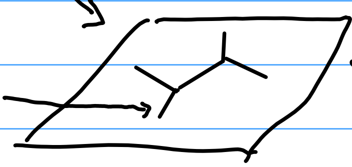
$$\prod_i \Gamma(1+\delta_i) = \exp\Big(\sum_{k\geq 2} (-1)^k \cdot \zeta(k)\cdot(k-1)!\cdot ch_k(TX)\Big)$$

Chern roots of TX.

Thm (Iritani): Holds if X, Y are mirror CYCI in Fano toric varieties.

Main result: Re-prove a certain part of this thm using ideas from section I and:

SYZ conjecture: X ← dual torus fibrations → Y



singular locus

B = common base = "tropical manifold"

"$\zeta(k)$ term arises from codim-k sing. locus".

Our techniques are 'local in SYZ base' $\Rightarrow$ expect to generalize to arbitrary Gross–Siebert mirrors (for which Γ conjecture is open).

Context:
$$\begin{tikzcd} D^b(Y) \arrow[rr, "\simeq"', "HMS"] \arrow[dr] & & \mathrm{Fuk}(X) \arrow[dl] \\ & \Delta^* & \end{tikzcd}$$

$$\Rightarrow \quad HP_\bullet(D^b(Y)) \simeq HP_\bullet(\mathrm{Fuk}(X))$$
$$\qquad \wr\| \qquad\qquad\qquad\qquad \wr\|$$
$$H^*(Y;\mathbb{C}) \simeq H^*(X;\mathbb{C}) \quad \underset{\simeq}{\mathbb{C}}\text{-VHS.}$$

Blanc/KKP propose $K^{top}(\mathcal{C})$, which gives rise to a $\mathbb{Q}$-lattice in $HP_\bullet(\mathcal{C})$, coinciding with $H^*(Y;\mathbb{Q})$

in above setting.

Conj: $HP_\cdot(\mathrm{Fuk}) \simeq H^*(X;\mathbb{C})$ sends

$$K^{top}(\mathrm{Fuk}) \mapsto H^*(X;\mathbb{Q}) \wedge T^{\omega} \wedge \hat{\Gamma}_X + O(T^{-\varepsilon})$$

Our approach is motivated by this conjecture.