Long tails in the long-time asymptotics of quasi-linear hyperbolic-parabolic systems of conservation laws

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Abstract

The long-time behavior of solutions of systems of conservation laws has been extensively studied. In particular, Liu and Zeng [6] have given a detailed exposition of the leading-order asymptotics of solutions close to a constant background state. In this paper, we extend the analysis of Liu and Zeng by examining higher-order terms in the asymptotics in the framework of the so-called two-dimensional *p-system*, though we believe that our methods and results also apply to more general systems. We give a constructive procedure for obtaining these terms, and we show that their structure is determined by the interplay of the parabolic and hyperbolic parts of the problem. In particular, we prove that the corresponding solutions develop *long tails*.

1 Introduction

In this paper, we consider the long-time behavior of solutions of systems of viscous conservation laws. This topic has been extensively studied. In particular, for the case of solutions close to a constant background state, [6] (building on work of [2]) contains a detailed exposition of the leading-order long-time behavior of such solutions. More precisely, it is shown in [6] that the leading-order asymptotics are given as a sum of contributions moving with the characteristic speeds of the undamped system of conservation laws and that each contribution evolves either as a Gaussian solution of the heat equation or as a self-similar solution of the viscous Burger's equation. Thus, with the exception of the translation along characteristics, these leading-order terms reflect primarily the dissipative aspects of the problem.

In this paper, in an effort to better understand the interplay between the hyperbolic and parabolic aspects of the problem, we examine higher-order terms in the asymptotics. We work with a specific two-dimensional system of equations–the *p-system*, but we believe that its behavior is prototypical. In particular, we think that our methods and results would extend to more complicated systems such as the 'full gas dynamics' and the equations of magnetohydrodynamics (MHD), as considered in [6].

The specific set of equations we consider is the following:

$$\begin{aligned} &\partial_t a = c_1 \partial_x b, && a(x,0) = a_0(x), \\ &\partial_t b = c_2 \partial_x a + \partial_x g(a,b) + \alpha\left(\partial_x^2 b + \partial_x (f(a,b)\partial_x b)\right), && b(x,0) = b_0(x). \end{aligned} \tag{1.1}$$

We will make precise the assumptions on the nonlinear terms f and g below; however, in order to describe our results informally, we basically assume that $|g(a,b)| \sim \mathcal{O}((|a|+|b|)^2)$ and $|f(a,b)| \sim \mathcal{O}((|a|+|b|))$. We also note that, without loss of generality, we can set $c_1 = 1 = c_2$ and $\alpha = 2$ in (1.1), which can be achieved by appropriate scalings of space, time and the dependent variables, and possible redefinition of the functions f and g.

Physically, (1.1) is a model for compressible, constant entropy flow, where a represents the volume fraction (i.e., the reciprocal of the density) and b is the fluid velocity. The first of the two equations in (1.1) is the consistency relation

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between these two physical quantities. In particular, it would not be physically reasonable to include a dissipative term in this equation, whereas such a term arises naturally in the second equation which is essentially Newton's law, in which internal frictional forces are often present. As a consequence of the form of the dissipation, the damping here is not 'diagonalizable' in the terminology of [6].

Next, we note that with the scaling $c_1 = 1 = c_2$ and $\alpha = 2$ in (1.1), the characteristic speeds are ± 1. If the initial conditions a_0 and b_0 in (1.1) decay sufficiently fast as $|x| \to \infty$, Liu and Zeng [6] showed that $a(x,t) \pm b(x,t) = \frac{1}{\sqrt{1+t}} g_0^{\pm}(\frac{x \pm t}{\sqrt{1+t}}) + \mathcal{O}((1+t)^{-\frac{3}{4}})$, where $g_0^{\pm}$ are self-similar solutions either of the heat equation or of Burger's equation, depending on the detailed form of the nonlinear terms. In this paper, we derive similar expressions for the higher-order terms in the asymptotics through a constructive procedure that can be carried out to arbitrary order.

More precisely, we show that, for any $N \geq 1$, there exist (universal) functions $\{g_n^{\pm}\}_{n=1}^N$ and constants $\{d_n^{\pm}\}_{n=1}^N$, determined by the initial conditions, such that

$$\begin{aligned} a(x,t) + b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^{+}\big(\tfrac{x+t}{\sqrt{1+t}}\big) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^{+} g_n^{+}\big(\tfrac{x+t}{\sqrt{1+t}}\big) + \mathcal{O}\bigg(\frac{1}{(1+t)^{1-\frac{1}{2^{N+2}}}}\bigg), \\ a(x,t) - b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^{-}\big(\tfrac{x-t}{\sqrt{1+t}}\big) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^{-} g_n^{-}\big(\tfrac{x-t}{\sqrt{1+t}}\big) + \mathcal{O}\bigg(\frac{1}{(1+t)^{1-\frac{1}{2^{N+2}}}}\bigg). \end{aligned} \tag{1.2}$$

We give explicit expressions for the functions $g_n^{\pm}$ below; however, focusing for the moment on the case $N = 1$ and the variable a, we have

$$a(x,t) = \frac{1}{2\sqrt{1+t}} \big[g_0^{+}\big(\tfrac{x+t}{\sqrt{1+t}}\big) + g_0^{-}\big(\tfrac{x-t}{\sqrt{1+t}}\big)\big] + \frac{1}{2(1+t)^{\frac{3}{4}}} \big[d_1^{+} g_1^{+}\big(\tfrac{x+t}{\sqrt{1+t}}\big) + d_1^{-} g_1^{-}\big(\tfrac{x-t}{\sqrt{1+t}}\big)\big] + \mathcal{O}\bigg(\frac{1}{(1+t)^{\frac{7}{8}}}\bigg),$$

where the functions $g_0^{\pm}(z)$ and $g_1^{\pm}(z)$ are solutions of the following ordinary differential equations:

$$\partial_z^2 g_0^{\pm}(z) + \frac{1}{2} z \partial_z g_0^{\pm}(z) + \frac{1}{2} g_0^{\pm}(z) + c_{\pm} \partial_z (g_0^{\pm}(z)^2) = 0, \tag{1.3}$$

$$\partial_z^2 g_1^{\pm}(z) + \frac{1}{2} z \partial_z g_1^{\pm}(z) + \frac{3}{4} g_1^{\pm}(z) + 2c_{\pm} \partial_z \left(g_0^{\pm}(z) g_1^{\pm}(z)\right) = 0. \tag{1.4}$$

Here, $c_{\pm}$ are constants that depend on the Hessian matrix of $g(a,b)$ at $a = 0 = b$ and that will be specified in the course of our analysis. We will prove that, while all solutions of (1.3) have Gaussian decay as $|x| \to \infty$, general solutions of the *linear* Equation (1.4) are linear combinations of two functions $g_{1,\pm}(z)$, where $g_{1,\pm}(z)$ decays like a Gaussian as $z \to \mp\infty$ but only like $|z|^{-\frac{3}{2}}$ as $z \to \pm\infty$; see also [5]. The graphs of the functions $g_0^{+}(z)$ and $g_1^{+}(z)$ are presented in Figure 1.

Thus, the higher-order terms in the asymptotics develop *long tails*. These tails are a manifestation of the hyperbolic part of the problem (or, perhaps more precisely, of the interplay between the parabolic and hyperbolic parts). Were we to consider just the asymptotic behavior of the viscous Burger's equation which gives the leading-order behavior of the solutions, we would find that, if the initial data are well localized, the higher-order terms in the long-time asymptotics decay rapidly in space and have temporal decay rates given by half-integers.

We also note one additional fact about the expansion in (1.2). Prior research [3, 9] has shown that for both parabolic equations and damped wave equations, the eigenfunctions of the operator

$$\mathcal{L}u(z) = \partial_z^2 u + \frac{1}{2} z \partial_z u$$

play an important role for the asymptotics. In particular, on appropriate function spaces, this operator has a sequence of isolated eigenvalues whose associated eigenfunctions can be used to construct an expansion for the long-time asymptotics. In this connection, we prove that the functions $g_n^{\pm}$ are closely approximated by eigenfunctions of $\mathcal{L}$ with eigenvalues $\lambda_n = -\frac{1}{2} + 2^{-(n+1)}$; more precisely, the functions $g_n^{\pm}$ are eigenfunctions of a compact perturbation of $\mathcal{L}$; see, e.g., (1.4). However, so far we have not succeeded in finding a function space which both contains these eigenfunctions (the functions $g_n^{\pm}$ decay slowly as $z \to \pm\infty$) and in which the corresponding eigenvalues are isolated points in the spectrum. We plan to investigate this point further in future research.

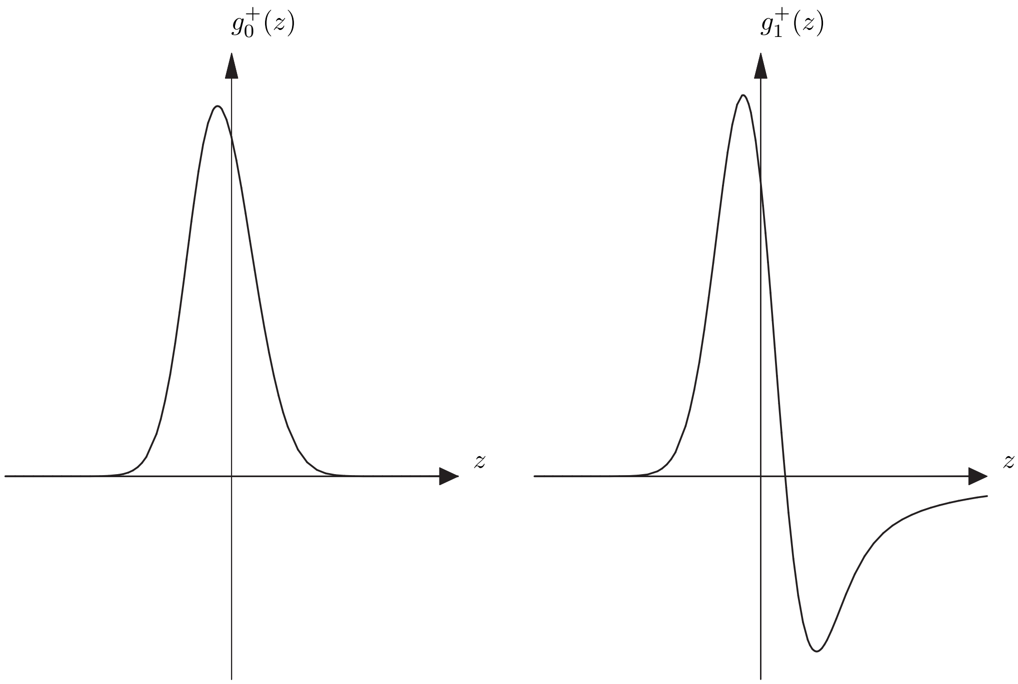

Figure 1: Graphs of the functions g_0^+ (left panel) and g_1^+ (right panel). Note the *long tail* of g_1^+ as $z \to \infty$.

Before moving to a precise statement of our results, we note that our approach makes no use of Kawashima's energy estimates for hyperbolic-parabolic conservation laws [4]. Instead, we prove existence by studying directly the integral form of (1.1).

We now state our results on the Cauchy problem (1.1). We begin by stating the precise assumptions we make on the nonlinearities f and g in (1.1).

Definition 1 *The maps $f, g : \mathbf{R}^2 \to \mathbf{R}$ are admissible nonlinearities for (1.1) if there is a quadratic map $g_0 : \mathbf{R}^2 \to \mathbf{R}$ and a constant C such that, for all $|\mathbf{z}|$, $|\mathbf{z}_1|$ and $|\mathbf{z}_2|$ small enough,*

$$
\begin{aligned}
|g(\mathbf{z})| \le C|\mathbf{z}|^2, &\qquad |g(\mathbf{z}_1) - g(\mathbf{z}_2)| \le C|\mathbf{z}_1 - \mathbf{z}_2|(|\mathbf{z}_1| + |\mathbf{z}_2|),\\
|\Delta g(\mathbf{z})| \le C|\mathbf{z}|^3, &\quad |\Delta g(\mathbf{z}_1) - \Delta g(\mathbf{z}_2)| \le C|\mathbf{z}_1 - \mathbf{z}_2|(|\mathbf{z}_1| + |\mathbf{z}_2|)^2,\\
|f(\mathbf{z})| \le C|\mathbf{z}| \quad \textit{and} &\quad |f(\mathbf{z}_1) - f(\mathbf{z}_2)| \le C|\mathbf{z}_1 - \mathbf{z}_2|,
\end{aligned}
$$

where $\Delta g(\mathbf{z}) \equiv g(\mathbf{z}) - g_0(\mathbf{z})$.

The main result of this paper can be formulated as follows:

Theorem 2 *Fix $N > 0$. There exists $\epsilon_0 > 0$ sufficiently small such that, if*

(i) $|a_0|_{\mathrm{H}^1(\mathbb{R})} + |a_0|_{\mathrm{L}^1(\mathbb{R})} < \epsilon_0$ *and* $|b_0|_{\mathrm{H}^2(\mathbb{R})} + |b_0|_{\mathrm{L}^1(\mathbb{R})} < \epsilon_0$,

(ii) $|x^2 a_0|_{\mathrm{L}^2(\mathbb{R})} + |x^2 b_0|_{\mathrm{L}^2(\mathbb{R})} < \infty$,

then (1.1) has a unique (mild) solution with initial conditions a_0 and b_0. Moreover, there exist functions $\{g_n^\pm\}_{n=0}^N$ (independent of initial conditions for $n \ge 1$) and constants C_N, $\{d_n^\pm\}_{n=1}^N$, determined by the initial conditions, such *that*

$$
\begin{aligned}
a(x,t)+b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^+\big(\tfrac{x+t}{\sqrt{1+t}}\big) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^+ g_n^+\big(\tfrac{x+t}{\sqrt{1+t}}\big) + R_u^N(x,t), \\
a(x,t)-b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^-\big(\tfrac{x-t}{\sqrt{1+t}}\big) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^- g_n^-\big(\tfrac{x-t}{\sqrt{1+t}}\big) + R_v^N(x,t),
\end{aligned}
\tag{1.5}
$$

where the remainders R_u^N and R_v^N satisfy the estimates

$$
\begin{aligned}
\sup_{t\geq 0}(1+t)^{\frac{3}{4}-\frac{1}{2^{N+2}}} \big\| R_{\{u,v\}}^N(\cdot,t)\big\|_{\mathrm{L}^2(\mathbb{R})} &\leq C_N, \\
\sup_{t\geq 0}(1+t)^{\frac{5}{4}-\frac{1}{2^{N+2}}} \big\| \partial_x R_{\{u,v\}}^N(\cdot,t)\big\|_{\mathrm{L}^2(\mathbb{R})} &\leq C_N.
\end{aligned}
\tag{1.6}
$$

Furthermore, for $n \geq 1$, the functions $g_n^\pm$ satisfy $g_n^\pm(z) \sim |z|^{-2+\frac{1}{2^n}}$ as $z \to \pm\infty$.

There is a slight incongruity in this result in that the norm in which we estimate the remainder term is weaker than the one we use on the initial data; namely, we do not give estimates for the remainder in $\mathrm{H}^2(\mathbb{R})$ or in the localization norms $\mathrm{L}^1(\mathbb{R})$ and the weighted $\mathrm{L}^2(\mathbb{R})$-norm (on that aspect of the problem, see Remark 3 below). Theorem 2 actually holds for slightly more general initial conditions than those satisfying (i)–(ii). Furthermore, we will prove that the estimates (1.6) hold for all initial conditions (a_0, b_0) in a subset $\mathcal{D}_2 \subset \mathrm{H}_1 \times \mathrm{H}_2$ that is *positively invariant* under the flow of (1.1). However, since the topology used to define the subset $\mathcal{D}_2$ is somewhat nonstandard, we have chosen to state the result initially in this slightly weaker, but hopefully more comprehensible, form to keep the introduction as simple as possible.

Remark 3 *It is interesting to note (see Proposition 7 below) that $\|x^2 a(\cdot,t)\|_{\mathrm{L}^2(\mathbb{R})} + \|x^2 b(\cdot,t)\|_{\mathrm{L}^2(\mathbb{R})}$ is finite for all finite $t > 0$, but that the terms with $n \geq 1$ in the asymptotic expansion do not satisfy this property due to the long tails of the functions $g_n^\pm$.*

Remark 4 *As the asymmetry in the degree of x-derivatives in (1.1) suggests, we require more spatial regularity from the second component (the b-variable) than from the first (the a-variable). It is then natural to expect that R_u^N or R_v^N are not necessarily in H^2, but that only their difference is.*

We conclude this section with a few remarks. Define $u_\pm(x,t) = a(x,t) \pm b(x,t)$. Then, the asymptotics of the solutions of (1.1) in the variables $u_\pm$ are the same as those of the two-dimensional (generalized) Burger's equation

$$
\begin{aligned}
\partial_t u_+ &= \partial_x^2 u_+ + \partial_x u_+ + \partial_x(c_+ u_+^2 - c_- u_-^2), \\
\partial_t u_- &= \partial_x^2 u_- - \partial_x u_- + \partial_x(c_- u_-^2 - c_+ u_+^2),
\end{aligned}
\tag{1.7}
$$

where the constants $c_\pm$ are determined by the Hessian of $g(a,b)$ at $a = 0 = b$ through

$$
c_\pm = \pm\frac{1}{8}(1,\pm 1)\cdot \begin{pmatrix} \partial_a^2 g & \partial_a\partial_b g \\ \partial_a\partial_b g & \partial_b^2 g \end{pmatrix}\Bigg|_{a=b=0} \cdot \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.
$$

We will show that the hyperbolic effects manifest themselves through the 'source' terms $-c_- u_-^2$ (respectively, $c_+ u_+^2$) in the first (respectively, second) equation in (1.7). In particular, none of the terms $g_n^\pm$ with $n \geq 1$ would be present in the asymptotic expansion if those terms were absent.

Finally, note that we have chosen to state Theorem 2 for finite N. As it turns out, the sums appearing in (1.5) converge in the limit as $N \to \infty$, in which case the estimates (1.6) hold with time weights replaced by $(1+t)^{\frac{3}{4}} \ln(2+t)^{-1}$ and $(1+t)^{\frac{5}{4}} \ln(2+t)^{-1}$. The proof can easily be done with the techniques used in this paper and is left to the reader.

The remainder of the paper is organized as follows: in Section 2, we discuss the well-posedness of the Cauchy problem (1.1) in an appropriately defined topology. In Section 3, we explain our strategy for proving our main result

(Theorem 2) on the long-time asymptotics of solutions of (1.1). Namely, we decompose that proof into a series of simpler subproblems which are then tackled in subsequent sections: in Sections 4 and 5, we investigate properties of solutions of Burger-type equations (respectively, of inhomogeneous heat equations), as they occur naturally in the asymptotic analysis. In Section 6, we collect some estimates that are used in the proof of the well-posedness of (1.1). Finally, in Section 7, we specify the sense in which the semigroup of the linearization of (1.1) is close to heat kernels translating along the characteristics, and we give estimates on the remainder terms occurring in Theorem 2.

2 Cauchy problem

To motivate our technical treatment of the problem and in particular our choice of function spaces, we first note that, upon taking the Fourier transform of the linearization of (1.1), it follows that

$$\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \mathrm{L} \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 0 & ik \\ ik & -2k^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.1}$$

We then find that the (Fourier transform of) the semigroup associated with (2.1) is

$$\mathrm{e}^{\mathrm{L}t} = \mathrm{e}^{-k^2 t} \begin{pmatrix} \cos(kt\Delta) + \frac{k}{\Delta}\sin(kt\Delta) & \frac{i}{\Delta}\sin(kt\Delta) \\ \frac{i}{\Delta}\sin(kt\Delta) & \cos(kt\Delta) - \frac{k}{\Delta}\sin(kt\Delta) \end{pmatrix}, \tag{2.2}$$

where $\Delta = \sqrt{1-k^2}$. The most important fact about the semigroup $\mathrm{e}^{\mathrm{L}t}$ is that it is close to $\mathrm{e}^{\mathrm{L}_0 t}$, the semigroup associated with the problem

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{L}_0 \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} \partial_x^2 + \partial_x & 0 \\ 0 & \partial_x^2 - \partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.3}$$

Formally, $\mathrm{e}^{\mathrm{L}_0 t}$ can be obtained by setting $\Delta = 1$ in $\mathrm{e}^{\mathrm{L}t}$ and by conjugating with the matrix

$$\mathcal{S} \equiv \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.4}$$

These two operations correspond to a long-wavelength expansion and a change of dependent variables to quantities that move along the characteristics. More precisely, we will prove that $\mathrm{e}^{\mathrm{L}t}$ satisfies the intertwining property

$$\mathcal{S}\mathrm{e}^{\mathrm{L}t} \approx \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S},$$

where the symbol $\approx$ means that the action of these two operators is the same in the large-scale–long-time limit; see Lemma 19 at the beginning of Section 7 for details.

Furthermore, $\mathrm{e}^{\mathrm{L}t}$ satisfies parabolic-like estimates

$$|\mathrm{e}^{\mathrm{L}t}| \le C\mathrm{e}^{-\min\{k^2,1\}\frac{t}{4}} \begin{pmatrix} 1 & \frac{1}{\sqrt{1+k^2}} \\ \frac{1}{\sqrt{1+k^2}} & 1 \end{pmatrix}, \tag{2.5}$$

$$\left| \mathrm{e}^{\mathrm{L}t} \begin{pmatrix} 0 \\ ik \end{pmatrix} \right| \le C \frac{\mathrm{e}^{-\min\{k^2,1\}\frac{t}{4}}}{\sqrt{t}} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{1+k^2}} \end{pmatrix} \tag{2.6}$$

uniformly in $t \ge 0$ and $k \in \mathbf{R}$.

Hence, to summarize, $\mathrm{e}^{\mathrm{L}t}$ behaves like a superposition of heat kernels translating along the characteristics of the underlying hyperbolic problem. In view of the above observations as well as of classical techniques for parabolic PDEs (see, e.g., [7, 1]), we will consider (1.1) in the following (somewhat non-standard) topology (cf. also [8]):

Definition 5 *We define $\mathcal{B}_0$ (respectively, $\mathcal{B}$) as the closure of $\mathcal{C}_0^\infty(\mathbf{R}, \mathbf{R}^2)$ (respectively, $\mathcal{C}_0^\infty(\mathbf{R} \times [0,\infty), \mathbf{R}^2)$) under the norm $|\cdot|$ (respectively, $\|\cdot\|$), where for $\mathbf{z}_0 = (a_0, b_0) : \mathbf{R} \to \mathbf{R}^2$ and $\mathbf{z} = (a,b) : \mathbf{R} \times [0,\infty) \to \mathbf{R}^2$, we define*

$$|\mathbf{z}_0| = \|\widehat{\mathbf{z}}_0\|_\infty + \|\mathbf{z}_0\|_2 + \|\mathrm{D}\mathbf{z}_0\|_2 + \|\mathrm{D}^2 b_0\|_2 \quad \textit{and} \quad \|\mathbf{z}\| = \|\hat{\mathbf{z}}\|_{\infty,0} + \|\mathbf{z}\|_{2,\frac{1}{4}} + \|\mathrm{D}\mathbf{z}\|_{2,\frac{3}{4}} + \|\mathrm{D}^2 b\|_{2,\frac{5}{4}}.$$

Here, $(Da)(x,t) \equiv \partial_x a(x,t)$*,* $\hat{a}(k,t)$ *is the Fourier transform of* $a(x,t)$*,*

$$\|f\|_{p,q} = \sup_{t\geq 0}(1+t)^q\|f(\cdot,t)\|_p, \quad \|f\|_{p,q^\star} = \sup_{t\geq 0}\frac{(1+t)^q}{\ln(2+t)}\|f(\cdot,t)\|_p$$

and $\|\cdot\|_p$ *is the standard* $\mathrm{L}^p(\mathbf{R})$*-norm.*

Before turning to the Cauchy problem with initial data in $\mathcal{B}_0$, we collect a few comments on our choice of function spaces.

Consider first the requirements on the initial conditions in (1.1): while the use of H^1-spaces is quite natural in this context, we choose to replace the L^1-norm by the (weaker) control of the L^∞-norm in Fourier space. This has the great advantage that all estimates can then be obtained in Fourier space, where the semigroup $\mathrm{e}^{\mathrm{L}t}$ has the simple, explicit form (2.2).

In turn, our choice of q-exponents in the norm $\|\cdot\|$ is motivated by the fact that these are the highest possible exponents for which the $\|\cdot\|$-norm of the leading-order asymptotic term $\frac{1}{\sqrt{1+t}}g_0^\pm(\frac{x\pm t}{\sqrt{1+t}})$ is bounded. Note also that, for the linear evolution (2.1), we have

$$\|\mathrm{e}^{\mathrm{L}t}\mathbf{z}_0\| \leq C|\mathbf{z}_0|, \tag{2.7}$$

since $\hat{j}(k,t) = \mathrm{e}^{-\min\{k^2,1\}t}u_0(k)$ satisfies

$$\|\mathrm{D}^n j(\cdot,t)\|_2 \leq C\big(\mathrm{e}^{-t}\|\mathrm{D}^n u_0\|_2 + \min\big\{t^{-\frac{1}{4}-\frac{n}{2}}\|\hat{u}_0\|_\infty, \|D^n u_0\|_2\big\}\big)$$

for all $n = 0, 1, \ldots$.

Finally, we note that, for admissible nonlinearities in the sense of Definition 1, the map $h(a,b) = f(a,b)\partial_x b + g(a,b) = h(\mathbf{z})$ satisfies

$$\|h(\mathbf{z})\|_{1,\frac{1}{2}} + \|h(\mathbf{z})\|_{2,\frac{3}{4}} + \|Dh(\mathbf{z})\|_{2,\frac{5}{4}} \leq C\|\mathbf{z}\|^2, \tag{2.8}$$

$$\|h(\mathbf{z}_1) - h(\mathbf{z}_2)\|_{1,\frac{1}{2}} + \|h(\mathbf{z}_1) - h(\mathbf{z}_2)\|_{2,\frac{3}{4}} \leq C\|\mathbf{z}_1 - \mathbf{z}_2\|(\|\mathbf{z}_1\| + \|\mathbf{z}_2\|), \tag{2.9}$$

$$\|D(h(\mathbf{z}_1) - h(\mathbf{z}_2))\|_{2,\frac{5}{4}} \leq C\|\mathbf{z}_1 - \mathbf{z}_2\|(\|\mathbf{z}_1\| + \|\mathbf{z}_2\|). \tag{2.10}$$

We are now fully equipped to study the Cauchy problem (1.1) in $\mathcal{B}$.

Theorem 6 *For all* $\mathbf{z}_0 \in \mathcal{B}_0$ *with* $|\mathbf{z}_0| = |(a_0,b_0)| \leq \epsilon_0$ *small enough, the Cauchy problem (1.1) is (locally) well posed in* $\mathcal{B}$ *if the nonlinearities are admissible in the sense of Definition 1. In particular, the solution satisfies* $\|\mathbf{z}\| \leq c\epsilon_0$ *for some* $c > 1$ *and is unique among functions in* $\mathcal{B}$ *satisfying this bound.*

Proof. Upon taking the Fourier transform of (1.1), we find

$$\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & ik \\ ik & -2k^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ ikh \end{pmatrix}, \tag{2.11}$$

which gives the following representation for the solution:

$$\mathbf{z}(t) \equiv \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \mathrm{e}^{\mathrm{L}t}\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{L}(t-s)}\begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix} \equiv \mathrm{e}^{\mathrm{L}t}\mathbf{z}_0 + \mathcal{N}[\mathbf{z}](t). \tag{2.12}$$

We will prove below that for all $\mathbf{z}_i \in \mathcal{B}$, $i = 1, 2$, we have

$$\|\mathcal{N}[\mathbf{z}]\| \leq C\|\mathbf{z}\|^2 \quad \text{and} \quad \|\mathcal{N}[\mathbf{z_1}] - \mathcal{N}[\mathbf{z_2}]\| \leq C\|\mathbf{z}_1 - \mathbf{z}_2\|(\|\mathbf{z}_1\| + \|\mathbf{z}_2\|) \tag{2.13}$$

for some constant C. The proof of Theorem 6 then follows from the fact that, for all $\mathbf{z}_0 \in \mathcal{B}_0$ with $|\mathbf{z}_0| \leq \epsilon_0$ small enough and $c > 1$, the right-hand side (r.h.s.) of (2.12) defines a contraction map from some (small) ball of radius $c\epsilon_0$ in $\mathcal{B}$ onto itself.

The general rule for proving the various estimates involved in (2.13) is to split the integration interval into two parts, with $s \in \mathcal{I}_1 \equiv [0, \frac{t}{2}]$ and $s \in \mathcal{I}_2 \equiv [\frac{t}{2}, t]$. In $\mathcal{I}_1$, we place as many derivatives (or, equivalently, factors of k) as possible on the semigroup $\mathrm{e}^{\mathrm{L}(t-s)}$, while in $\mathcal{I}_2$, (most of) these derivatives need to act on h, since the integral would otherwise be divergent at $s = t$.

Additional difficulties arise from the fact that $\mathrm{e}^{\mathrm{L}t}$ has very few smoothing properties (slow or no decay in k as $|k| \to \infty$), so that in some cases we need to consider separately the large-k part and the small-k part of the L^2-norm, say. This is done through the use of $\mathbb{P}$, defined as the Fourier multiplier with the characteristic function on $[-1, 1]$.

We decompose the proof of $\|\mathcal{N}[\mathbf{z}]\| \leq C\|\mathbf{z}\|^2$ into that of

$$\begin{aligned}\|\mathcal{N}[\mathbf{z}]\| &\leq \|\widehat{\mathcal{N}[\mathbf{z}]}\|_{\infty,0} + \|\mathcal{N}[\mathbf{z}]\|_{2,\frac{1}{4}} + \|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} + \|(1-\mathbb{P})\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} \\ &\quad + \|(1-\mathbb{P})\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} + \|(1-\mathbb{Q})\mathbb{P}\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} + \|\mathbb{Q}\mathbb{P}\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} \\ &\leq C\|\mathbf{z}\|^2, \end{aligned} \tag{2.14}$$

where $\mathbb{Q}$ is the characteristic function for $t \geq 1$ and $\mathcal{N}[\mathbf{z}]_2$ denotes the second component of $\mathcal{N}[\mathbf{z}]$.

We now consider $\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}}$ as an example of how we will prove the above estimates: we have

$$\begin{aligned}\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}](\cdot,t)\|_2 &\leq \|h(\mathbf{z})\|_{2,\frac{3}{4}} \left(\sup_{|k|\leq 1, \tau \geq 0} |k|\sqrt{\tau}\mathrm{e}^{-\frac{k^2\tau}{4}} \right) \int_0^{\frac{t}{2}} \mathrm{d}s \, \frac{(1+s)^{-\frac{3}{4}}}{t-s} \\ &\quad + \|\mathrm{D}h(\mathbf{z})\|_{2,\frac{5}{4}} \left(\sup_{|k|\leq 1, \tau\geq 0} \mathrm{e}^{-\frac{k^2\tau}{4}} \right) \int_{\frac{t}{2}}^t \mathrm{d}s \, \frac{(1+s)^{-\frac{5}{4}}}{\sqrt{t-s}} \\ &\leq C\|\mathbf{z}\|^2 \left(\frac{2}{t} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{(1+s)^{\frac{3}{4}}} + \frac{1}{(1+\frac{t}{2})^{\frac{5}{4}}} \int_{\frac{t}{2}}^t s \frac{\mathrm{d}s}{\sqrt{t-s}} \right) \leq C\|\mathbf{z}\|^2(1+t)^{-\frac{3}{4}} \end{aligned} \tag{2.15}$$

for all $t \geq 0$, which shows that $\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} \leq C\|\mathbf{z}\|^2$. All other estimates in (2.14) can be obtained in a similar manner; we postpone their proof to Section 6 below.

Finally, we note that the Lipschitz-type estimate in (2.13) can be obtained in the same manner, *mutatis mutandis*, due to the similarity between (2.9) and (2.10) with (2.8); we omit the details. ■

We can now turn to the question of the asymptotic structure of the solutions of (1.1) provided by Theorem 6. Note that already if we wanted to prove that $\mathrm{e}^{\mathrm{L}t}\mathbf{z}_0$ satisfies 'Gaussian asymptotics,' we would need more localization properties on $\mathbf{z}_0$ than those provided by the $\mathcal{B}_0$-topology. It will turn out to be sufficient to require $\mathbf{z}_0 \in \mathcal{B}_0 \cap \mathrm{L}^2(\mathbb{R}, x^m\mathrm{d}x)$ for (some) $m \geq 2$. We now prove that this requirement is *forward invariant* under the flow of (1.1).

Proposition 7 *Let $\rho_m(x) = |x|^m$, and define*

$$\mathcal{D}_m = \{\mathbf{z}_0 \in \mathcal{B}_0 \quad \textit{such that} \quad |\mathbf{z}_0| + \|\rho_m\mathbf{z}_0\|_2 < \infty\}\,.$$

If $\mathbf{z}_0 \in \mathcal{D}_m$ and $|\mathbf{z}_0| \leq \epsilon_0$ such that Theorem 6 holds, then the corresponding solution $\mathbf{z}(t)$ of (1.1) satisfies $\mathbf{z}(t) \in \mathcal{D}_m$ for all finite $t > 0$. Furthermore, there holds $|\mathbf{z}(t)| \leq (1+\delta)\epsilon_0$ for some (small) constant δ.

Proof. Note first that, by Theorem 6, $|\mathbf{z}(t)| \leq \|\mathbf{z}\| \leq (1+\delta)\epsilon_0$, since $\mathbf{z}_0 \in \mathcal{B}_0$ and $|\mathbf{z}_0| \leq \epsilon_0$. Then, fix $m \in \mathbf{N}$, $m \geq 1$. The proof of Theorem 6 can easily be adapted to show that (1.1) is *locally* (in time) well posed in $\mathcal{D}_m$. Global existence then follows from the fact that the quantity

$$N(t) = \frac{1}{2}\|\rho_m\mathbf{z}(\cdot,t)\|^2 = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \, |x|^m \left(a(x,t)^2 + b(x,t)^2\right)$$

grows *at most exponentially* as $t \to \infty$. Namely, we have

$$\begin{aligned}\partial_t N(t) &= \int_{-\infty}^{\infty} \mathrm{d}x \, |x|^m \left(\partial_x(ab) + 2b\partial_x^2 b + b\partial_x\left(f(a,b)\partial_x b + g(a,b)\right)\right) \\ &= -\int_{-\infty}^{\infty} \mathrm{d}x \, m|x|^{m-1}\mathrm{sign}(x)\left(b(a+g(a,b)) + (2+f(a,b))b\partial_x b\right)\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{-\infty}^{\infty} \mathrm{d}x \, |x|^m (\partial_x b)^2 \, (2 + f(a,b)) \\
&\le \int_{-\infty}^{\infty} \mathrm{d}x \, \left((m-1)^{m-1} + |x|^m\right) |b(a + g(a,b)) + (2 + f(a,b)) b \partial_x b| \\
&\quad - \int_{-\infty}^{\infty} \mathrm{d}x \, |x|^m (\partial_x b)^2 \, (2 + f(a,b)) \\
&\le \int_{-\infty}^{\infty} \mathrm{d}x \, \left((m-1)^{m-1} + |x|^m\right) \left(|b(a + g(a,b))| + 2^{-1}|2 + f(a,b)|b^2\right) \\
&\le C_1(m, \epsilon_0) + C_2(\epsilon_0) N(t),
\end{aligned}
$$

due to the estimates $\|f(a,b)\|_\infty \le C\epsilon_0 \ll 2$ and $\|\frac{g(a,b)}{\sqrt{a^2+b^2}}\|_\infty \le C\epsilon_0$. ∎

3 Asymptotic structure–Proof of Theorem 2

We can now state our main result on the asymptotic structure of solutions of (1.1) in a definitive manner.

Theorem 8 *Let $\mathcal{D}_m$ be as in Proposition 7, with $m \ge 2$, let $\mathbf{z}_0 \in \mathcal{D}_m$ with $|\mathbf{z}_0| \le \epsilon_0$ such that Theorem 6 holds, and write $\mathbf{z}(t) = (a(t), b(t))$ for the corresponding solution of (1.1). Then, there exist functions $\{g_n^\pm\}_{n=0}^N$ (independent of $\mathbf{z}_0$ for $n \ge 1$) and constants C_N, $\{d_n^\pm\}_{n=1}^N$, determined by $\mathbf{z}_0$, such that*

$$
\begin{aligned}
a(x,t) + b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^+\left(\tfrac{x+t}{\sqrt{1+t}}\right) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^+ g_n^+\left(\tfrac{x+t}{\sqrt{1+t}}\right) + R_u^N(x,t), \\
a(x,t) - b(x,t) &= \frac{1}{\sqrt{1+t}} g_0^-\left(\tfrac{x-t}{\sqrt{1+t}}\right) + \sum_{n=1}^{N} \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^- g_n^-\left(\tfrac{x-t}{\sqrt{1+t}}\right) + R_v^N(x,t),
\end{aligned} \tag{3.1}
$$

where the remainders R_u^N and R_v^N satisfy the estimates

$$
\begin{aligned}
\sup_{t \ge 0} (1+t)^{\frac{3}{4} - \frac{1}{2^{N+2}}} \left\| R_{\{u,v\}}^N(\cdot, t) \right\|_{\mathrm{L}^2(\mathbb{R})} &\le C_N, \\
\sup_{t \ge 0} (1+t)^{\frac{5}{4} - \frac{1}{2^{N+2}}} \left\| \partial_x R_{\{u,v\}}^N(\cdot, t) \right\|_{\mathrm{L}^2(\mathbb{R})} &\le C_N.
\end{aligned} \tag{3.2}
$$

Furthermore, for $n \ge 1$, the functions $g_n^\pm$ satisfy $g_n^\pm(z) \sim |z|^{-2+\frac{1}{2^n}}$ as $z \to \pm\infty$.

Remark 9 *As will be apparent from the proof of Theorem 8, any hyperbolic-parabolic system of the form*

$$
\partial_t \mathbf{z} + f(\mathbf{z})_x = (B(\mathbf{z}) \mathbf{z}_x)_x
$$

with admissible nonlinearities in the sense of (the natural extension of) Definition 1 gives rise to solutions that have the same asymptotic structure as those of the p-system as long as the following two conditions are satisfied:

1. *There exist two matrices $\mathcal{S}$ and A, with $\mathcal{S}$ non-singular and A diagonal and with eigenvalues of multiplicity 1, for which $\mathcal{S}\mathrm{e}^{\mathrm{L}t} \approx \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}$ in the sense of Lemma 19 (see Section 7), where $\mathrm{L}_0 = \partial_x^2 + \mathrm{A}\partial_x$ and $\mathrm{L} = B(0)\partial_x^2 - f'(0)\partial_x$.*

2. *The Cauchy problem with initial conditions in the corresponding function space (the natural extension of $\mathcal{B}_0$ to the problem considered) is well posed and satisfies the analogues of Theorem 6 and Proposition 7.*

We now briefly comment on the above assumptions for specific systems such as the 'full gas dynamics' and the MHD system. The intertwining property of item 1 above is proven in [6] for quite general systems, though not in exactly the same topology as that used in Lemma 19. As for item 2, local well-posedness for initial data in $\mathcal{B}_0$ is certainly not an issue, the only difficulty is to prove that the various norms of Definition 5 exhibit 'parabolic-like' decay as $t \to \infty$. This is very likely to hold, particularly for systems satisfying item 1.

While the variables (a, b) are adapted to the study of the Cauchy problem because of the inherent asymmetry of spatial regularity in (1.1), they are not the best framework for studying the asymptotic structure of the solutions to (1.1). It turns out to be more convenient to change variables to quantities that move along the characteristics. We thus define

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{T}^{-1} & 0 \\ 0 & \mathcal{T} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a(x,t) \\ b(x,t) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{T}^{-1} & 0 \\ 0 & \mathcal{T} \end{pmatrix} \mathcal{S}\mathbf{z}(x,t),$$

where $\mathcal{T}$ is the translation operator defined by

$$(\mathcal{T}f)(x,t) = f(x+t,t) \quad \text{or, equivalently, by} \quad \widehat{\mathcal{T}f}(k,t) = \mathrm{e}^{ikt}\hat{f}(k,t). \tag{3.3}$$

Note in passing that

$$a(x,t) = \frac{1}{2}\left(u(x+t,t) + v(x-t,t)\right) \quad \text{and} \quad b(x,t) = \frac{1}{2}\left(u(x+t,t) - v(x-t,t)\right).$$

We then use the fact that $\mathbf{z}$ satisfies the integral equation

$$\begin{aligned} \mathcal{S}\mathbf{z}(t) &= \mathcal{S}\mathrm{e}^{\mathrm{L}t}\mathbf{z}_0 + \int_0^t \mathrm{d}s\, \mathcal{S}\mathrm{e}^{\mathrm{L}(t-s)} \begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix} \\ &= \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\mathbf{z}_0 + \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{L}_0(t-s)}\mathcal{S} \begin{pmatrix} 0 \\ \partial_x g_0(\mathbf{z}(s)) \end{pmatrix} + \mathcal{R}[\mathbf{z}](t), \end{aligned} \tag{3.4}$$

where

$$\mathcal{R}[\mathbf{z}](t) = \left(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right)\mathbf{z}_0 + \int_0^t \mathrm{d}s \left[\mathcal{S}\mathrm{e}^{\mathrm{L}(t-s)} \begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix} - \mathrm{e}^{\mathrm{L}_0(t-s)}\mathcal{S} \begin{pmatrix} 0 \\ \partial_x g_0(\mathbf{z}(s)) \end{pmatrix}\right].$$

To justify the notation, which suggests that $\mathcal{R}[\mathbf{z}] = (\mathcal{R}_u[\mathbf{z}], \mathcal{R}_v[\mathbf{z}])$ is a remainder term, we will prove in Section 7 that $\mathcal{R}$ satisfies the improved decay rates

$$\|\mathcal{R}_{\{u,v\}}[\mathbf{z}]\|_{2,\frac{3}{4}^\star} + \|\mathrm{D}\mathcal{R}_{\{u,v\}}[\mathbf{z}]\|_{2,\frac{5}{4}^\star} \le C\epsilon_0, \tag{3.5}$$

which follow from the intertwining relation $\mathcal{S}\mathrm{e}^{\mathrm{L}t} \approx \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}$ (see Lemma 19) and the fact that $h(\mathbf{z}) = g_0(\mathbf{z}) + h.o.t.$

Recalling that g_0 is quadratic (cf. Definition 1), we will write

$$\begin{aligned} g_0(\mathbf{z}) &= c_+(a+b)^2 - c_-(a-b)^2 + c_3(a+b)(a-b) \\ &= c_+(\mathcal{T}u)^2 - c_-(\mathcal{T}^{-1}v)^2 + c_3(\mathcal{T}u)(\mathcal{T}^{-1}v) \end{aligned}$$

for $\mathbf{z} = (a, b)$. We thus find from (3.4) that u and v satisfy

$$\begin{aligned} u(t) &= \mathrm{e}^{\partial_x^2 t}(a_0 + b_0) + \partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} \left(c_+ u(s)^2 - c_- \mathcal{T}^{-2} v(s)^2\right) \\ &\quad + \mathcal{T}^{-1}\mathcal{R}_u[\mathbf{z}](t) + c_3\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)}\mathcal{T}^{-1}\left((\mathcal{T}u(s))(\mathcal{T}^{-1}v(s))\right), \end{aligned} \tag{3.6}$$

$$\begin{aligned} v(t) &= \mathrm{e}^{\partial_x^2 t}(a_0 - b_0) + \partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} \left(c_- v(s)^2 - c_+ \mathcal{T}^{2} u(s)^2\right) \\ &\quad + \mathcal{T}\mathcal{R}_v[\mathbf{z}](t) - c_3\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)}\mathcal{T}\left((\mathcal{T}u(s))(\mathcal{T}^{-1}v(s))\right). \end{aligned} \tag{3.7}$$

Note that, but for the presence of the second lines in (3.6) and (3.7), these expressions are precisely Duhamel's formula for the solution of the model problem (1.7), written in terms of $u = \mathcal{T}^{-1}u_+$ and $v = \mathcal{T}u_-$. The next step is to write

$$u = u_\star + R_u^N = u_0 + u_1 + R_u^N \quad \text{and} \quad v = v_\star + R_v^N = v_0 + v_1 + R_v^N,$$

considering R_u^N and R_v^N as new 'unknowns' and

$$\begin{aligned} u_0(x,t) &= \frac{1}{\sqrt{1+t}} g_0^+\big(\tfrac{x}{\sqrt{1+t}}\big), & u_1(x,t) &= \sum_{n=1}^N \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^+ g_n^+\big(\tfrac{x}{\sqrt{1+t}}\big), \\ v_0(x,t) &= \frac{1}{\sqrt{1+t}} g_0^-\big(\tfrac{x}{\sqrt{1+t}}\big), \quad \text{and} & v_1(x,t) &= \sum_{n=1}^N \frac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} d_n^- g_n^-\big(\tfrac{x}{\sqrt{1+t}}\big) \end{aligned} \tag{3.8}$$

for some coefficients $\{d_n^\pm\}_{n=1}^N$ and functions $\{g_n^\pm\}_{n=0}^N$, to be determined later on.

We now use

$$\begin{aligned} u^2 &= (u-u_\star)(u+u_\star) + u_\star^2 = R_u^N(u+u_\star) + u_1^2 + 2u_0u_1 + u_0^2, \\ v^2 &= (v-v_\star)(v+v_\star) + v_\star^2 = R_v^N(v+v_\star) + v_1^2 + 2v_0v_1 + v_0^2, \\ (\mathcal{T}u)(\mathcal{T}^{-1}v) &= (\mathcal{T}R_u^N)\mathcal{T}^{-1}\Big(\frac{v+v_\star}{2}\Big) + (\mathcal{T}^{-1}R_v^N)\mathcal{T}\Big(\frac{u+u_\star}{2}\Big) + (\mathcal{T}u_\star)(\mathcal{T}^{-1}v_\star). \end{aligned}$$

Since

$$\begin{aligned} g_0^+(x) &= u_0(x,0), & u_1(x,0) &= \sum_{n=1}^N d_n^+ g_n^+(x), \\ g_0^-(x) &= v_0(x,0), \quad \text{and} & v_1(x,0) &= \sum_{n=1}^N d_n^- g_n^-(x), \end{aligned}$$

we find that R_u^N and R_v^N satisfy

$$\begin{aligned} R_u^N(t) &= \mathrm{e}^{\partial_x^2 t}(a_0 + b_0 - g_0^+) \\ &+ \Big[\mathrm{e}^{\partial_x^2 t} u_0(0) + c_+\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} u_0(s)^2\Big] - u_0(t) \\ &+ \Big[\mathrm{e}^{\partial_x^2 t} u_1(0) + 2c_+\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} u_0(s)u_1(s)\Big] - u_1(t) \\ &- c_-\Big[\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} \mathcal{T}^{-2}\left(v_0(s)^2 + 2v_0(s)v_1(s)\right)\Big] - \sum_{n=1}^N \mathrm{e}^{\partial_x^2 t} d_n^+ g_n^+ \\ &+ \widetilde{\mathcal{R}}_u[\mathbf{z}, \mathbf{R}^N](t) + \mathcal{T}^{-1}\mathcal{R}_u[\mathbf{z}](t), \end{aligned} \tag{3.9}$$

$$\begin{aligned} R_v^N(t) &= \mathrm{e}^{\partial_x^2 t}(a_0 - b_0 - g_0^-) \\ &+ \Big[\mathrm{e}^{\partial_x^2 t} v_0(0) + c_-\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} v_0(s)^2\Big] - v_0(t) \\ &+ \Big[\mathrm{e}^{\partial_x^2 t} v_1(0) + 2c_-\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} v_0(s)v_1(s)\Big] - v_1(t) \\ &- c_+\Big[\partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)} \mathcal{T}^{2}\left(u_0(s)^2 + 2u_0(s)u_1(s)\right)\Big] - \sum_{n=1}^N \mathrm{e}^{\partial_x^2 t} d_n^- g_n^- \\ &+ \widetilde{\mathcal{R}}_v[\mathbf{z}, \mathbf{R}^N](t) + \mathcal{T}\mathcal{R}_v[\mathbf{z}](t), \end{aligned} \tag{3.10}$$

where

$$\widetilde{\mathcal{R}}_u[\mathbf{z},\mathbf{R}^N](t) = c_+ \mathrm{E}_0[h_{1,u}+h_{3,u}](t) - c_- \mathrm{E}_{-2}[h_{1,v}+h_{3,v}](t) + c_3 \mathrm{E}_{-1}[h_2+h_4](t),$$
$$\widetilde{\mathcal{R}}_v[\mathbf{z},\mathbf{R}^N](t) = c_- \mathrm{E}_0[h_{1,v}+h_{3,v}](t) - c_+ \mathrm{E}_{2}[h_{1,u}+h_{3,u}](t) - c_3 \mathrm{E}_{1}[h_2+h_4](t),$$

with $\mathbf{R}^N = (R_u^N, R_v^N)$,

$$\mathrm{E}_\sigma[h](t) = \partial_x \int_0^t \mathrm{d}s \, \mathrm{e}^{\partial_x^2 (t-s)} \, \mathcal{T}^\sigma h(s),$$

and

$$h_{1,u} = R_u^N(u+u_\star), \quad h_{3,u} = u_1^2, \quad h_2 = (\mathcal{T} R_u^N)\mathcal{T}^{-1}\Big(\frac{v+v_\star}{2}\Big) + (\mathcal{T}^{-1} R_v^N)\mathcal{T}\Big(\frac{u+u_\star}{2}\Big),$$
$$h_{1,v} = R_v^N(v+v_\star), \quad h_{3,v} = v_1^2, \quad h_4 = (\mathcal{T} u_\star)(\mathcal{T}^{-1} v_\star).$$

Note that we can write (3.9) and (3.10) as $\mathbf{R}^N = \mathcal{F}[\mathbf{z},\mathbf{R}^N]$. If we now consider $\mathbf{z}$ fixed, we can interpret $\mathbf{R}^N = \mathcal{F}[\mathbf{z},\mathbf{R}^N]$ as an equation for $\mathbf{R}^N$ which can be solved via a contraction-mapping argument. Namely, we will prove that if $\|\mathbf{z}\| \le C\epsilon_0$, $\mathbf{R}^N \mapsto \mathcal{F}[\mathbf{z},\mathbf{R}^N]$ defines a contraction map inside the ball

$$\|R_u^N\|_{2,\frac{3}{4}-\epsilon} + \|\mathrm{D}R_u^N\|_{2,\frac{5}{4}-\epsilon} + \|R_v^N\|_{2,\frac{3}{4}-\epsilon} + \|\mathrm{D}R_v^N\|_{2,\frac{5}{4}-\epsilon} \le C \tag{3.11}$$

for $\epsilon = 2^{-N-2}$, provided $\{g_n^\pm\}_{n=0}^N$ and $\{d_n^\pm\}_{n=1}^N$ are appropriately chosen.

Basically, we will choose u_0, v_0, u_1, and v_1 in such a way that the second and third lines in (3.9) and (3.10) vanish. Note that if, for instance, we set the second (respectively, third) lines of (3.9) and (3.10) equal to zero, the resulting equalities are nothing but Duhamel's formulae for Burger's equations for u_0 and v_0 (respectively, for linearized Burger's equations for u_1 and v_1). Properties of solutions to these types of equations are studied in detail in Section 4 below.

Once u_0, v_0, u_1, and v_1 are fixed, the time convolutions in the fourth lines of (3.9) and (3.10) can then be viewed as the solution of inhomogeneous heat equations with very specific inhomogeneous terms. Properties of solutions to this type of equations are studied in detail in Section 5 below.

Assuming all results of Sections 4 and 5, we now explain how to proceed to prove that $\mathcal{F}[\mathbf{z},\mathbf{R}^N]$ defines a contraction map.

Obviously, the requirement on $\{g_n^\pm\}_{n=0}^N$ and $\{d_n^\pm\}_{n=1}^N$ is that the first four lines in (3.9) and (3.10) satisfy (3.11). This is achieved in the following way:

1. The first line of (3.9) (respectively, of (3.10)) satisfies (3.11) for any $g_0^\pm$ such that the total mass of $g_0^\pm$ is equal to that of $a_0 \pm b_0$, provided $a_0 \pm b_0$ and $g_0^\pm$ satisfy $\|\, x^2(a_0 \pm b_0)\|_2 < \infty$ and $\|\, x^2 g_0^\pm\|_2 < \infty$. This fixes the total mass of $g_0^\pm$. Note also that we need the estimate $\|\, x^2(a_0 \pm b_0)\|_2 < \infty$. There is no smallness assumption here, which is to be expected, since, generically, $\|\, x^2(a(\cdot,t) \pm b(\cdot,t))\|_2$ will grow as $t \to \infty$. Note, on the other hand, that Proposition 7 shows that $\|\, x^2(a(\cdot,t) \pm b(\cdot,t))\|_2$ remains finite for all $t < \infty$; thus, requiring $\|\, x^2(a_0 \pm b_0)\|_2 < \infty$ is acceptable.

2. We can set the second lines in (3.9) and (3.10) equal to zero by picking for u_0 and v_0 any solution of Burger's equations

$$\partial_t u_0 = \partial_x^2 u_0 + c_+ \partial_x (u_0)^2 \quad \text{and} \quad \partial_t v_0 = \partial_x^2 v_0 + c_- \partial_x (v_0)^2$$

(or of the corresponding heat equations if either c_+ or c_- happen to be zero). In Proposition 12, we will prove that there exist unique functions u_0 and v_0 of the form given in (3.8) that satisfy the conditions of item 1 above (total mass and decay properties). This uniquely determines u_0 and v_0.

3. We can also set the third lines in (3.9) and (3.10) equal to zero, by picking any solutions u_1 and v_1 of the linearized Burger's equations

$$\partial_t u_1 = \partial_x^2 u_1 + 2c_+ \partial_x (u_0 u_1) \quad \text{and} \quad \partial_t v_1 = \partial_x^2 v_1 + 2c_- \partial_x (v_0 v_1). \tag{3.12}$$

In Proposition 12, we will also prove that there is a choice of functions $\{g_n^{\pm}\}_{n=1}^N$ such that u_1 and v_1 in (3.8) satisfy (3.12) for any choice of the coefficients $\{d_n^{\pm}\}_{n=1}^N$. Furthermore, in Proposition 12, we will show that the choice of functions can be made in such a way that $g_n^{\pm}(x)$ have Gaussian tails as $x \to \mp\infty$ and algebraic tails as $x \to \pm\infty$, which actually completely determines $g_n^{\pm}(x)$, up to multiplicative constants. (This last indeterminacy will be removed when the coefficients $\{d_n^{\pm}\}_{n=1}^N$ are fixed.)

4. We then further decompose the terms involving $g_n^{\pm}$ in the fourth lines in (3.9) and (3.10) as $g_n^{\pm}(x) = f_n(\mp x) + R_n^{\pm}(x)$. The definition and properties of $f_n(x)$ are given in Lemma 10. In particular, in Proposition 12, we will prove that $R_n^{\pm}(x)$ have zero total mass and Gaussian tails as $|x| \to \infty$, which implies that $\mathrm{e}^{\partial_x^2 t} R_n^{\pm}$ also satisfy (3.11).

5. Finally, in Section 5, we will show that the time convolution part of the fourth lines in (3.9) and (3.10) can be split into linear combinations of $\mathrm{e}^{\partial_x^2 t} f_n(\mp x)$ with $n = 1 \dots N+1$, plus a remainder that satisfies (3.11). The coefficients $\{d_n^{\pm}\}_{n=1}^N$ can then be defined recursively by requiring that all the terms with $n = 1 \dots N$ coming from the time convolution are canceled by those coming from item 4 above. This can always be done, as the coefficient of $\mathrm{e}^{\partial_x^2 t} f_m(\mp x)$ in the time convolution part in the fourth lines in (3.9) and (3.10) depends only on $g_0^{\pm}$ if $m = 1$ and on $d_{m-1}^{\pm}$ if $m > 1$. The only term that cannot be set to zero is the last term in the linear combination (the one with $n = N+1$), which is the one that 'drives' the equations and fixes $\epsilon = 2^{-N-2}$.

The procedure outlined in 1–5 takes care of the first four lines in (3.9) and (3.10). We will then prove in Section 7 that the terms $\mathcal{R}_{\{u,v\}}[\mathbf{z}]$ satisfy (3.11) and that

$$\sum_{\alpha=0}^{1} \left\| \mathrm{D}^{\alpha} \widetilde{\mathcal{R}}_{\{u,v\}}[\mathbf{z}, \mathbf{R}^N] \right\|_{2, \frac{3}{4}+\frac{\alpha}{2}-\epsilon} \leq C\epsilon_0 \sum_{\alpha=0}^{1} \|\mathrm{D}^{\alpha} \mathbf{R}^N\|_{2, \frac{3}{4}+\frac{\alpha}{2}-\epsilon} + C, \tag{3.13}$$

$$\sum_{\alpha=0}^{1} \left\| \mathrm{D}^{\alpha} \big(\widetilde{\mathcal{R}}_{\{u,v\}}[\mathbf{z}, \mathbf{R}_1^N] - \widetilde{\mathcal{R}}_{\{u,v\}}[\mathbf{z}, \mathbf{R}_2^N] \big) \right\|_{2, \frac{3}{4}+\frac{\alpha}{2}-\epsilon} \leq C\epsilon_0 \sum_{\alpha=0}^{1} \|\mathrm{D}^{\alpha} (\mathbf{R}_1^N - \mathbf{R}_2^N)\|_{2, \frac{3}{4}+\frac{\alpha}{2}-\epsilon}. \tag{3.14}$$

This finally proves that $\mathcal{F}[\mathbf{z}, \mathbf{R}^N]$ defines a contraction map and that the solution of $\mathbf{R}^N = \mathcal{F}[\mathbf{z}, \mathbf{R}^N]$ satisfies (3.11), which completes the proof of Theorems 2 and 8. ■

4 Burger-type equations

In this section, we consider particular solutions of Burger-type equations

$$\partial_t u_0 = \partial_x^2 u_0 + \gamma \partial_x u_0^2, \tag{4.1}$$
$$\partial_t u_n^{\pm} = \partial_x^2 u_n^{\pm} + 2\gamma \partial_x (u_0 u_n^{\pm}) \tag{4.2}$$

of the form

$$u_0(x,t) = \tfrac{1}{\sqrt{1+t}} g_0\big(\tfrac{x}{\sqrt{1+t}}\big) \quad \text{and} \quad u_n^{\pm}(x,t) = \tfrac{1}{(1+t)^{1-\frac{1}{2^{n+1}}}} g_n^{\pm}\big(\tfrac{x}{\sqrt{1+t}}\big). \tag{4.3}$$

We will show that, for fixed $\mathrm{M}(u_0) = \int_{-\infty}^{\infty} \mathrm{d}x \; u_0(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x \; g_0(x)$ small enough, there is a unique choice of g_0 and $g_n^{\pm}$ such that $g_n^{\pm}(x) = f_n(\mp x) + R_n^{\pm}(x)$, where

$$f_n(z) = \int_z^{\infty} \mathrm{d}\xi \; \frac{\xi \mathrm{e}^{-\frac{\xi^2}{4}}}{(\xi - z)^{1-\frac{1}{2^n}}} \tag{4.4}$$

and $R_n^{\pm}$ has zero mean and Gaussian tails as $|x| \to \infty$. In particular, $g_n^{\pm}(x)$ decays algebraically as $x \to \pm\infty$, as is apparent from (4.4).

Before proceeding to our study of (4.1) and (4.2), we prove key properties of the functions f_n.

Lemma 10 *Fix $1 \le n < \infty$. The function f_n is the unique solution of*

$$\begin{aligned}
&\partial_z^2 f_n(z) + \tfrac{1}{2} z \partial_z f_n(z) + \big(1 - \tfrac{1}{2^{n+1}}\big) f_n(z) = 0, \quad \textit{with} \\
&f_n(0) = 2^{\frac{1}{2^n}} \Gamma\big(\tfrac{1+2^{-n}}{2}\big) \quad \textit{and} \quad \lim_{z\to\infty} z^{-1+\frac{1}{2^n}} \mathrm{e}^{\frac{z^2}{4}} f_n(z) < \infty.
\end{aligned} \tag{4.5}$$

Moreover, f_n satisfies $\int_{-\infty}^{\infty} \mathrm{d}z\, f_n(z) = 0$, and there exists a constant $C(n)$ such that

$$\begin{aligned}
\sup_{z\in\mathbf{R}} \sum_{m=0}^{2} \rho_{\frac{1}{2^n}-m, 1+m-\frac{1}{2^n}}(z) |\partial_z^m \left(z f_n(z) + 2\partial_z f_n(z)\right)| &\le C(n), \\
\sup_{z\in\mathbf{R}} \sum_{m=0}^{3} \rho_{\frac{1}{2^n}-1-m, 2+m-\frac{1}{2^n}}(z) |\partial_z^m f_n(z)| &\le C(n),
\end{aligned} \tag{4.6}$$

where

$$\rho_{p,q}(z) = \begin{cases} (1+z^2)^{\frac{p}{2}} \mathrm{e}^{\frac{z^2}{4}} & \textit{if} \quad z \ge 0, \\ (1+z^2)^{\frac{q}{2}} & \textit{if} \quad z \le 0. \end{cases}$$

Proof. We first note that f_n can be written as

$$f_n(z) = \int_0^\infty \mathrm{d}\xi\, \frac{(\xi+z)\mathrm{e}^{-\frac{(\xi+z)^2}{4}}}{\xi^{1-\frac{1}{2^n}}} = -2 \int_0^\infty \mathrm{d}\xi\, \xi^{\frac{1}{2^n}-1} \partial_\xi \big(\mathrm{e}^{-\frac{(z+\xi)^2}{4}}\big). \tag{4.7}$$

This shows that f_n solves (4.5), since, by defining $\mathcal{L}f \equiv \partial_z^2 f + \frac{1}{2} z \partial_z f + (1 - \frac{1}{2^{n+1}}) f$, we find

$$\mathcal{L} f_n(z) = \int_0^\infty \mathrm{d}\xi \left[\xi^{\frac{1}{2^n}} \partial_\xi^2 \big(\mathrm{e}^{-\frac{(z+\xi)^2}{4}}\big) - \tfrac{1}{2^{n+1}}(-2) \xi^{\frac{1}{2^n}-1} \partial_\xi \big(\mathrm{e}^{-\frac{(z+\xi)^2}{4}}\big) \right] = 0.$$

As $f_n(z)$ is obviously finite for all finite z, we only need to prove that f_n satisfies the correct decay properties as $|z| \to \infty$ so that (4.6) holds. It is apparent from (4.4) that f_n decays like a (modified) Gaussian as $z \to \infty$ and algebraically as $z \to -\infty$. Furthermore, substituting $f(z) = C|z|^{p_1}$ and $f(z) = C|z|^{p_2} \mathrm{e}^{-\frac{z^2}{4}}$ into $\mathcal{L}f = 0$ shows that the only decay rates compatible with $\mathcal{L}f = 0$ are $p_1 = -2 + \frac{1}{2^n}$ and $p_2 = 1 - \frac{1}{2^n}$.

We now complete the proof of the decay estimates in (4.6). Let $F_{n,m}(\xi, z) = \partial_z^m ((\xi+z)\mathrm{e}^{-\frac{(\xi+z)^2}{4}})$ and $G_{n,m}(\xi, z) = \partial_z^m (z F_n(\xi, z) + 2\partial_z F_n(\xi, z))$.

We first consider the case when $z > 0$ and note that $F_{n,m}$ and $G_{n,m}$ satisfy

$$|F_{n,m}(\xi, z)| \le |F_{n,m}(0, z)| \quad \text{and} \quad |G_{n,m}(\xi, z)| \le |G_{n,m}(0, z)|$$

for all $\xi \ge 0$ if $z \ge z_0$ for some z_0 large enough. We thus find, e.g.,

$$|f_n(z)| = \left| \int_0^\infty \mathrm{d}\xi\, F_{n,0}(\xi, z) \xi^{\frac{1}{2^n}-1} \right| \le |F_{n,0}(0, z)| \int_0^{z^{-1}} \mathrm{d}\xi\, \xi^{\frac{1}{2^n}-1} + z^{1-\frac{1}{2^n}} \int_{z^{-1}}^\infty \mathrm{d}\xi\, |F_{n,0}(\xi, z)| \le C z^{1-\frac{1}{2^n}} \mathrm{e}^{-\frac{z^2}{4}}.$$

The estimates on $|\partial_z^m (z f_n(z) + 2\partial_z f_n(z))|$ and $|\partial_z^{1+m} f_n(z)|$ for $z > 0$ and $m \ge 1$ can be obtained in exactly the same way; hence, we omit the details.

We now consider the case when $z < 0$ and note that $F_{n,m}$ and $G_{n,m}$ satisfy

$$|F_{n,m}(\xi, z)| \le \left|F_{n,m}\left(-\tfrac{z}{2}, z\right)\right| \quad \text{and} \quad |G_{n,m}(\xi, z)| \le \left|G_{n,m}\left(-\tfrac{z}{2}, z\right)\right|,$$

respectively, for all $0 \le \xi \le -\frac{z}{2}$ if $z \le -z_0$ for some z_0 large enough. We thus have (integrating by parts in the second integral below)

$$|f_n(z)| = \left| \int_0^\infty \mathrm{d}\xi\, F_{n,0}(\xi, z) \xi^{\frac{1}{2^n}-1} \right| \le \left|F_{n,0}\left(-\tfrac{z}{2}, z\right)\right| \int_0^{-\frac{z}{2}} \mathrm{d}\xi\, \xi^{\frac{1}{2^n}-1} + \left| \int_{-\frac{z}{2}}^\infty \mathrm{d}\xi\, F_{n,0}(\xi, z) \xi^{\frac{1}{2^n}-1} \right|$$

$$\leq C|z|^{\frac{1}{2^n}-1}\mathrm{e}^{-\frac{z^2}{16}} + 2\left(1-\tfrac{1}{2^n}\right)\int_{-\frac{z}{2}}^{\infty}\mathrm{d}\xi\, \mathrm{e}^{-\frac{(\xi+z)^2}{4}}\xi^{\frac{1}{2^n}-2} \leq C|z|^{\frac{1}{2^n}-2}.$$

Since the remaining estimates can again be obtained in exactly the same way, we omit the details. It only remains to show that $f_n(z)$ has zero total mass; this follows from

$$\int_{-\infty}^{\infty}\mathrm{d}z\, f_n(z) = \left(\tfrac{1}{2}-\tfrac{1}{2^{n+1}}\right)^{-1}\int_{-\infty}^{\infty}\mathrm{d}z\,\mathcal{L}f_n(z) = 0,$$

since $\partial_z^2 f_n, z\partial_z f_n$, and f_n are all integrable over $\mathbf{R}$. ∎

Remark 11 *Using the representation in (4.7), splitting the integration interval into* $[0, 2^{-\frac{n}{2}})$ *and* $[2^{-\frac{n}{2}}, \infty)$*, integrating by parts, and letting* $n \to \infty$*, one can prove that*

$$\lim_{n\to\infty} 2^{-n} f_n(z) = z\mathrm{e}^{-\frac{z^2}{4}},$$

which shows that the constant $C(n)$ *in (4.6) grows at most like* 2^n.

We can now study in detail the solutions of (4.1) and (4.2) that are of the form (4.3).

Proposition 12 *Fix* $1 \leq n < \infty$. *For all* $\alpha, \gamma \in \mathbf{R}$ *with* $|\alpha\gamma|$ *small enough, there exist unique functions* u_0 *and* $u_n^{\pm}$ *of the form (4.3) that solve (4.1) and (4.2), with* g_0 *satisfying*

$$\int_{-\infty}^{\infty}\mathrm{d}z\, g_0(z) = \alpha \quad \textit{and} \quad \sum_{m=0}^{3}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^m}|\partial_z^m g_0(z)| \leq C|\alpha|$$

and with $g_n^{\pm}(z) = f_n(\mp z) + R_n^{\pm}(z)$*, where* $R_n^{\pm}$ *satisfy*

$$\int_{-\infty}^{\infty}\mathrm{d}z\, R_n^{\pm}(z) = 0 \quad \textit{and} \quad \sup_{z\in\mathbf{R}}\sum_{m=0}^{3}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^{1+m-\frac{1}{2^n}}}|\partial_z^m R_n^{\pm}(z)| \leq C|\alpha\gamma|.$$

Proof. The (unique) solution of (4.1) of the form $u_0(x,t) = \frac{1}{\sqrt{1+t}}g_0(\frac{x}{\sqrt{1+t}})$ that satisfies $\int_{-\infty}^{\infty}\mathrm{d}z\, g_0(z) = \alpha$ is given by

$$g_0(z) = \frac{\tanh(\frac{\alpha\gamma}{2})\mathrm{e}^{-\frac{z^2}{4}}}{\gamma\sqrt{\pi}(1+\tanh(\frac{\alpha\gamma}{2})\mathrm{erf}(\frac{z}{2}))}.$$

In particular, we have

$$\sum_{m=0}^{3}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^m}|\partial_z^m g_0(z)| \leq C|\alpha|. \tag{4.8}$$

Next, we note that substitution of (4.3) into (4.2) gives

$$\begin{aligned} 0 &= \partial_z^2 g_n^{\pm}(z) + \tfrac{1}{2}z\partial_z g_n^{\pm}(z) + (1-\tfrac{1}{2^{n+1}})g_n^{\pm}(z) + 2\gamma\partial_z(g_0(z)g_n^{\pm}(z)) \\ &\equiv \mathcal{L}g_n^{\pm}(z) + 2\gamma\partial_z(u_0(z)g_n^{\pm}(z)). \end{aligned} \tag{4.9}$$

Formally (using integration by parts), we find

$$\int_{-\infty}^{\infty}\mathrm{d}z\, g_n^{\pm}(z) = \left(\tfrac{1}{2}-\tfrac{1}{2^{n+1}}\right)^{-1}\int_{-\infty}^{\infty}\mathrm{d}z\,\mathcal{L}g_n^{\pm}(z) + 2\gamma\partial_z(u_0(z)g_n^{\pm}(z)) = 0, \tag{4.10}$$

which shows that $g_n^\pm$ have zero total mass, *provided the formal manipulations above are justified*, i.e., provided $g_n^\pm$ and its derivatives decay fast enough so that the integrals are convergent.

As is easily seen, $f_n(z)$ and $f_n(-z)$ are two linearly independent solutions of $\mathcal{L}f = 0$, whose general solution can thus be written as $c_1 f_n(z) + c_2 f_n(-z)$. Using the variation of constants formula, we find that the solution of (4.9) satisfies the integral equation

$$g_n^\pm(z) = f_n(z)\left(c_1^\pm + 2\gamma\int_0^z \mathrm{d}\xi\ \tfrac{f_n(-\xi)\partial_\xi(g_0(\xi)g_n^\pm(\xi))}{W(\xi)}\right) + f_n(-z)\left(c_2^\pm - 2\gamma\int_0^z \mathrm{d}\xi\ \tfrac{f_n(\xi)\partial_\xi(g_0(\xi)g_n^\pm(\xi))}{W(\xi)}\right),$$

where the Wronskian $W(z)$ is given by $W(z) = f_n(z)\partial_z f_n(-z) - f_n(-z)\partial_z f_n(z)$ and $c_1^\pm$ and $c_2^\pm$ are free parameters. Note that $W(z)$ satisfies $\partial_z W(z) = -\frac{z}{2}W(z)$ and, hence, $W(z) = W(0)\mathrm{e}^{-\frac{z^2}{4}}$ for some $W(0) \neq 0$. We now define $c_1^\pm$ and $c_2^\pm$ in such a way that (after integration by parts), we have

$$\begin{aligned} g_n^\pm(z) &= f_n(\mp z) + R[g_n^\pm](z), \\ R[g_n^\pm](z) &= \tfrac{\gamma}{W(0)} f_n(z)\int_{-\infty}^z \mathrm{d}\xi\ \mathrm{e}^{\frac{\xi^2}{4}}(\xi f_n(-\xi) + 2\partial_\xi f_n(-\xi))g_0(\xi)g_n^\pm(\xi) \\ &\quad + \tfrac{\gamma}{W(0)} f_n(-z)\int_z^\infty \mathrm{d}\xi\ \mathrm{e}^{\frac{\xi^2}{4}}(\xi f_n(\xi) + 2\partial_\xi f_n(\xi))g_0(\xi)g_n^\pm(\xi). \end{aligned} \tag{4.11}$$

By using Lemma 10 and (4.8), it is then easy to show that, for $|\alpha\gamma|$ small enough, (4.11) defines a contraction map in the norm

$$|f|_{2-\frac{1}{2^n}} \equiv \sup_{z\in\mathbf{R}}(\sqrt{1+z^2})^{2-\frac{1}{2^n}}|f(z)|.$$

Namely, we have the improved decay rates

$$\sup_{z\in\mathbf{R}}\sum_{m=0}^{1}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^{1+m-\frac{1}{2^n}}}|\partial_z^m R[g_n^\pm](z)| \le C|\alpha\gamma|\ |g_n^\pm|_{2-\frac{1}{2^n}}.$$

This shows that (4.11) has a (locally) unique solution among functions with $|f|_{2-\frac{1}{2^n}} \le c_0$ if $|\alpha\gamma|$ is small enough. In particular, there holds

$$\sup_{z\in\mathbf{R}}\sum_{m=0}^{1}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^{1+m-\frac{1}{2^n}}}|\partial_z^m R[g_n^\pm](z)| \le C|\alpha\gamma|,$$

from which we deduce, using again (4.11) and Lemma 10, that $|\mathrm{D}g_n^\pm|_{3-\frac{1}{2^n}} \le c_1$ and, thus, that

$$\sup_{z\in\mathbf{R}}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^{3-\frac{1}{2^n}}}|\partial_z^2 R[g_n^\pm](z)| \le C|\alpha\gamma|.$$

Iterating this procedure shows that $|\mathrm{D}^m g_n^\pm|_{2+m-\frac{1}{2^n}} \le c_m$ and that

$$\sup_{z\in\mathbf{R}}\sum_{m=0}^{3}\frac{\mathrm{e}^{\frac{z^2}{4}}}{(\sqrt{1+z^2})^{1+m-\frac{1}{2^n}}}|\partial_z^m R[g_n^\pm](z)| \le C|\alpha\gamma|,$$

as claimed. In turn, this proves that the formal manipulations in (4.10) are justified; hence, the functions $g_n^\pm(z)$ have zero total mass, which shows that the remainders $R[g_n^\pm](z)$ have zero total mass, as claimed, since $R[g_n^\pm](z) = g_n^\pm(z) - f_n(\pm z)$ and since both $g_n^\pm(z)$ and $f_n(z)$ have zero total mass. ■

5 Inhomogeneous heat equations

In this section, we consider solutions of inhomogeneous heat equations of the form

$$\partial_t u = \partial_x^2 u + \partial_x\big((1+t)^{\frac{1}{2^n}-\frac{3}{2}} f\big(\tfrac{x-2\sigma t}{\sqrt{1+t}}\big)\big), \qquad u(x,0)=0, \tag{5.1}$$

where f is a regular function having Gaussian decay at infinity. Solutions of (5.1) satisfy the following theorem.

Theorem 13 *Let $1 \le n < \infty$, $\sigma = \pm 1$, $\Xi(x) = \mathrm{e}^{\frac{x^2}{8}}$, $\mathrm{M}(f) = \int_{-\infty}^{\infty} \mathrm{d}z\, f(z)$, and*

$$u_n(x,t) = \frac{\sigma}{(1+t)^{1-\frac{1}{2^{n+1}}}} \frac{2^{-1-\frac{1}{2^n}}}{\sqrt{4\pi}} f_n\big(\tfrac{-\sigma x}{\sqrt{1+t}}\big), \quad \textit{with} \quad f_n(z) = \int_z^{\infty} \mathrm{d}\xi \, \frac{\xi \mathrm{e}^{-\frac{\xi^2}{4}}}{(\xi - z)^{1-\frac{1}{2^n}}}. \tag{5.2}$$

The solution u of (5.1) satisfies

$$\|u - \mathrm{M}(f)\, u_n\|_{2,\frac{3}{4}^\star} + \|\mathrm{D}(u - \mathrm{M}(f)\, u_n)\|_{2,\frac{5}{4}^\star} \le C \sum_{m=0}^{2} \|\Xi \mathrm{D}^m f\|_\infty \tag{5.3}$$

for all f such that the r.h.s. of (5.3) is finite.

Remark 14 *Note that, while $u \to \mathrm{M}(f) u_n$ as $t \to \infty$ in the Sobolev norm (5.3), it does not converge in spatially weighted norms such as $\mathrm{L}^2(\mathbf{R}, x^2 \mathrm{d}x)$, as u_n has infinite spatial moments for all times, while all moments of u are bounded for finite time.*

Proof. We first define

$$F(\xi) = \int_{-\infty}^{\xi} \mathrm{d}z \, \Big(f(z) - \mathrm{M}(f)\, \frac{\mathrm{e}^{-\frac{z^2}{4}}}{\sqrt{4\pi}}\Big), \quad \text{with} \quad \mathrm{M}(f) = \int_{-\infty}^{\infty} \mathrm{d}z\, f(z), \tag{5.4}$$

and note that F satisfies

$$\|\mathrm{D}^3 F\|_1 + \sum_{m=0}^{2} \|\rho \mathrm{D}^m F\|_1 + \sum_{m=1}^{2} \|\mathrm{D}^m F\|_2 \le C \sum_{m=0}^{2} \|\Xi \mathrm{D}^m f\|_\infty, \tag{5.5}$$

where $\rho(x) = \sqrt{1+x^2}$. Namely, we first note that $\|\rho F\|_1 \le \|\hat{F}\|_2 + \|\hat{F}''\|_2$ and $\hat{F}(k) = (ik)^{-1}(\hat{f}(k) - \hat{f}(0)\mathrm{e}^{-k^2})$. Then, $\hat{F}$ is regular near $k=0$, since $\|\Xi f\|_\infty < \infty$ implies that $\hat{f}$ is analytic. The proof of (5.5) now follows from elementary arguments.

Finally, it follows from (5.4) that

$$\begin{aligned} (1+t)^{\frac{1}{2^n}-\frac{3}{2}} f\big(\tfrac{x-2\sigma t}{\sqrt{1+t}}\big) &= \mathrm{M}(f)\; A(x,t) + \partial_x B(x,t), \quad \text{where} \\ A(x,t) &= \frac{(1+t)^{\frac{1}{2^n}-\frac{3}{2}}}{\sqrt{4\pi}} \mathrm{e}^{-\frac{(x-2\sigma t)^2}{4(1+t)}}, \\ B(x,t) &= (1+t)^{\frac{1}{2^n}-1} \partial_x F\big(\tfrac{x-2\sigma t}{\sqrt{1+t}}\big). \end{aligned} \tag{5.6}$$

The proof of (5.3) is then completed by considering separately the solutions of heat equations with inhomogeneous terms given by $\partial_x A(x,t)$ and $\partial_x^2 B(x,t)$. This is done in Propositions 15 and 16 below. ■

Proposition 15 *Let $\sigma = \pm 1$, $1 \le n < \infty$, and let u_n be defined as in (5.2). Then, the solution u of*

$$\partial_t u = \partial_x^2 u + \partial_x A, \qquad u(x,0) = 0, \tag{5.7}$$

with A defined as in (5.6), satisfies

$$\|u - u_n\|_{2,\frac{3}{4}} + \|\mathrm{D}(u - u_n)\|_{2,\frac{5}{4}} \le C. \tag{5.8}$$

Proof. The solution of (5.7) is given by

$$u(x,t)=\partial_x\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{\mathrm{e}^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\,\frac{\mathrm{e}^{-\frac{(y-2\sigma s)^2}{4(1+s)}}}{\sqrt{4\pi}(1+s)^{\frac{3}{2}-\frac{1}{2^n}}}. \tag{5.9}$$

To motivate our result, we note that, performing the y-integration and changing variables from s to $\xi\equiv\frac{2s-\sigma x}{\sqrt{1+t}}$ in (5.9), we find

$$\lim_{t\to\infty}(1+t)^{1-\frac{1}{2^{n+1}}}u(-\sigma z\sqrt{1+t},t)=\lim_{t\to\infty}\frac{\sigma 2^{-1-\frac{1}{2^n}}}{\sqrt{4\pi}}\int_z^{\frac{2t}{\sqrt{1+t}}+z}\mathrm{d}\xi\,\frac{\xi\mathrm{e}^{-\frac{\xi^2}{4}}}{(\xi-z+\frac{2}{\sqrt{1+t}})^{1-\frac{1}{2^n}}}=\frac{\sigma 2^{-1-\frac{1}{2^n}}}{\sqrt{4\pi}}f_n(z).$$

More formally, taking the Fourier transform of (5.9), we obtain

$$\hat{u}(k,t)=ik\mathrm{e}^{-k^2(1+t)}\int_0^t\mathrm{d}s\,\frac{\mathrm{e}^{2ik\sigma s}}{(1+s)^{1-\frac{1}{2^n}}}.$$

We now make use of

$$\left|\int_0^t\mathrm{d}s\,\frac{\mathrm{e}^{2ik\sigma s}}{(1+s)^{1-\frac{1}{2^n}}}-\int_0^t\mathrm{d}s\,\frac{\mathrm{e}^{2ik\sigma s}}{s^{1-\frac{1}{2^n}}}\right|\le C(n),$$
$$\int_0^t\mathrm{d}s\,\frac{\mathrm{e}^{2ik\sigma s}}{s^{1-\frac{1}{2^n}}}=|k|^{-\frac{1}{2^n}}\big(\theta(\sigma k)J_n(|k|t)+\theta(-\sigma k)\overline{J_n(|k|t)}\big),$$

where $\theta(k)$ is the Heaviside step function and we defined

$$J_n(z)=\int_0^z\mathrm{d}s\,\frac{\mathrm{e}^{2is}}{s^{1-\frac{1}{2^n}}}$$

for $z\ge 0$. The function J_n satisfies

$$\sup_{z\ge 0}z^{1-\frac{1}{2^n}}|J_n(z)-J_{n,\infty}|\le\frac{1}{2}\quad\text{for}\quad J_{n,\infty}=\lim_{z\to\infty}J_n(z).$$

Now, defining

$$\widehat{u_n}(k,t)=ik\mathrm{e}^{-k^2(1+t)}|k|^{-\frac{1}{2^n}}\left(\theta(\sigma k)J_{n,\infty}+\theta(-\sigma k)\overline{J_{n,\infty}}\right), \tag{5.10}$$

we have

$$|\hat{u}(k,t)-\widehat{u_n}(k,t)|\le\big(C(n)|k|+t^{-1+\frac{1}{2^n}}\big)\mathrm{e}^{-k^2(1+t)}\le\frac{\widetilde{C}(n)}{\sqrt{t}}\mathrm{e}^{-\frac{k^2(1+t)}{2}}, \tag{5.11}$$

from which (5.8) follows by direct integration. The proof is completed by showing that the inverse Fourier transform of the function $\widehat{u_n}(k,t)$ defined in (5.10) satisfies

$$u_n(x,t)=\frac{\sigma}{(1+t)^{1-\frac{1}{2^{n+1}}}}\frac{2^{-1-\frac{1}{2^n}}}{\sqrt{4\pi}}f_n\big(\tfrac{-\sigma x}{\sqrt{1+t}}\big)\quad\text{for}\quad f_n(z)=\int_z^\infty\mathrm{d}\xi\,\frac{\xi\mathrm{e}^{-\frac{\xi^2}{4}}}{(\xi-z)^{1-\frac{1}{2^n}}}. \tag{5.12}$$

This follows easily from the fact that

$$\widehat{u_n}(k,t)=(1+t)^{-\frac{1}{2}+\frac{1}{2^{n+1}}}\widehat{u_n}(k\sqrt{1+t},0),$$

and that, with

$$f_n(z)=\int_0^\infty\mathrm{d}\xi\,\frac{(z+\xi)\mathrm{e}^{-\frac{(z+\xi)^2}{4}}}{\xi^{1-\frac{1}{2^n}}},$$

we obtain

$$\tfrac{\sigma 2^{-1-\frac{1}{2^n}}}{\sqrt{4\pi}}\widehat{f_n}(-\sigma k) = 2^{-\frac{1}{2^n}}ik\mathrm{e}^{-k^2}\int_0^\infty \mathrm{d}\xi\,\frac{\mathrm{e}^{ik\sigma\xi}}{\xi^{1-\frac{1}{2^n}}} = ik\mathrm{e}^{-k^2}|k|^{-\frac{1}{2^n}}\int_0^\infty \mathrm{d}\xi\,\frac{\mathrm{e}^{2i\mathrm{sign}(k\sigma)\xi}}{\xi^{1-\frac{1}{2^n}}}$$
$$= ik\mathrm{e}^{-k^2}|k|^{-\frac{1}{2^n}}\left(\theta(k\sigma)J_{n,\infty} + \theta(-k\sigma)\overline{J_{n,\infty}}\right) = \widehat{u_n}(k,0),$$

as claimed. ■

Proposition 16 *Let $\sigma = \pm 1$, $1 \leq n < \infty$, and let $\rho(x) = \sqrt{1+x^2}$. Then, the solution u of*

$$\partial_t u = \partial_x^2 u + \partial_x^2 B, \qquad u(x,0) = 0, \tag{5.13}$$

with B defined as in (5.6), satisfies

$$\|u\|_{2,\frac{3}{4}^\star} + \|\mathrm{D}u\|_{2,\frac{5}{4}^\star} \leq C\left(\|\mathrm{D}^3F\|_1 + \sum_{m=0}^{2}\|\rho\mathrm{D}^mF\|_1 + \sum_{m=1}^{2}\|\mathrm{D}^mF\|_2\right) \tag{5.14}$$

for all F for which the r.h.s. of (5.14) is finite.

Proof. We first note that the Fourier transform of u is given by

$$\hat{u}(k,t) = -k^2\int_0^t \mathrm{d}s\,\mathrm{e}^{-k^2(t-s)-2ik\sigma s}\hat{F}(k\sqrt{1+s})(1+s)^{\frac{1}{2^n}-\frac{1}{2}},$$

which implies that

$$\|(1-\mathbb{Q})u\|_{2,\frac{3}{4}} + \|(1-\mathbb{Q})\mathrm{D}u\|_{2,\frac{5}{4}} \leq C\left(\|\mathrm{D}F\|_2 + \|\mathrm{D}^2F\|_2\right)\sup_{0\leq t\leq 1}\int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}}.$$

Here, $\mathbb{Q}$ is again defined as the characteristic function for $t \geq 1$. Next, integrating by parts, we find

$$\hat{u}(k,t) = \frac{ik\hat{F}(k)\mathrm{e}^{-k^2t}}{2\sigma} - \frac{ik\hat{F}(k\sqrt{1+t})\mathrm{e}^{-2ik\sigma t}}{2\sigma(1+t)^{\frac{1}{2}-\frac{1}{2^n}}} + \hat{N}(k,t),$$

where

$$\hat{N}(k,t) = \frac{ik}{2\sigma}\int_0^t \mathrm{d}s\,\mathrm{e}^{-k^2(t-s)-2ik\sigma s}\left(k^2 + \partial_s\right)\left(\frac{\hat{F}(k\sqrt{1+s})}{(1+s)^{\frac{1}{2}-\frac{1}{2^n}}}\right).$$

We then note that

$$\|u-N\|_{2,\frac{3}{4}} + \|\mathrm{D}(u-N)\|_{2,\frac{5}{4}} \leq C\left(\|F\|_1 + \|\mathrm{D}F\|_2 + \|\mathrm{D}^2F\|_2\right)$$

and that, by defining $\hat{G}(k) = \frac{1}{2}\partial_k\hat{F}(k)$, we have $\hat{N}(k,t) = \hat{N}_0(k,t) + \hat{N}_1(k,t) + \hat{N}_2(k,t)$, where

$$\hat{N}_0(k,t) = \frac{ik^3}{2\sigma}\int_0^t \mathrm{d}s\,\mathrm{e}^{-k^2(t-s)-2ik\sigma s}\left(\frac{\hat{F}(k\sqrt{1+s})}{(1+s)^{\frac{1}{2}-\frac{1}{2^n}}}\right),$$
$$\hat{N}_1(k,t) = \frac{ik^2}{2\sigma}\int_0^t \mathrm{d}s\,\mathrm{e}^{-k^2(t-s)-2ik\sigma s}\left(\frac{\hat{G}(k\sqrt{1+s})}{(1+s)^{1-\frac{1}{2^n}}}\right),$$
$$\hat{N}_2(k,t) = \frac{ik}{2\sigma}\left(\tfrac{1}{2^n} - \tfrac{1}{2}\right)\int_0^t \mathrm{d}s\,\mathrm{e}^{-k^2(t-s)-2ik\sigma s}\left(\frac{\hat{F}(k\sqrt{1+s})}{(1+s)^{\frac{3}{2}-\frac{1}{2^n}}}\right).$$

The procedure is now similar to that outlined in the proof of Theorem 6: we split the integration intervals into $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$, and distribute the derivatives (k-factors) either on the functions F and G or on the Gaussian. By introducing the notation

$$B_1\big[\begin{smallmatrix} p_1, q_1 \\ p_2, q_2 \end{smallmatrix}\big](t) \equiv \int_0^{\frac{t}{2}} \mathrm{d}s\, \frac{(1+s)^{-q_1}}{(t-s)^{p_1}} + \int_{\frac{t}{2}}^{t} \mathrm{d}s\, \frac{(1+s)^{-q_2}}{(t-s)^{p_2}}, \tag{5.15}$$

we then find that

$$\begin{aligned} \|\mathbb{Q}\mathrm{D}^\alpha N_0\|_{2,\frac{3}{4}+\frac{\alpha}{2}} &\leq C\big(\|F\|_1 + \|\mathrm{D}^{2+\alpha}F\|_1\big) \sup_{t\geq 1} t^{\frac{3}{4}+\frac{\alpha}{2}}\, B_1\big[\begin{smallmatrix} \frac{7}{4}+\frac{\alpha}{2}, 0 \\ \frac{3}{4}, 1+\frac{\alpha}{2} \end{smallmatrix}\big](t), \\ \|\mathbb{Q}\mathrm{D}^\alpha N_1\|_{2,\frac{3}{4}+\frac{\alpha}{2}} &\leq C\big(\|G\|_1 + \|\mathrm{D}^{1+\alpha}G\|_1\big) \sup_{t\geq 1} t^{\frac{3}{4}+\frac{\alpha}{2}}\, B_1\big[\begin{smallmatrix} \frac{5}{4}+\frac{\alpha}{2}, \frac{1}{2} \\ \frac{3}{4}, 1+\frac{\alpha}{2} \end{smallmatrix}\big](t), \\ \|\mathbb{Q}\mathrm{D}^\alpha N_2\|_{2,\frac{3}{4}+\frac{\alpha}{2}^\star} &\leq C\big(\|F\|_1 + \|\mathrm{D}^{\alpha}F\|_1\big) \sup_{t\geq 1} \frac{t^{\frac{3}{4}+\frac{\alpha}{2}}}{\ln(2+t)}\, B_1\big[\begin{smallmatrix} \frac{3}{4}+\frac{\alpha}{2}, 1 \\ \frac{3}{4}, 1+\frac{\alpha}{2} \end{smallmatrix}\big](t) \end{aligned}$$

for $\alpha = 0, 1$. The proof is completed by a straightforward application of Lemma 18 below, where we consider generalizations of the function B_1 in (5.15), since those will occur later on, in Sections 6 and 7 (see Definition 17 below). ■

6 Proof of Theorem 6, continued

In view of the estimates on $\mathrm{e}^{\mathrm{L}t}$ and h in (2.6) and (2.8), respectively, the estimates needed to conclude the proof of Theorem 6 will naturally involve the functions B_0 and B, which are defined as follows.

Definition 17 *We define*

$$\begin{aligned} B_0[q](t) &= \int_0^t \mathrm{d}s \frac{\mathrm{e}^{-\frac{t-s}{8}}}{\sqrt{t-s}(1+s)^q}, \\ B\big[\begin{smallmatrix} p_1, q_1, r_1 \\ p_2, q_2, r_2, r_3 \end{smallmatrix}\big](t) &= \int_0^{\frac{t}{2}} \mathrm{d}s\, \frac{(1+s)^{-q_1}}{(t-s)^{p_1}(t-1+s)^{r_1}} + \int_{\frac{t}{2}}^{t} \mathrm{d}s\, \frac{(1+s)^{-q_2}\ln(2+s)^{r_3}}{(t-s)^{p_2}(t-1+s)^{r_2}}. \end{aligned} \tag{6.1}$$

These functions satisfy the following estimates.

Lemma 18 *Let* $0 \leq p_2 < 1$, $0 \leq r_2 \leq 1 - p_2$, $p_1, q_1, q_2, r_1 \geq 0$, *and* $r_3 \in \{0, 1\}$. *There exists a constant* C *such that for all* $t \geq 0$, *there holds*

$$\begin{aligned} B_0[q_1](t) &\leq C(1+t)^{-q_1}, \\ B\big[\begin{smallmatrix} p_1, q_1, r_1 \\ p_2, q_2, r_2, r_3 \end{smallmatrix}\big](t) &\leq C\ \ln(2+t)^\alpha \begin{cases} \frac{1}{(1+t)^\beta} & \text{if} \quad 0 \leq p_1 \leq 1, \\ \frac{1}{t^{p_1-1}\ (1+t)^{\beta-p_1+1}} & \text{if} \quad p_1 > 1, \end{cases} \end{aligned} \tag{6.2}$$

where $\beta = \min\{p_1 + \min\{q_1-1, 0\} + r_1, p_2 + q_2 + r_2 - 1\}$, $\alpha = \max(\delta_{q_1,1}, \delta_{p_2+r_2,1} + r_3)$, *and* $\delta_{i,j}$ *is the Kronecker delta. Furthermore, since*

$$B_1\big[\begin{smallmatrix} p_1, q_1 \\ p_2, q_2 \end{smallmatrix}\big](t) = B\big[\begin{smallmatrix} p_1, q_1, 0 \\ p_2, q_2, 0, 0 \end{smallmatrix}\big](t),$$

the estimate in (6.2) applies for B_1*, as well.*

Proof. The proof follows immediately from

$$B_0[q_1](t) \leq \mathrm{e}^{-\frac{t}{16}} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{\sqrt{t-s}} + \frac{1}{(\frac{t}{2}+1)^{q_1}} \int_0^{\frac{t}{2}} \mathrm{d}s\, \frac{\mathrm{e}^{-\frac{s}{8}}}{\sqrt{s}},$$

$$B[^{p_1,q_1,r_1}_{p_2,q_2,r_2,r_3}](t) \leq \frac{1}{(\frac{t}{2})^{p_1}(\frac{t}{2}+1)^{r_1}} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{(1+s)^{q_1}} + \frac{\ln(2+t)^{r_3}}{(\frac{t}{2}+1)^{q_2}} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{s^{p_2}(1+s)^{r_2}},$$

and straightforward integrations. ■

We can now complete the proof of Theorem 6.

Proof of Theorem 6, continued. First, we recall that our goal is to prove that the map $\mathcal{N}$ defined by

$$\mathcal{N}[\mathbf{z}](t) = \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{L}(t-s)} \begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix} \tag{6.3}$$

satisfies $\|\mathcal{N}[\mathbf{z}]\| \leq C$ for all $\mathbf{z} \in \mathcal{B}$ with $\|\mathbf{z}\| = 1$. The estimate $\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} \leq C$ has already been proven. The other necessary estimates are obtained as follows:

$$\begin{aligned}
\|\widehat{\mathcal{N}[\mathbf{z}]}\|_{\infty,0} &\leq C \sup_{t\geq 0} B_1[^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}](t) \leq C, \\
\|\mathcal{N}[\mathbf{z}]\|_{2,\frac{1}{4}} &\leq C \sup_{t\geq 0}(1+t)^{\frac{1}{4}} B_1[^{\frac{1}{2},\frac{3}{4}}_{\frac{1}{2},\frac{3}{4}}](t) \leq C, \\
\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} &\leq C \sup_{t\geq 0}(1+t)^{\frac{3}{4}} B_1[^{1,\frac{3}{4}}_{\frac{1}{2},\frac{5}{4}}](t) \leq C, \\
\|(1-\mathbb{P})\mathrm{D}\mathcal{N}[\mathbf{z}]\|_{2,\frac{3}{4}} &\leq \sup_{t\geq 0}(1+t)^{\frac{3}{4}} B_0[\tfrac{5}{4}](t) \leq C, \\
\|(1-\mathbb{Q})\mathbb{P}\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} &\leq C\|(1-\mathbb{Q})\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{3}{4}} \leq C\|\mathbb{P}\mathrm{D}\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{3}{4}} \leq C, && (6.4)\\
\|\mathbb{Q}\mathbb{P}\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} &\leq C \sup_{t\geq 1} \tfrac{(1+t)^{\frac{5}{4}}}{\ln(2+t)}\, B[^{\frac{3}{2},\frac{3}{4},0}_{\frac{1}{2},\frac{5}{4},\frac{1}{2},0}](t) \leq C, && (6.5)\\
\|(1-\mathbb{P})\mathrm{D}^2\mathcal{N}[\mathbf{z}]_2\|_{2,\frac{5}{4}^\star} &\leq \sup_{t\geq 0}(1+t)^{\frac{5}{4}} B_0[\tfrac{5}{4}](t) \leq C. && (6.6)
\end{aligned}$$

In (6.4), we applied the obvious estimates $\|\mathbb{P}\mathrm{D}f\|_2 \leq \|\mathbb{P}f\|_2$ and $\|(1-\mathbb{Q})f\|_{2,p} \leq 2^{p-q}\|(1-\mathbb{Q})f\|_{2,q}$ for $q < p$, while in (6.5), we made use of $\sup_{|k|\leq 1, t\geq 0} |k|\sqrt{1+t}\mathrm{e}^{-k^2t} \leq 1$, and finally, in (6.6), we used $\sup_{k\in\mathbf{R}} |k|(1+k^2)^{-\frac{1}{2}} = 1$. Incidentally, (6.6) is the only place in the above estimates where the (crucial) presence of the extra factor $(1+k^2)^{-\frac{1}{2}}$ in the second component of the r.h.s. of (2.6) is used. This concludes the proof of Theorem 6. ■

7 Remainder estimates

We now make precise the sense in which the semigroup $\mathrm{e}^{\mathrm{L}t}$ is *close* to that of (2.3), whose Fourier transform is given by

$$\mathrm{e}^{\mathrm{L}_0 t} \equiv \begin{pmatrix} \mathrm{e}^{-k^2t+ikt} & 0 \\ 0 & \mathrm{e}^{-k^2t-ikt} \end{pmatrix}. \tag{7.1}$$

Lemma 19 *Let $\mathbb{P}$ be the Fourier multiplier with the characteristic function on $[-1,1]$, let $\mathrm{e}^{\mathrm{L}t}$ (respectively, $\mathrm{e}^{\mathrm{L}_0 t}$) be as in (2.2) (respectively, (7.1)), and let $\mathcal{S}$ be defined as in (2.4). Then, one has the estimates*

$$\sup_{t\geq 0, k\in\mathbf{R}} \sqrt{1+t}\mathrm{e}^{\frac{k^2t}{2}} \left|\left(\mathbb{P}\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right)_{i,j}\right| \leq C, \tag{7.2}$$

where $(\mathbb{P}\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S})_{i,j}$ denotes the (i,j)-entry in the matrix $\mathbb{P}\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}$.

Proof. The proof follows by considering separately $|k| \leq 1$ and $|k| > 1$. We first write

$$\mathbb{P}\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S} = \mathbb{P}\left(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right) + (1-\mathbb{P})\mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}.$$

We then have

$$\sup_{t\geq 0, k\in\mathbf{R}} \sqrt{1+t}\mathrm{e}^{\frac{k^2 t}{2}} \left| \big((1-\mathbb{P})\mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)_{i,j} \right| \leq \sup_{t\geq 0, |k|\geq 1} \sqrt{1+t}\mathrm{e}^{-\frac{k^2 t}{2}} \leq C.$$

For $|k| \leq 1$, we first compute

$$\begin{aligned}
\mathrm{e}^{\mathrm{L}_0 t}\mathcal{S} &= \mathrm{e}^{-k^2 t} \begin{pmatrix} \mathrm{e}^{ikt} & \mathrm{e}^{ikt} \\ \mathrm{e}^{-ikt} & -\mathrm{e}^{-ikt} \end{pmatrix}, \\
\mathcal{S}\mathrm{e}^{\mathrm{L}t} &= \mathrm{e}^{-k^2 t} \begin{pmatrix} \cos(kt\Delta) + \frac{1-ik}{\Delta}\, i\, \sin(kt\Delta) & \cos(kt\Delta) + \frac{1+ik}{\Delta}\, i\, \sin(kt\Delta) \\ \cos(kt\Delta) - \frac{1+ik}{\Delta}\, i\, \sin(kt\Delta) & -\big(\cos(kt\Delta) - \frac{1-ik}{\Delta}\, i\, \sin(kt\Delta)\big) \end{pmatrix},
\end{aligned}$$

where we recall that $\Delta = \sqrt{1-k^2}$. Next, we note that

$$\begin{aligned}
\mathbb{P}|\sin(kt\Delta) - \sin(kt)| + \mathbb{P}|\cos(kt\Delta) - \cos(kt)| &\leq \mathbb{P}|\cos(kt(\Delta-1)) - 1| + \mathbb{P}|\sin(kt(\Delta-1))| \\
&\leq \mathbb{P}\big|\sqrt{1-k^2} - 1\big|\; |k|t \leq \mathbb{P}|k|^3 t, \\
\mathbb{P}\left|\big(\tfrac{1}{\Delta} - 1\big)\sin(kt\Delta)\right| &\leq \mathbb{P}\big|\sqrt{1-k^2} - 1\big|\; |k|t \leq \mathbb{P}|k|^3 t.
\end{aligned}$$

The proof is completed by noting that

$$\sup_{|k|\leq 1, t\geq 0} t^{\frac{m}{2}} |k|^n \mathrm{e}^{-\frac{k^2 t}{2}} \leq C(n)$$

for any (finite) $0 \leq m \leq n$. ■

We are now in a position to prove that the remainder

$$\mathcal{R}[\mathbf{z}](t) = \big(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)\,\mathbf{z}_0 + \int_0^t \mathrm{d}s\; \left[\mathcal{S}\mathrm{e}^{\mathrm{L}(t-s)} \begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix} - \mathrm{e}^{\mathrm{L}_0(t-s)}\mathcal{S}\; \begin{pmatrix} 0 \\ \partial_x g_0(\mathbf{z}(s)) \end{pmatrix}\right]$$

satisfies improved estimates, as stated in (3.5):

Theorem 20 *Let ϵ_0 be again the (small) constant provided by Theorem 6. Then, for all $\mathbf{z}_0 \in \mathcal{B}_0$ with $|\mathbf{z}_0| \leq \epsilon_0$, the solution $\mathbf{z}$ of (1.1) satisfies*

$$\|\mathcal{R}[\mathbf{z}]\|_{2,\frac{3}{4}^\star} + \|\mathrm{D}\mathcal{R}[\mathbf{z}]\|_{2,\frac{5}{4}^\star} \leq C\epsilon_0. \tag{7.3}$$

Proof. We first note that

$$\big(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)\,\mathbf{z}_0 = \big(\mathcal{S}\mathbb{P}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)\,\mathbf{z}_0 + \mathcal{S}(1-\mathbb{P})\mathrm{e}^{\mathrm{L}t}\mathbf{z}_0 \equiv L_1[\mathbf{z}_0](t) + L_2[\mathbf{z}_0](t)$$

and then use the fact that, by Lemma 19, we have

$$\|\mathrm{D}^\alpha L_1[\mathbf{z}_0]\|_{2,\frac{3}{4}+\frac{\alpha}{2}} \leq C \sup_{t\geq 0}(1+t)^{\frac{1}{4}+\frac{\alpha}{2}} \min\left\{\|\mathrm{D}^\alpha \mathbf{z}_0\|_2,\; t^{-\frac{1}{4}-\frac{\alpha}{2}}\|\widehat{\mathbf{z}_0}\|_\infty\right\} \leq C|\mathbf{z}_0|$$

for $\alpha = 0, 1$, and, finally,

$$\|L_2[\mathbf{z}_0]\|_{2,\frac{3}{4}} + \|\mathrm{D}L_2[\mathbf{z}_0]\|_{2,\frac{5}{4}} \leq C\big(\|\mathbf{z}_0\|_2 + \|\mathrm{D}\mathbf{z}_0\|_2\big) \sup_{t\geq 0}(1+t)^{\frac{5}{4}}\mathrm{e}^{-\frac{t}{4}} \leq C|\mathbf{z}_0|.$$

This proves

$$\big\|\big(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)\,\mathbf{z}_0\big\|_{2,\frac{3}{4}} + \big\|\mathrm{D}\,\big(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\big)\,\mathbf{z}_0\big\|_{2,\frac{5}{4}} \leq C|\mathbf{z}_0|$$

for all $\mathbf{z}_0 \in \mathcal{B}_0$. We then show that

$$\left\|\mathcal{R}[\mathbf{z}](t) - \left(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right)\mathbf{z}_0\right\|_{2,\frac{3}{4}^\star} + \left\|\mathrm{D}\left(\mathcal{R}[\mathbf{z}](t) - \left(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right)\mathbf{z}_0\right)\right\|_{2,\frac{5}{4}^\star} \leq C\|\mathbf{z}\|^2$$

for all $\mathbf{z} \in \mathcal{B}$. We only need to prove the estimates for $\|\mathbf{z}\| = 1$. We first decompose

$$\mathcal{R}[\mathbf{z}](t) - \left(\mathcal{S}\mathrm{e}^{\mathrm{L}t} - \mathrm{e}^{\mathrm{L}_0 t}\mathcal{S}\right)\mathbf{z}_0 = \mathcal{S}\mathcal{N}_1[\mathbf{z}](t) + \mathcal{S}\mathcal{N}_2[\mathbf{z}](t) + \mathcal{N}_3[\mathbf{z}](t), \tag{7.4}$$

where

$$\mathcal{N}_1[\mathbf{z}](t) = (1-\mathbb{P})\int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{L}(t-s)}\begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) \end{pmatrix},$$
$$\mathcal{N}_2[\mathbf{z}](t) = \mathbb{P}\int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{L}(t-s)}\begin{pmatrix} 0 \\ \partial_x h(\mathbf{z}(s)) - \partial_x g_0(\mathbf{z}(s)) \end{pmatrix},$$
$$\mathcal{N}_3[\mathbf{z}](t) = \int_0^t \mathrm{d}s\, \left(\mathbb{P}\mathcal{S}\mathrm{e}^{\mathrm{L}(t-s)} - \mathrm{e}^{\mathrm{L}_0(t-s)}\mathcal{S}\right)\begin{pmatrix} 0 \\ \partial_x g_0(\mathbf{z}(s)) \end{pmatrix}.$$

We then recall that $h(\mathbf{z})$ satisfies

$$\|h(\mathbf{z})\|_{2,\frac{3}{4}} + \|\mathrm{D}h(\mathbf{z})\|_{2,\frac{5}{4}} \leq C\|\mathbf{z}\|^2,$$

which implies that

$$\|\mathcal{N}_1[\mathbf{z}]\|_{2,\frac{3}{4}} \leq C\sup_{t\geq 0}(1+t)^{\frac{3}{4}}B_0[\tfrac{3}{4}](t) \leq C \quad \text{and} \quad \|\mathrm{D}\mathcal{N}_1[\mathbf{z}]\|_{2,\frac{5}{4}} \leq C\sup_{t\geq 0}(1+t)^{\frac{5}{4}}B_0[\tfrac{5}{4}](t) \leq C.$$

Moreover, $h_0(a,b) \equiv f(a,b)\partial_x b + g(a,b) - g_0(a,b)$ satisfies

$$\|h_0(\mathbf{z})\|_{1,1} + \|\mathrm{D}h_0(\mathbf{z})\|_{1,\frac{3}{2}^\star} \leq C\|\mathbf{z}\|^2.$$

Here, we need to consider separately $t \in [0,1]$ and $t \geq 1$ when estimating $\|\mathbb{P}\mathrm{D}\mathcal{N}_2[\mathbf{z}]\|_{2,\frac{5}{4}^\star}$. Writing again $\mathbb{Q}$ for the characteristic function for $t \geq 1$, we find that

$$\|\mathbb{P}\mathcal{N}_2[\mathbf{z}]\|_{2,\frac{3}{4}^\star} \leq C\sup_{t\geq 0}\tfrac{(1+t)^{\frac{3}{4}}}{\ln(2+t)}B_1[{}^{\frac{3}{4},1}_{\frac{3}{4},1}](t) \leq C,$$
$$\|(1-\mathbb{Q})\mathbb{P}\mathrm{D}\mathcal{N}_2[\mathbf{z}]\|_{2,\frac{5}{4}^\star} \leq C\sup_{0\leq t\leq 1}(1+t)^{\frac{5}{4}}\, B_1[{}^{\frac{3}{4},\frac{3}{2}}_{\frac{3}{4},\frac{3}{2}}](t) \leq C,$$
$$\|\mathbb{Q}\mathbb{P}\mathrm{D}\mathcal{N}_2[\mathbf{z}]\|_{2,\frac{5}{4}^\star} \leq C\sup_{t\geq 1}\tfrac{(1+t)^{\frac{5}{4}}}{\ln(2+t)}\, B[{}^{\frac{5}{4},1,0}_{\frac{3}{4},\frac{3}{2},0,1}](t) \leq C.$$

We finally note that

$$\|g_0(\mathbf{z})\|_{2,\frac{3}{4}} + \|\mathrm{D}g_0(\mathbf{z})\|_{2,\frac{5}{4}} \leq C\|\mathbf{z}\|^2;$$

hence, using Lemma 19, we find

$$\|\mathcal{N}_3[\mathbf{z}]\|_{2,\frac{3}{4}^\star} \leq \sup_{t\geq 0}\tfrac{(1+t)^{\frac{3}{4}}}{\ln(2+t)}\, B[{}^{\frac{1}{2},\frac{3}{4},\frac{1}{2}}_{\frac{1}{2},\frac{3}{4},\frac{1}{2},0}](t) \leq C,$$
$$\|\mathrm{D}\mathcal{N}_3[\mathbf{z}]\|_{2,\frac{5}{4}^\star} \leq \sup_{t\geq 0}\tfrac{(1+t)^{\frac{5}{4}}}{\ln(2+t)}\, B[{}^{1,\frac{3}{4},\frac{1}{2}}_{\frac{1}{2},\frac{5}{4},\frac{1}{2},0}](t) \leq C.$$

This completes the proof. ■

It now only remains to prove the estimates (3.13) and (3.14) on the maps $\widetilde{\mathcal{R}}_{\{u,v\}}$, where we recall that

$$\widetilde{\mathcal{R}}_u[\mathbf{z},\mathbf{R}^N](t) = c_+\mathrm{E}_0[h_{1,u} + h_{3,u}](t) - c_-\mathrm{E}_{-2}[h_{1,v} + h_{3,v}](t) + c_3\mathrm{E}_{-1}[h_2 + h_4](t),$$

$$\widetilde{\mathcal{R}}_v[\mathbf{z}, \mathbf{R}^N](t) = c_- \mathrm{E}_0[h_{1,v} + h_{3,v}](t) - c_+ \mathrm{E}_2[h_{1,u} + h_{3,u}](t) - c_3 \mathrm{E}_1[h_2 + h_4](t),$$

with

$$\mathrm{E}_\sigma[h](t) = \partial_x \int_0^t \mathrm{d}s\, \mathrm{e}^{\partial_x^2(t-s)}\, \mathcal{T}^\sigma h(s)$$

and

$$\begin{aligned} h_{1,u} &= R_u^N(u+u_\star), \quad h_{3,u} = u_1^2, \quad h_2 = (\mathcal{T} R_u^N)\mathcal{T}^{-1}\Big(\frac{v+v_\star}{2}\Big) + (\mathcal{T}^{-1} R_v^N)\mathcal{T}\Big(\frac{u+u_\star}{2}\Big), \\ h_{1,v} &= R_v^N(v+v_\star), \quad h_{3,v} = v_1^2, \quad h_4 = (\mathcal{T} u_\star)(\mathcal{T}^{-1} v_\star). \end{aligned}$$

Here, we will only prove

$$\sum_{\alpha=0}^{1} \big\|\mathrm{D}^\alpha \widetilde{\mathcal{R}}_{\{u,v\}}[\mathbf{z}, \mathbf{R}^N]\big\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} \le C\epsilon_0 \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathbf{R}^N\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} + C. \tag{7.5}$$

It is then straightforward to show (3.14), namely, that the maps $\widetilde{\mathcal{R}}_{\{u,v\}}$ are Lipschitz in their second argument; we omit the details.

To prove (7.5), we first need estimates on $\mathbf{h}_1 = (h_{1,u}, h_{1,v})$, h_2, $\mathbf{h}_3 = (h_{3,u}, h_{3,v})$, and h_4. We note that $\mathbf{u}_0 = (u_0, v_0)$ and $\mathbf{u}_1 = (u_1, v_1)$ satisfy

$$\begin{aligned} \|\mathbf{u}_0\|_{1,0} + \|\mathbf{u}_1\|_{1,0} + \|\mathrm{D}\mathbf{u}_0\|_{1,\frac{1}{2}} + \|\mathrm{D}\mathbf{u}_1\|_{1,\frac{1}{2}} &\le C, \\ \sup_{t\ge 0}(1+t)^{\frac{3}{2}}\big(|\mathbf{u}_0(\pm t,t)| + |\mathbf{u}_1(\pm t,t)|\big) + (1+t)^2\big(|\mathrm{D}\mathbf{u}_0(\pm t,t)| + |\mathrm{D}\mathbf{u}_1(\pm t,t)|\big) &\le C \end{aligned}$$

for some constant C; see Proposition 12. We thus find that

$$\begin{aligned} \|\mathbf{h}_1\|_{1,1-\epsilon} + \|\mathrm{D}\mathbf{h}_1\|_{1,\frac{3}{2}-\epsilon} + \|h_2\|_{1,1-\epsilon} + \|\mathrm{D}h_2\|_{1,\frac{3}{2}-\epsilon} &\le C\epsilon_0 \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathbf{R}^N\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon}, \\ \|\mathbf{h}_3\|_{1,1} + \|\mathrm{D}\mathbf{h}_3\|_{1,\frac{3}{2}} + \|h_4\|_{1,\frac{3}{2}} + \|\mathrm{D}h_4\|_{2,2} &\le C. \end{aligned} \tag{7.6}$$

The proof of (7.5) then follows from Proposition 21, which implies that

$$\begin{aligned} \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathrm{E}_\sigma[\mathbf{h}_1]\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} + \|\mathrm{D}^\alpha \mathrm{E}_\sigma[h_2]\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} &\le C\epsilon_0 \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathbf{R}^N\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon}, \\ \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathrm{E}_\sigma[\mathbf{h}_3]\|_{2,\frac{3}{4}+\frac{\alpha}{2}\star} + \|\mathrm{D}^\alpha \mathrm{E}_\sigma[h_4]\|_{2,\frac{3}{4}+\frac{\alpha}{2}\star} &\le C \end{aligned}$$

for any $\sigma \in \{-2,-1,0,1,2\}$ if the estimates in (7.6) are satisfied.

Proposition 21 *Let $\epsilon > 0$ and $\sigma \in \{-2,-1,0,1,2\}$. Then, there holds*

$$\begin{aligned} \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathrm{E}_\sigma[h_1]\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} &\le C \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha h_1\|_{1,1+\frac{\alpha}{2}-\epsilon}, \\ \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha \mathrm{E}_\sigma[h_2]\|_{2,\frac{3}{4}+\frac{\alpha}{2}\star} &\le C \sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha h_2\|_{1,1+\frac{\alpha}{2}}. \end{aligned}$$

Proof. Let $u_i = \mathrm{E}_\sigma[h_i]$. By taking the Fourier transform, we find that

$$\widehat{u_i}(k,t) = ik \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)+i\sigma ks} \widehat{h_i}(k,s).$$

We can restrict ourselves to $\sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha h_1\|_{1,1+\frac{\alpha}{2}-\epsilon} = 1$ and $\sum_{\alpha=0}^{1} \|\mathrm{D}^\alpha h_2\|_{1,1+\frac{\alpha}{2}} = 1$. Then, it follows that

$$\|\mathrm{D}^\alpha u_1\|_{2,\frac{3}{4}+\frac{\alpha}{2}-\epsilon} \le C \sup_{t\ge 0} (1+t)^{\frac{3}{4}+\frac{\alpha}{2}-\epsilon}\, B_1\Big[{}^{\frac{3}{4}+\frac{\alpha}{2},1-\epsilon}_{\frac{3}{4},1+\frac{\alpha}{2}-\epsilon}\Big](t) \le C,$$
$$\|\mathrm{D}^\alpha u_2\|_{2,\frac{3}{4}+\frac{\alpha}{2}\star} \le C \sup_{t\ge 0} \frac{(1+t)^{\frac{3}{4}+\frac{\alpha}{2}}}{\ln(2+t)}\, B_1\Big[{}^{\frac{3}{4}+\frac{\alpha}{2},1}_{\frac{3}{4},1+\frac{\alpha}{2}}\Big](t) \le C$$

for $\alpha = 0, 1$ as claimed. ■

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