The Effect of a Cut-Off on a Model of Invasion with Dispersive Variability

Nikola Popović and Zhouqian Miao

Abstract. In 1994, a population model of invasion with dispersive variability which represents an extension of the classical Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) reaction-diffusion equation was proposed by Cook, whereby the population is partitioned into dispersers and non-dispersers. In 1997, an alternative modification to the FKPP equation was suggested by Brunet and Derrida, who included a cut-off in the reaction kinetics to account for the fact that, in many applications, population growth cannot reasonably occur when the population density is below a certain threshold. Here, we combine these two modifications by studying the effect of a Heaviside cut-off in Cook's extended model. We prove the existence of travelling front solutions; moreover, we determine the correction to the "critical" front propagation speed that is due to the cut-off. Our analysis is based on a combination of geometric singular perturbation theory and the desingularisation technique known as blow-up.

1. Introduction

The colonisation of new territory by all forms of life, including animals, plants, and disease, is of great ecological importance. A classical model for these and similar processes is given by the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) reaction-diffusion equation [**6**] which, in non-dimensionalised form, reads

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{1.1}$$

Equation (1.1) was suggested by Fisher in 1937 as a deterministic version of a stochastic model for the spatial spread of a favoured gene in a population [**6**]. The FKPP equation has been widely studied since; see, *e.g.*, [**13**] for an extensive discussion and further references. Of particular interest has been the study of travelling front solutions that connect the two homogeneous rest states at $u = 0$ and $u = 1$ in (1.1). As is well-known, such fronts exist only for propagation speeds that exceed a "critical" speed $c_{\mathrm{crit}} = 2$, as can be seen by linearisation at the zero rest state; see again [**13**] for an exposition. Propagating fronts are particularly relevant in applications, as general solutions to Equation (1.1) tend to them under relatively mild conditions; thus, Kolmogorov *et al.* [**9**] showed that if $u(x, 0)$ has compact support and is continuous thereon, then the solution $u(x, t)$ of (1.1) converges to the front with propagation speed c_{crit}.

One significant limitation of Equation (1.1), as recognised by Cook [**13**, Section 13.7], is that it applies only to populations within which all individuals disperse alike. Cook hence differentiates between individuals who will disperse and those who will not by partitioning the population into two distinct subpopulations, namely, of dispersers and non-dispersers. Cook's reaction-diffusion model is then given, in dimensional form, by

$$\frac{\partial A}{\partial t} = D\frac{\partial^2 A}{\partial x^2} + r_1(A+B)\Big[1 - \frac{A+B}{K}\Big], \tag{1.2a}$$

$$\frac{\partial B}{\partial t} = r_2(A+B)\Big[1 - \frac{A+B}{K}\Big]. \tag{1.2b}$$

Here, we have followed the notation in [**13**, Section 13.7], where A and B denote the populations of dispersers and non-dispersers, respectively, D is the diffusion coefficient for the dispersing subpopulation, K is the carrying capacity of the environment, and r_1 and r_2 are the intrinsic growth rates of the two subpopulations.

2000 *Mathematics Subject Classification.* Primary 35K57, 34E15, 35C07; Secondary 34E05, 92D40, 92D25.

Equation (1.2) can be non-dimensionalised via the following rescaling:

$$u=\frac{A}{K},\quad v=\frac{B}{K},\quad T=Rt,\quad\text{and}\quad X=\left(\frac{R}{D}\right)^{\frac{1}{2}}x, \tag{1.3}$$

where $R=r_1+r_2$. We consider R to be the overall intrinsic rate of growth; then,

$$p=\frac{r_1}{r_1+r_2}=\frac{r_1}{R} \tag{1.4}$$

represents the probability that a given individual is a disperser. (As we are not interested in the degenerate cases where $p=0$ or $p=1$, *i.e.*, where either no or all individuals disperse, we will henceforth assume $p\in(0,1)$.) With these notations, we may write (1.2) as

$$\frac{\partial u}{\partial T}=\frac{\partial^2 u}{\partial X^2}+p(u+v)[1-(u+v)], \tag{1.5a}$$

$$\frac{\partial v}{\partial T}=(1-p)(u+v)[1-(u+v)]. \tag{1.5b}$$

While front propagation in Equation (1.5) is discussed in some detail in [**13**, Section 13.7], we propose to study propagating fronts in a further modification of (1.5), which is obtained by introduction of a so-called "cut-off".

Reaction-diffusion equations, such as (1.1) or (1.5), typically arise as the result of mean-field approximations and large-scale limits in discrete N-particle models as the number N of agents, or particles, tends to infinity [**13**]. Consequently, the resulting model will deteriorate when N is not sufficiently large; in particular, the propagation speed of front solutions is frequently misestimated [**12**]. To remedy these discrepancies, Brunet and Derrida [**1**] suggested to "cut off" the reaction kinetics in Equation (1.1) whenever the density u is below some (small) threshold value ε, as one would only expect diffusion of agents in that case. They hence considered the FKPP equation with cut-off,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(1-u)\psi(u,\varepsilon), \tag{1.6}$$

for a wide class of cut-off functions $\psi(u,\varepsilon)$, where

$$\psi(u,\varepsilon)=1\quad\text{for } u>\varepsilon\quad\text{and}\quad \psi(u,\varepsilon)\ll 1\quad\text{for } u<\varepsilon.$$

(In the simplest case, ψ can be taken to be the Heaviside cut-off H, with $H(u,\varepsilon)\equiv 0$ when $u<\varepsilon$.) In particular, they explored the effects of a cut-off on the critical propagation speed c_{crit}, showing that a Heaviside cut-off decelerates the "critical" front solution due to $c_{\text{crit}}(\varepsilon)=2-\frac{\pi^2}{(\ln\varepsilon)^2}+\mathcal{O}[(\ln\varepsilon)^{-2}]$ as $\varepsilon\to 0$. Moreover, they conjectured that the asymptotics of $c_{\text{crit}}(\varepsilon)$ is, to leading order, independent of the choice of cut-off function ψ. That asympotics and their conjecture were later proven rigorously by Dumortier *et al.* [**4**] via *geometric singular perturbation theory (GSPT)* and the desingularisation technique known as *blow-up* [**2**]; see also [**3**] for a related study of a different modification of the FKPP equation with cut-off.

The aim of the present paper is to derive analogous results for Cook's model, Equation (1.5). For the sake of exposition, these results will be restricted to a Heaviside cut-off in (1.5). While our analysis closely follows that in [**4**], we have aimed for our presentation to be as self-contained as possible in order to convey the philosophy behind our geometric approach.

The paper is organised as follows: in Section 2, we present background on front propagation in Cook's model of invasion in the form of the non-dimensionalised Equation (1.5); we give necessary and sufficient conditions for the existence of front solutions, and we define the critical speed in the context of (1.5). In Section 3, we study Cook's model augmented with a cut-off in the total population: we introduce the blow-up transformation that will allow us to desingularise the flow near the degenerate origin, and we construct a singular front solution by combining the dynamics in two appropriately defined coordinate charts. Then, in Section 4, we prove the existence and uniqueness of a persistent "critical" front in the cut-off model, and we derive the corresponding correction to the front propagation speed in dependence of the cut-off parameter. We conclude briefly in Section 5.

2. Background on Cook's model

In this section, we summarise various results on Cook's model, Equation (1.5), in preparation for our analysis of the modified model with cut-off studied in Section 3. While some of these results are known [**13**], others, such as a proof for the existence of front solutions, are new to the best of our knowledge.

2.1. Front propagation. In order to describe travelling front solutions to (1.5), we revert to a co-moving frame, making the ansatz

$$u(X,T) = U(Z) \quad \text{and} \quad v(X,T) = V(Z), \quad \text{with} \quad Z = X - CT. \tag{2.1}$$

Equation (2.1) characterises the form of a front moving with positive speed C in the direction of increasing X, where the new variable Z is referred to as the travelling wave variable. Substitution of (2.1) into (1.5) yields the following systems of ordinary differential equations (ODEs):

$$-CU_Z = U_{ZZ} + p(U+V)[1-(U+V)], \tag{2.2a}$$

$$-CV_Z = (1-p)(U+V)[1-(U+V)], \tag{2.2b}$$

where the subscript denotes differentiation with respect to Z. We then introduce a new variable $W = U_Z$, which reduces (2.2) to the following first-order system,

$$U_Z = W, \tag{2.3a}$$

$$V_Z = \frac{p-1}{C}(U+V)[1-(U+V)], \tag{2.3b}$$

$$W_Z = -p(U+V)[1-(U+V)] - CW. \tag{2.3c}$$

The steady states of (2.3) are located on the lines

$$q^+ := \big\{(U,-U,0)\,\big|\, U \in \mathbb{R}\big\} \quad \text{and} \quad q^- := \big\{(U,1-U,0)\,\big|\, U \in \mathbb{R}\big\}; \tag{2.4}$$

these states are the equivalent of the homogeneous rest states, at $u+v=0$ and $u+v=1$, of the underlying partial differential equation (PDE) in (1.5). The sought-after travelling front solutions will hence correspond to heteroclinic connections between points $(U^+,V^+,W^+) \in q^+$ and $(U^-,V^-,W^-) \in q^-$ that satisfy $U^+ + V^+ = 0$ and $U^- + V^- = 1$, respectively, as well as $W^\pm = 0$; we note that, for these solutions to be ecologically realistic, U and V must remain non-negative for all Z. Under that assumption, the only relevant steady state on q^+ is the origin O, with $(U^+,V^+,W^+) = (0,0,0)$; moreover, we must take $U \in [0,1]$ in the definition of q^-. In sum, front solutions must therefore manifest as heteroclinic orbits in (U,V,W)-phase space that connect an appropriately chosen point $(U^-,1-U^-,0)$ on q^- to O, for a suitable propagation speed C. (While C is clearly a function of the dispersal probability p, we will suppress that dependence in our notation.)

2.2. Total population. Given the dependence of both Equation (2.3) and of the corresponding steady states on the total population – the sum of dispersers (U) and non-dispersers (V) – it seems natural to introduce the new variable $Q = U + V$, following also Murray [**13**, Section 13.7]. Writing $U_Z = P$, we thus obtain

$$U' = P, \tag{2.5a}$$

$$P' = -CP - pQ(1-Q), \tag{2.5b}$$

$$Q' = P + \frac{p-1}{C}Q(1-Q) \tag{2.5c}$$

from (2.3), where the prime now denotes differentiation with respect to Z.

We note that Equation (2.5a) decouples, *i.e.*, that U can be determined by integration once P is known; correspondingly, the (P,Q)-subsystem in (2.5) is U-independent, as is also observed in [**13**, Section 13.7]. In fact, our analysis of Equation (2.5) will be further facilitated by the following observation:

LEMMA 2.1. *The plane* Π*, with*

$$U = \frac{p-1}{C}P + pQ \tag{2.6}$$

for $(P,Q) \in \mathbb{R}^2$*, is invariant under the flow of Equation* (2.5).

Proof. We differentiate (2.6) with respect to Z, which gives $U' = \frac{p-1}{C}P' + pQ'$, and substitute in (2.5) to find

$$P = \frac{p-1}{C}\big[-CP - pQ(1-Q)\big] + p\Big[P + \frac{p-1}{C}Q(1-Q)\Big],$$

which is easily seen to be a true statement. $\square$

Lemma 2.1 will allow us to restrict to the (P,Q)-subsystem $\{(2.5\text{b}),(2.5\text{c})\}$ in the following. In particular, we will show that the latter can be studied in analogy to the classical FKPP equation, with or without a cut-off.

The steady states of $\{(2.5\text{b}),(2.5\text{c})\}$ are now located at the origin $\mathcal{Q}^+ : (P,Q) = (0,0)$ and the point $\mathcal{Q}^- : (P,Q) = (0,1)$. Regarding the stability of these steady states, we have the following result:

Lemma 2.2.

(1) *The point $\mathcal{Q}^+$ is a hyperbolic steady state for $\{(2.5\text{b}),(2.5\text{c})\}$, with eigenvalues*

$$\lambda_\pm^+ = -\frac{1-p+C^2}{2C} \pm \sqrt{\frac{(1-p+C^2)^2}{4C^2} - 1}. \tag{2.7}$$

Specifically, $\mathcal{Q}^+$ is

(a) *a stable spiral for $C \in (1-\sqrt{p}, 1+\sqrt{p})$;*
(b) *a stable node for $C \in (0, 1-\sqrt{p}) \cup (1+\sqrt{p}, \infty)$; and*
(c) *a stable degenerate node for $C = 1 \mp \sqrt{p}$.*

(2) *The point $\mathcal{Q}^-$ is a hyperbolic saddle of $\{(2.5\text{b}),(2.5\text{c})\}$ for any $C \in (0,\infty)$, with eigenvalues*

$$\lambda_\pm^- = \frac{1-p-C^2}{2C} \pm \sqrt{\frac{(1-p-C^2)^2}{4C^2} + 1}. \tag{2.8}$$

Proof. The eigenvalues $\lambda_\pm^+$ and $\lambda_\pm^-$ at $\mathcal{Q}^+$ and $\mathcal{Q}^-$, respectively, are obtained by straightforward linearisation.

The stability classification of $\mathcal{Q}^+$ follows from $1-p>0$ due to $p \in (0,1)$ and from the fact that the discriminant in (2.7) is negative for $C \in (1-\sqrt{p}, 1+\sqrt{p})$, zero for $C = 1 \mp \sqrt{p}$, and positive otherwise.

Similarly, the classification of $\mathcal{Q}^-$ is due to $\frac{(1-p-C^2)^2}{4C^2} + 1 > \frac{(1-p-C^2)^2}{4C^2} \geq 0$. $\square$

Lemma 2.2 implies, in particular, that the saddle point at $\mathcal{Q}^-$ will always admit a one-dimensional unstable manifold, $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ say, corresponding to the eigenvalue λ_+^-. By contrast, the origin $\mathcal{Q}^+$ will be fully stable under the flow of $\{(2.5\text{b}),(2.5\text{c})\}$; hence, travelling front solutions to Equation (1.5) will correspond to heteroclinic orbits that leave $\mathcal{Q}^-$ along $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ and then terminate in $\mathcal{Q}^+$. It follows that such fronts satisfy

$$\lim_{Z\to-\infty} Q(Z) = 1 \qquad \text{and} \qquad \lim_{Z\to\infty} Q(Z) = 0 \tag{2.9}$$

in $\{(2.5\text{b}),(2.5\text{c})\}$; moreover, as the eigenvector $\mathbf{v}_+^- = \left(\frac{p}{\lambda_+^- + C}, 1\right)^T$ corresponding to λ_+^- has positive slope, they must pass through the negative P-plane.

Remark 2.3. By Lemma 2.1, the U-coordinate of the steady state $\mathcal{Q}^-$ is given by $U^- = p$. It is instructive to verify that U^- can be found directly, following reasoning in [**13**, Section 13.7]: one integrates the change in U along a given orbit of (2.5), noting that $Q = U + (1-U) = 1$ on q^-, which gives

$$U^- = \int_0^1 \frac{dU}{dQ} dQ = \int_0^1 \frac{P\,dQ}{P + \frac{p-1}{C}Q(1-Q)}. \tag{2.10}$$

To evaluate (2.10), one divides (2.5b) formally by (2.5c) to find

$$\frac{dP}{dQ} = \frac{-CP - pQ(1-Q)}{P + \frac{p-1}{C}Q(1-Q)} = -\frac{pC}{p-1} + \frac{C}{p-1} \cdot \frac{P}{P + \frac{p-1}{C}Q(1-Q)}, \tag{2.11}$$

which can be rewritten as

$$\frac{P}{P + \frac{p-1}{C}Q(1-Q)} = p + \frac{p-1}{C}\frac{dP}{dQ}. \tag{2.12}$$

Substituting (2.12) into (2.10), one finally obtains

$$U^- = \int_0^1 \left(p + \frac{p-1}{C}\frac{dP}{dQ}\right) dQ = p, \tag{2.13}$$

as before, since $P|_{Q=1} = 0 = P|_{Q=0}$ independently of C. However, the above reasoning only gives a necessary, and not a sufficient, condition for the existence of a corresponding heteroclinic connection in (2.5), in contrast to Proposition 2.5 below.

2.3. Critical front speed. Next, we argue that there will exist a "critical" front speed C_{crit} such that Equation (1.5) admits monotonic travelling front solutions that satisfy (2.9) for any $C \geq C_{\mathrm{crit}}$. To that end, we note that, for ecological reasons, we are not interested in the case where the origin $\mathcal{Q}^+$ is a (stable) spiral for $\{(2.5b),(2.5c)\}$, as non-monotonic oscillation would arise about $\mathcal{Q}^+$ in that case. By Lemma 2.2, it hence suffices to consider $C \in (0, 1-\sqrt{p}] \cup [1+\sqrt{p}, \infty)$, in which case $\mathcal{Q}^+$ is a stable (degenerate) node.

As in [**13**, Section 13.7], we claim that we may restrict to $C \in [1+\sqrt{p}, \infty)$ in the following. To see that, we show that the lower range of potential front speeds, with $C \in (0, 1-\sqrt{p}]$, equally leads to ecologically unrealistic solutions. To that end, we expand the eigenvalues $\lambda_\pm^+$ from (2.7), and the corresponding eigenvectors $\mathbf{v}_\pm^+$, for $C > 0$ small, which gives

$$\lambda_+^+ = -\frac{1}{1-p}C - \frac{p}{(1-p)^3}C^3 + \mathcal{O}(C^5) \quad \text{and} \quad \lambda_-^+ = -\frac{1-p}{C} + \frac{p}{1-p}C + \mathcal{O}(C^3),$$

as well as

$$\mathbf{v}_+^+ = \Big(\frac{1-p}{C} - \frac{1}{1-p}C + \mathcal{O}(C^3), 1\Big)^T \quad \text{and} \quad \mathbf{v}_-^+ = \Big(\frac{p}{1-p}C + \frac{p}{(1-p)^3}C^3 + \mathcal{O}(C^5), 1\Big)^T,$$

respectively. Hence, it follows that the weak eigendirection $\mathbf{v}_+^+$ at $\mathcal{Q}^+$ is near-horizontal, while the strong eigendirection $\mathbf{v}_-^+$ is near-vertical, as well as that both of those have positive slope in the (P, Q)-plane.

Similarly, a small-C expansion of the eigenvalues and corresponding eigenvectors at $\mathcal{Q}^-$ yields

$$\lambda_+^- = \frac{1-p}{C} + \frac{p}{p-1}C + \mathcal{O}(C^3) \quad \text{and} \quad \lambda_-^- = -\frac{1}{1-p}C + \frac{p}{(1-p)^3}C^3 + \mathcal{O}(C^5),$$

as well as

$$\mathbf{v}_+^- = \Big(\frac{p}{1-p}C - \frac{p}{(1-p)^3}C^3 + \mathcal{O}(C^5), 1\Big)^T \quad \text{and} \quad \mathbf{v}_-^- = \Big(-\frac{1-p}{C} - \frac{1}{1-p}C + \mathcal{O}(C^3), 1\Big)^T,$$

respectively; in particular, the unstable eigendirection $\mathbf{v}_+^-$ has positive slope. Therefore, it is easy to see that $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ must leave a neighbourhood of $\mathcal{Q}^-$ in the negative P-direction, as before, as well as that the saturation of $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ under the flow of $\{(2.5b),(2.5c)\}$ must enter the negative Q-plane in a neighbourhood of $\mathcal{Q}^+$ for C small, which is unecological. By continuity, it follows as in [**13**] that the phase plane geometry cannot change topologically with increasing C and that we may hence exclude $C \in (0, 1-\sqrt{p}]$. (Naturally, the above expansions may not remain valid over that interval, *i.e.*, for $C = \mathcal{O}(1)$.)

We therefore take $C = 1+\sqrt{p} =: C_{\mathrm{crit}}$ as the minimum propagation speed of realistic front solutions to (1.5); see again [**13**] for details. Correspondingly, we refer to C_{crit} as the *critical front speed*.

Remark 2.4. As $p \to 1$, $C_{\mathrm{crit}} = 1+\sqrt{p} \to 2$ evaluates to the minimum speed in the (non-dimensionalised) FKPP equation, (1.1).

Consequently, it is reasonable to assume that monotonic travelling front solutions to (1.5) exist for all $C \geq C_{\mathrm{crit}}$, as is the case for Equation (1.1), with $c_{\mathrm{crit}} = 2$. That assumption can be verified by constructing a trapping region for the flow of $\{(2.5b),(2.5c)\}$, which will prove the existence of a heteroclinic connection between $\mathcal{Q}^-$ and $\mathcal{Q}^+$ for $C \geq C_{\mathrm{crit}}$:

Proposition 2.5. *For any $C \in [C_{\mathrm{crit}}, \infty)$, with $C_{\mathrm{crit}} = 1+\sqrt{p}$, $\{(2.5b),(2.5c)\}$ admits a (monotonic) heteroclinic connection between the steady states at $\mathcal{Q}^- :\ (0,1)$ and $\mathcal{Q}^+ :\ (0,0)$.*

Proof. We construct a trapping region $\mathcal{T}$ for the flow of the (P, Q)-subsystem in (2.5) with $C \geq C_{\mathrm{crit}}$, as follows.

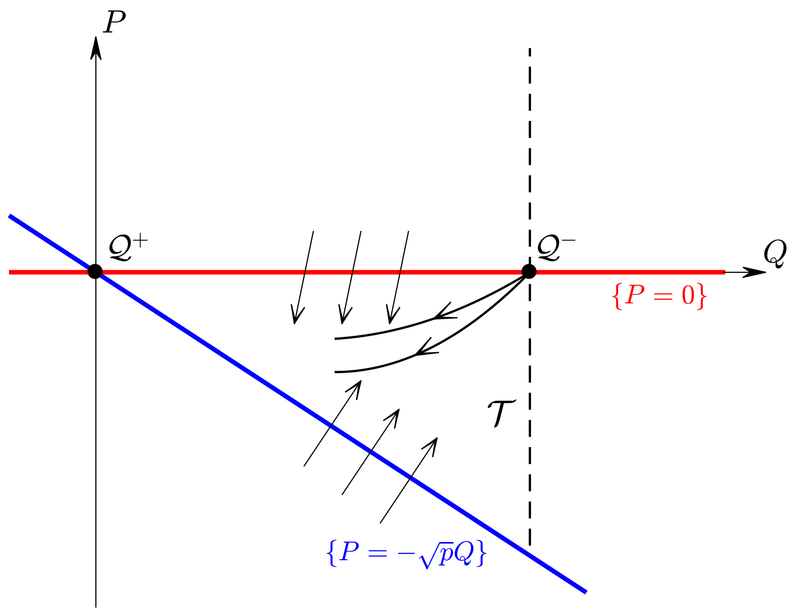

FIGURE 1. The trapping region $\mathcal{T}$ for $\{$(2.5b),(2.5c)$\}$, the (P,Q)-subsystem in Equation (2.5).

- On the Q-axis $\{P=0\}$, we have

$$\begin{aligned} P' &= -pQ(1-Q), \\ Q' &= \frac{p-1}{C}Q(1-Q) \end{aligned}$$

 and $\mathbf{n}=(-1,0)^T$ for the inward-pointing normal vector, as we consider negative P only. Therefore, $(-1,0)\cdot\big(-pQ(1-Q),\frac{p-1}{C}Q(1-Q)\big)^T=pQ(1-Q)>0$, since $Q\in(0,1)$.
- Similarly, on $\{P=-\sqrt{p}Q\}$, we calculate

$$\begin{aligned} &(1,\sqrt{p})\cdot\Big(C\sqrt{p}Q-pQ(1-Q),-\sqrt{p}Q+\frac{p-1}{C}Q(1-Q)\Big)^T \\ &=\sqrt{p}(C-\sqrt{p})Q+\sqrt{p}\Big(\frac{p-1}{C}-\sqrt{p}\Big)Q(1-Q) \\ &\geq\sqrt{p}\,1\,Q+\sqrt{p}(-1)Q(1-Q)=\sqrt{p}Q^2>0, \end{aligned} \tag{2.14}$$

 since the first term in the second line of (2.14) is always positive due to $C\geq C_{\text{crit}}(=1+\sqrt{p})$ and $Q>0$, while the second term is always negative.
- Finally, on $\{Q=1\}$, we have $(0,-1)\cdot(-CP,P)^T=-P>0$, as P is again negative.

Hence, the flow of the (P,Q)-subsystem in Equation (2.5) is trapped in the wedge $\mathcal{T}$ that is bounded by the lines $\{P=0\}$, $\{P=-\sqrt{p}Q\}$, and $\{Q=1\}$; see Figure 1 for an illustration. (Here, we remark that, crucially, the two steady states $\mathcal{Q}^+$ and $\mathcal{Q}^-$ lie on the boundary $\partial\mathcal{T}$ of $\mathcal{T}$.) Since heteroclinic orbits that are initiated in $\mathcal{Q}^-$ must follow $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ into the negative P-plane in order to connect monotonically to $\mathcal{Q}^+$, recall Section 2.2, the proof is complete. $\square$

Recalling that any solution to the FKPP equation, (1.1), tends asymptotically to the travelling front solution with minimum speed c_{crit} under mild assumptions on initial conditions [**9**], we conjecture that analogous convergence behaviour is observed for Cook's model, Equation (1.5). Hence, we will be concerned with the front solution corresponding to $C=C_{\text{crit}}$ in the following.

3. Cook's model with cut-off

In Section 2, we showed that travelling front solutions to Cook's model, Equation (1.5), correspond to heteroclinic orbits, in (U, P, Q)-space, of the first-order system in (2.5). We first reduced that system to the invariant plane Π; recall Lemma 2.1. We then showed that heteroclinic connections between the steady states at $\mathcal{Q}^-:\ (0,1)$ and $\mathcal{Q}^+:\ (0,0)$ in the resulting reduced system exist for all propagation speeds that are greater than the minimum ("critical") speed $C_{\mathrm{crit}} = 1 + \sqrt{p}$. Finally, we conjectured that solutions to (1.5) will typically tend to the "critical" front solution with propagation speed C_{crit}.

In this section, we consider the reduction of Cook's model in $\{(2.5b),(2.5c)\}$ with a cut-off in the total population Q; for simplicity, we restrict to a Heaviside cut-off here, with

$$H(Q-\varepsilon) \equiv 0 \quad \text{for } Q < \varepsilon \quad \text{and} \quad H(Q-\varepsilon) \equiv 1 \quad \text{for } Q > \varepsilon.$$

(Other choices of cut-off can be studied in a similar fashion [**4**].) Then, Equation (2.5) implies

$$P' = -CP - pQ(1-Q)H(Q-\varepsilon), \tag{3.1a}$$

$$Q' = P - \frac{1-p}{C}Q(1-Q)H(Q-\varepsilon). \tag{3.1b}$$

We emphasise that there is no immediate guarantee that travelling front solutions will persist after inclusion of a cut-off in (2.5).

Remark 3.1. It is easy to see that one can alternatively introduce a Heaviside cut-off in the original Equation (1.5) and then transform the result to a co-moving frame, as in (2.1), which again yields Equation (3.1).

For ε positive and fixed, we may consider Equation (3.1) in the $\{Q < \varepsilon\}$-regime, where

$$P' = -CP, \tag{3.2a}$$

$$Q' = P \tag{3.2b}$$

due to $H \equiv 0$. The steady states of (3.2) are located on the line $\{(0, Q) \,|\, Q \in [0, \varepsilon)\}$; however, we will again restrict to the origin $\mathcal{Q}^+$ here, as it is the only state that corresponds to the zero rest state of Equation (1.5) for any value of ε.

Remark 3.2. It is straightforward to verify that Lemma 2.1 also applies in the presence of a cut-off, *i.e.*, that the plane Π defined in (2.6) is still invariant for $\{U' = P, (3.1)\}$. In particular, it follows that the U-coordinate of the steady state corresponding to $\mathcal{Q}^-$ would again be $U^- = p$, which can also be seen directly, as before: we integrate the change in U along a given orbit, which gives

$$U^- = \int_0^1 \frac{dU}{dQ}dQ = \int_0^\varepsilon 1dQ + \int_\varepsilon^1 \frac{PdQ}{P + \frac{p-1}{C}Q(1-Q)}. \tag{3.3}$$

To evaluate the right-hand side in (3.3), we recall (2.12) to find

$$\begin{aligned} U^- &= \int_0^\varepsilon 1dQ + \int_\varepsilon^1 \left(p + \frac{p-1}{C}\frac{dP}{dQ}\right)dQ \\ &= \varepsilon + \int_\varepsilon^1 pdQ + \int_{-C\varepsilon}^0 \frac{p-1}{C}dP = \varepsilon + p - p\varepsilon + (p-1)\varepsilon = p, \end{aligned} \tag{3.4}$$

as $P|_{Q=1} = 0$ and $P|_{Q=\varepsilon} = -C\varepsilon$, independently of C.

Clearly, Equation (3.2) is linear and can be solved analytically; however, we also need to describe the flow in $\{Q > \varepsilon\}$, where (3.1) reduces to $\{(2.5b),(2.5c)\}$, giving the unique steady state $\mathcal{Q}^-$, as before. Moreover, and more importantly, we need to investigate the transition between these two regimes to determine whether they can be "patched" in $\{Q = \varepsilon\}$ to yield "full" heteroclinic orbits, as were constructed for Cook's model without cut-off in Section 2. The limit as $\varepsilon \to 0$ in Equation (3.1) is non-uniform; that non-uniformity can be removed via the blow-up technique. Details can be found in [**4**], where a similar analysis was performed in the context of the classical FKPP equation, (1.1); for a general introduction to blow-up, the reader is referred to [**2**] and [**10**].

The following theorem is the main result of this paper:

THEOREM 3.3. *There exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in [0, \varepsilon_0)$, Equation* (3.1) *with propagation speed*

$$C = C_{\rm crit}(\varepsilon) = 1 + \sqrt{p} - \frac{1+\sqrt{p}}{2\sqrt{p}}\frac{\pi^2}{(\ln\varepsilon)^2} + \mathcal{O}\big[(\ln\varepsilon)^{-2}\big]$$

admits a solution that satisfies (2.9). *(That solution corresponds to a monotonic propagating front for Cook's model, Equation* (1.5)*, with a Heaviside cut-off in the total population $u + v$.)*

Clearly, $C_{\rm crit}(0) = C_{\rm crit}(= 1 + \sqrt{p})$ in the singular limit of $\varepsilon = 0$, which will define a singular heteroclinic connection Γ_0 between $\mathcal{Q}^-$ and $\mathcal{Q}^+$ in (3.1).

3.1. Blow-up transformation. To desingularise the flow of Equation (3.1) in a neighbourhood of the degenerate origin $\mathcal{Q}^+$, we define the (homogeneous) blow-up transformation [**5**]

$$P = r\bar{P}, \quad Q = r\bar{Q}, \quad \text{and} \quad \varepsilon = r\bar{\varepsilon}; \tag{3.5}$$

here, $r \in [0, r_0]$ for r_0 positive and small, while $(\bar{P}, \bar{Q}, \bar{\varepsilon}) \in \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the unit sphere in $(\bar{P}, \bar{Q}, \bar{\varepsilon})$-space, with $\bar{P}^2 + \bar{Q}^2 + \bar{\varepsilon}^2 = 1$. Hence, the effect of the transformation in (3.5) is to "blow up" $(P, Q, \varepsilon) = (0, 0, 0)$ to $\mathbb{S}^2$; however, since we are only interested in non-negative ε, we may restrict our attention to the upper half-sphere $\mathbb{S}^2_+ = \mathbb{S}^2 \cap \{\bar{\varepsilon} \geq 0\}$.

To analyse the induced vector field on $\mathbb{S}^2_+$, we will require two coordinate charts, which we denote by K_1 and K_2. These charts are realised for $\bar{Q} = 1$ and $\bar{\varepsilon} = 1$ in (3.5), respectively:

$$K_1: \quad P = r_1 P_1, \quad Q = r_1, \quad \text{and} \quad \varepsilon = r_1\varepsilon_1; \tag{3.6a}$$

$$K_2: \quad P = r_2 P_2, \quad Q = r_2 Q_2, \quad \text{and} \quad \varepsilon = r_2. \tag{3.6b}$$

The relevant changes of coordinates between charts K_1 and K_2, which we denote by κ_{12} and $\kappa_{21} = \kappa_{12}^{-1}$, respectively, can be obtained by straightforward calculation:

$$\kappa_{12}: \quad (P_1, r_1, \varepsilon_1) = \big(P_2 Q_2^{-1}, r_2 Q_2, Q_2^{-1}\big); \tag{3.7a}$$

$$\kappa_{21}: \quad (P_2, Q_2, r_2) = \big(P_1 \varepsilon_1^{-1}, \varepsilon_1^{-1}, r_1\varepsilon_1\big). \tag{3.7b}$$

As will become apparent, the singular orbit Γ_0 from $\mathcal{Q}^-$ to $\mathcal{Q}^+$ can be constructed by combining the dynamics in charts K_1 and K_2; roughly speaking, the "outer" regime where $\{Q > \varepsilon\}$ is studied in the phase-directional chart K_1, while the rescaling chart K_2 covers the "inner" regime, with $\{Q < \varepsilon\}$. In the subsequent subsections, we will study the dynamics in these two charts, which we will then combine to conclude the existence of Γ_0 for $C = C_{\rm crit}$ in (3.1). Then, we will show in Proposition 4.1 that, for $\varepsilon > 0$ sufficiently small in (3.1), that singular orbit will persist as an orbit Γ_ε and, hence, that Cook's model will admit a "critical" front solution after inclusion of a cut-off. Furthermore, the persistent solution will be found for $C_{\rm crit}(\varepsilon) = C_{\rm crit} + \mathcal{O}[(\ln\varepsilon)^{-2}]$ in (3.1); see Proposition 4.2 below. In particular, and as was also the case for the FKPP equation with a (Heaviside) cut-off [**4**], that solution will hence be unique.

REMARK 3.4. Given any object $\square$ in the original (P, Q, ε)-variables, we will denote the corresponding blown-up object by $\overline{\square}$; in charts K_i $(i = 1, 2)$, the same object will be denoted by $\square_i$.

3.1.1. *Dynamics in chart K_2.* Substituting the coordinates in chart K_2, which are defined in (3.6b), into (3.1), we obtain

$$P_2' = -CP_2 - pQ_2(1 - r_2Q_2)H(Q_2 - 1), \tag{3.8a}$$

$$Q_2' = P_2 - \frac{1-p}{C}Q_2(1 - r_2Q_2)H(Q_2 - 1), \tag{3.8b}$$

$$r_2' = 0 \tag{3.8c}$$

for the governing equations in that chart, since $\varepsilon' = 0 = r_2'$ implies $P' = r_2P_2'$ and $Q' = r_2Q_2'$. Restricting our attention to $\{Q < \varepsilon\}$, which is equivalent to $Q_2 < 1$ in K_2, and noting that $H(Q_2 - 1) \equiv 0$ then, we may simplify Equation (3.8) to

$$P_2' = -CP_2, \tag{3.9a}$$

$$Q_2' = P_2, \tag{3.9b}$$

$$r_2' = 0. \tag{3.9c}$$

Equilibria of (3.9) lie in the plane $\{P_2 = 0\}$. Here, we are interested in the line $\ell_2^+ = \{(0,0,r_2) \mid r_2 \in [0,r_0]\}$, with r_0 positive and small: for each $r_2 = \varepsilon$ fixed, the associated point on ℓ_2^+ corresponds to the point $\mathcal{Q}^+$ before blow up. In particular, in the singular limit of $r_2 = 0$, we recover the origin on ℓ_2^+, which we will denote by $\mathcal{Q}_2^+$ in chart K_2.

A direct calculation yields

LEMMA 3.5. *Any point $(0,0,r_2) \in \ell_2^+$ is a partially hyperbolic steady state for Equation* (3.9)*, with eigenvalues $-C$ and 0 (double). The corresponding eigenspaces are spanned by $(-C,1,0)^T$ and $\{(0,1,0)^T, (0,0,1)^T\}$, respectively.*

It follows, in particular, that $\mathcal{Q}_2^+$ admits one stable eigendirection corresponding to the eigenvalue $-C_{\rm crit}$, and consequently, a one-dimensional stable manifold $\mathcal{W}_2^{\rm s}(\mathcal{Q}_2^+)$, where we have made use of the fact that $C = C_{\rm crit}(= 1+\sqrt{p})$ for $r_2 = 0$. To describe that manifold, we rewrite Equation (3.9) by introducing Q_2 as the independent variable:

$$\frac{dP_2}{dQ_2} = -(1+\sqrt{p}). \tag{3.10}$$

Equation (3.10) can be solved explicitly with $P_2(0) = 0$ to give

$$P_2(Q_2) = -(1+\sqrt{p})Q_2; \tag{3.11}$$

hence, the manifold $\mathcal{W}_2^{\rm s}(\mathcal{Q}_2^+)$ can be expressed as in (3.11), with Q_2 non-negative.

We now introduce the following section in (P_2, Q_2, r_2)-space:

$$\Sigma_2^{\rm in} = \{(P_2, 1, r_2) \mid (P_2, r_2) \in [-P_0, 0] \times [0, r_0]\}, \tag{3.12}$$

where P_0 is an appropriately defined, positive constant. We emphasise that $\Sigma_2^{\rm in}$ corresponds to the hyperplane $\{Q = \varepsilon\}$ before blow-up, and that it hence represents an entry face through which orbits of (3.9) enter the "inner" regime.

Given the definition of $\Sigma_2^{\rm in}$, we can define the portion Γ_2 of the singular orbit Γ_0 that is located in chart K_2 as the segment of the invariant line $\mathcal{W}_2^{\rm s}(\mathcal{Q}_2^+)$ that is obtained for $Q_2 \in [0,1]$. Correspondingly, we define the point $\mathcal{P}_2^{\rm in} = \Gamma_2 \cap \Sigma_2^{\rm in}$ as the intersection of the orbit Γ_2 with the section $\Sigma_2^{\rm in}$; from Equations (3.11) and (3.12), it follows that $\mathcal{P}_2^{\rm in} = (-1-\sqrt{p}, 1, 0)$.

Finally, for $r_2 \in [0, r_0]$, with r_0 sufficiently small, and $C \sim C_{\rm crit}$, the stable manifold $\mathcal{W}_2^{\rm s}(\ell_2)$ of the line ℓ_2^+ in K_2 will still be a regular perturbation of Γ_2, with $P_2(Q_2) = -CQ_2$. The geometry in chart K_2 is illustrated in Figure 2.

3.1.2. *Dynamics in chart K_1.* Substituting the coordinates in chart K_1, which are defined in (3.6a), into (3.1), we find

$$Q' = r_1' = r_1 P_1 - \frac{1-p}{C} r_1(1-r_1)H(1-\varepsilon_1) \tag{3.13}$$

and $P' = r_1' P_1 + r_1 P_1'$, as well as $0 = \varepsilon' = r_1'\varepsilon_1 + r_1\varepsilon_1'$. Making use of (3.13), solving for P_1' and ε_1', respectively, and rearranging, we have

$$P_1' = -CP_1 - p(1-r_1)H(1-\varepsilon_1) - \left[P_1 - \tfrac{1-p}{C}(1-r_1)H(1-\varepsilon_1)\right]P_1, \tag{3.14a}$$

$$r_1' = \left[P_1 - \tfrac{1-p}{C}(1-r_1)H(1-\varepsilon_1)\right]r_1, \tag{3.14b}$$

$$\varepsilon_1' = -\left[P_1 - \tfrac{1-p}{C}(1-r_1)H(1-\varepsilon_1)\right]\varepsilon_1 \tag{3.14c}$$

for the governing equations in that chart. Restricting to the regime where $\{Q > \varepsilon\}$ in K_1 and noting that $H(1-\varepsilon_1) \equiv 1$ then, we may write (3.14) as

$$P_1' = -CP_1 - p(1-r_1) + F_1(P_1, r_1)P_1, \tag{3.15a}$$

$$r_1' = -F_1(P_1, r_1)r_1, \tag{3.15b}$$

$$\varepsilon_1' = F_1(P_1, r_1)\varepsilon_1, \tag{3.15c}$$

where we have defined $F_1(P_1, r_1) = \frac{1-p}{C}(1-r_1) - P_1$.

The principal steady state of Equation (3.15) is located at $\mathcal{Q}_1 := (-\sqrt{p}, 0, 0)$; here, we note that $r_1 = 0 = \varepsilon_1$ implies $\varepsilon = 0$ and, hence, again $C = C_{\rm crit}$, from which the P_1-coordinate of $\mathcal{Q}_1$ is obtained. An

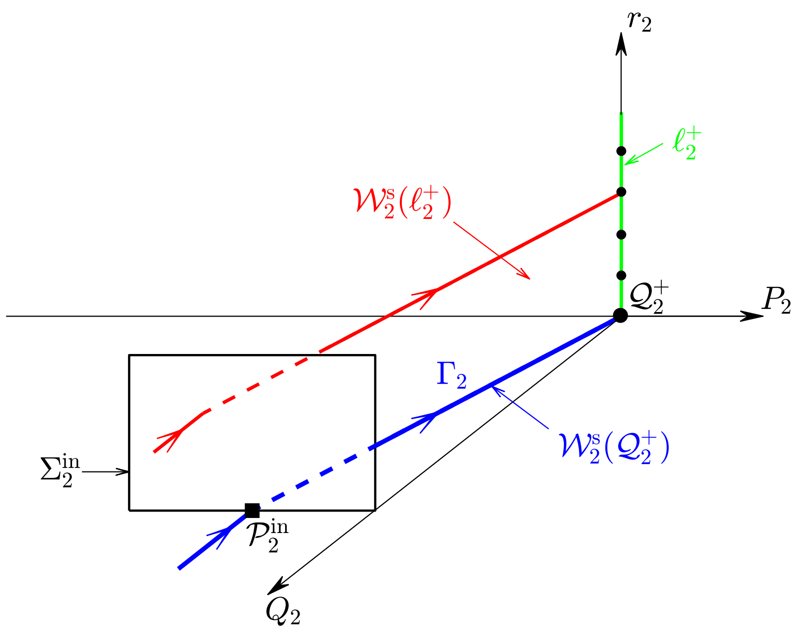

FIGURE 2. The geometry in (P_2, Q_2, r_2)-space of chart K_2: singular orbit Γ_2 (blue) and perturbed orbit (red).

additional line of steady states is found at $P_1 = 0$ and $r_1 = 1$, with ε_1 arbitrary; for $\varepsilon(=\varepsilon_1)$ fixed, points on that line, which we denote by ℓ_1^-, correspond precisely to the steady state at $\mathcal{Q}^-$. In particular, for $\varepsilon_1 = 0$, we write $\mathcal{Q}_1^-$ for the corresponding point.

Simple linearisation of (3.15) about $\mathcal{Q}_1$ shows

LEMMA 3.6. *The point $\mathcal{Q}_1$ is a partially hyperbolic steady state for Equation* (3.15)*, with eigenvalues* -1*,* 0*, and* 1*. The corresponding eigenspaces are spanned by* $(-\sqrt{p}, 1, 0)^T$*,* $(1, 0, 0)^T$*, and* $(0, 0, 1)^T$*, respectively.*

For future reference, we define the following two sections for the flow of Equation (3.15):

$$\Sigma_1^{\mathrm{in}} = \big\{(P_1, r_0, \varepsilon_1) \,\big|\, (P_1, \varepsilon_1) \in [-P_0, 0] \times [0, 1]\big\} \quad \text{and} \tag{3.16a}$$

$$\Sigma_1^{\mathrm{out}} = \big\{(P_1, r_1, 1) \,\big|\, (P_1, r_1) \in [-P_0, 0] \times [0, r_0]\big\}; \tag{3.16b}$$

here, we note that Σ_1^{out} corresponds to the section Σ_2^{in} under the change of coordinates defined in (3.7a), and that it hence represents the boundary between the "outer" and the "inner" regime.

Next, we consider the dynamics of Equation (3.15) in the singular limit as $\varepsilon \to 0$. Since $\varepsilon = r_1\varepsilon_1$, that limit corresponds to either $r_1 \to 0$ or $\varepsilon_1 \to 0$ in chart K_1. The hyperplanes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ are invariant under the flow of Equation (3.15); to construct the portion Γ_1 of the singular orbit Γ that lies in K_1, we study the dynamics of (3.15) separately in these two hyperplanes.

The portion of Γ_1 that is located in $\{\varepsilon_1 = 0\}$ is labelled Γ_1^-. Since the governing equations there are equivalent to the unperturbed Equation (2.5), Γ_1^- corresponds precisely to the unstable manifold $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ of $\mathcal{Q}^-$ for $\varepsilon = 0$ or, equivalently, to the "tail" of the associated heteroclinic orbit after blow-down. The situation is summarised in Figure 3. An adaptation of the proof of Lemma 2.5 in [**4**] yields the following result on the asymptotics of Γ_1^-.

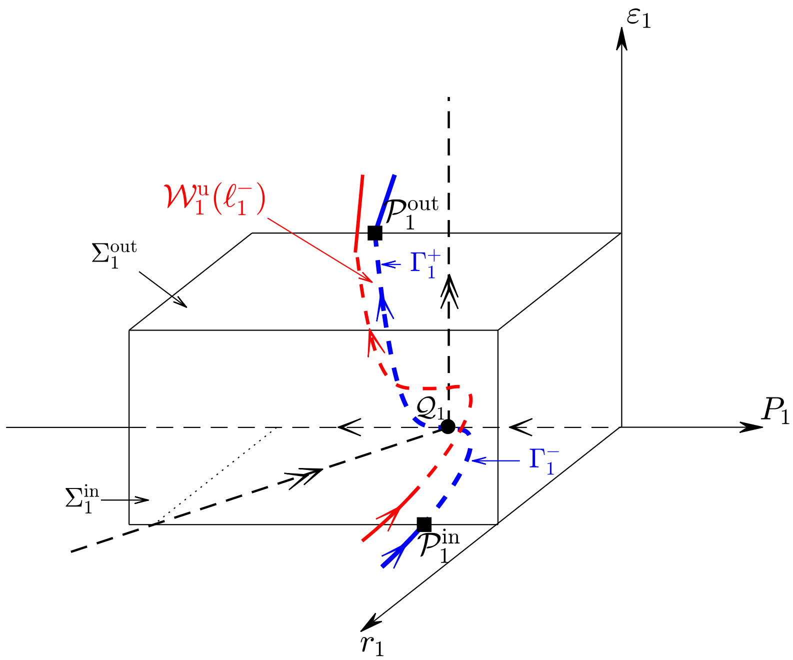

FIGURE 3. The geometry in $(P_1, r_1, \varepsilon_1)$-space of chart K_1: singular orbit $\Gamma_1^{\mp}$ (blue) and perturbed orbit for $\varepsilon(= r_1\varepsilon_1) > 0$ small (red). Dashed lines indicate eigendirections at $\mathcal{Q}_1$.

LEMMA 3.7. *The orbit* Γ_1^- *is tangent to the* P_1*-axis – i.e., to the vector* $(1, 0, 0)^T$ *– as* $\Gamma_1^- \to \mathcal{Q}_1$*.*

PROOF. As in the proof of [**4**, Lemma 2.5], the assertion follows from a phase plane argument. We consider the original first-order system, Equation (2.5), which corresponds to Cook's model without cut-off; recall (1.5). Moreover, we recall that for $C = C_{\mathrm{crit}}(= 1 + \sqrt{p})$, the point $\mathcal{Q}^+$ is a degenerate node for the (P, Q)-subsystem therein, with a unique, smooth, one-dimensional strong stable manifold $\mathcal{W}^{\mathrm{ss}}(\mathcal{Q}^+)$. Finally, we note that the manifold $\mathcal{W}^{\mathrm{ss}}(\mathcal{Q}^+)$ agrees with the stable manifold $\mathcal{W}^{\mathrm{s}}(\mathcal{Q}_1)$ of $\mathcal{Q}_1$ after transformation to chart K_1; expanding $\mathcal{W}^{\mathrm{ss}}(\mathcal{Q}^+)$ about $\mathcal{Q}^+$, we find $P(Q) = -\sqrt{p}Q - \sqrt{p}Q^2 + \mathcal{O}(Q^3)$ and, hence, $P_1^{\mathrm{s}} < -\sqrt{p}$, where P_1^{s} denotes the P_1-coordinate of the point of intersection of $\mathcal{W}^{\mathrm{s}}(\mathcal{Q}_1)$ with the section Σ_1^{in}.

To determine where Γ_1^- will lie with respect to $\mathcal{W}^{\mathrm{ss}}(\mathcal{Q}^+)$ after blow-up, we recall the proof of Lemma 2.2: we can consider the trapping region $\mathcal{T}$ constructed there for $C = C_{\mathrm{crit}}$, which implies directly that the flow of the (P, Q)-subsystem in Equation (2.5) is trapped in $\mathcal{T}$ for that C-value.

Hence, we can conclude that the singular orbit Γ_1^- corresponding to $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ must enter the equivalent of $\mathcal{T}$ in K_1 under the flow of (3.15), *i.e.*, that it must intersect Σ_1^{in} in a point $\mathcal{P}_1^{\mathrm{in}} = (P_1^{\mathrm{in}}, r_0, 0)$ with $P_1^{\mathrm{in}} > P_1^{\mathrm{s}}$. In conclusion, Γ_1^- must be tangent to the P_1-axis as $\Gamma_1^- \to \mathcal{Q}_1$. □

The portion Γ_1^+ of Γ_1 that lies in the hyperplane $\{r_1 = 0\}$ is backward asymptotic to $\mathcal{Q}_1$ and can be obtained explicitly, as follows. Equation (3.15) reduces to

$$P_1' = -(\sqrt{p} + P_1)^2, \tag{3.17a}$$

$$\varepsilon_1' = (1 - \sqrt{p} - P_1)\varepsilon_1 \tag{3.17b}$$

in $\{r_1 = 0\}$, as $C = C_{\mathrm{crit}}(= 1 + \sqrt{p})$ and, hence, $F_1(P_1, 0) = 1 - \sqrt{p} - P_1$. Rewriting Equation (3.17) with ε_1 as the independent variable, we find

$$\frac{dP_1}{d\varepsilon_1} = -\frac{(\sqrt{p} + P_1)^2}{(1 - \sqrt{p} - P_1)\varepsilon_1}. \tag{3.18}$$

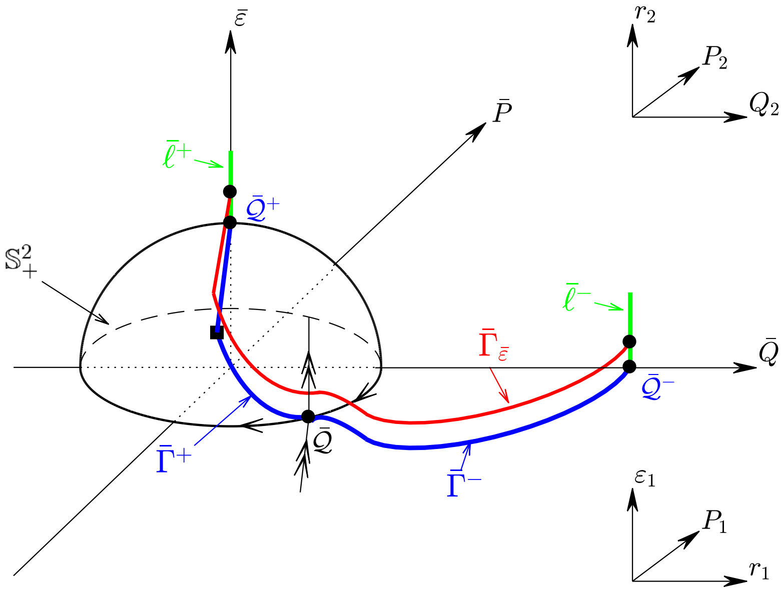

FIGURE 4. The global geometry in $(\bar{P}, \bar{Q}, \bar{\varepsilon})$-space: singular orbit $\bar{\Gamma}_0$ (blue) and perturbed orbit $\bar{\Gamma}_{\bar{\varepsilon}}$ for $\bar{\varepsilon} > 0$ small (red).

Equation (3.18) is separable and can be solved explicitly; imposing the condition that $P_1(1) = -1 - \sqrt{p}$ in Σ_1^{out}, we find

$$P_1(\varepsilon_1) = -\sqrt{p} - \frac{1}{\text{Lambert W}(\frac{\mathrm{e}}{\varepsilon_1})}. \tag{3.19}$$

Here, the Lambert W-function is defined as the (principal branch of the) solution of

$$\text{Lambert W}(z) \cdot \mathrm{e}^{\text{Lambert W}(z)} = z;$$

see [**4**] and the references therein for details. Hence, the orbit Γ_1^+ can be defined for $\varepsilon_1 \in (0, 1]$ and $r_1 = 0$, with $P_1(\varepsilon_1)$ as in (3.19). In analogy to [**4**, Lemma 2.6], we have the following result:

LEMMA 3.8. *The orbit* Γ_1^+ *is tangent to the* P_1*-axis as* $\Gamma_1^+ \to \mathcal{Q}_1$.

PROOF. The assertion follows by expanding (3.19) for ε_1 small, as

$$\text{Lambert W}(x) \sim \ln x + \mathcal{O}(\ln \ln x) \qquad \text{for } x \to \infty$$

when x is real; cf. [**14**, Eqn. 4.13.10]. $\square$

The global geometry in $(\bar{P}, \bar{Q}, \bar{\varepsilon})$-space is illustrated in Figure 4; in particular, the sought-after singular heteroclinic connection $\bar{\Gamma}_0$ is defined as the union of the orbits Γ_1^-, Γ_1^+, and Γ_2, as well as of the singularities $\mathcal{Q}_1^-$, $\mathcal{Q}_1$, and $\mathcal{Q}_2^+$. Details can be found in [**4**].

4. Proof of Theorem 3.3

In this section, we give the proof of our main result, Theorem 3.3.

4.1. Existence and uniqueness. We begin by proving that, for $\varepsilon > 0$ sufficiently small in (3.1), the unstable manifold $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ intersects the stable manifold $\mathcal{W}^{\mathrm{s}}(\mathcal{Q}^+)$ for a unique value $C_{\mathrm{crit}}(\varepsilon)$ of C. While the proof is analogous to that of [**4**, Proposition 3.1], we include it here for completeness:

Proposition 4.1. *For $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, and $C \sim C_{\mathrm{crit}}(= 1 + \sqrt{p})$, there exists a unique speed $C_{\mathrm{crit}}(\varepsilon)$ such that for $C = C_{\mathrm{crit}}(\varepsilon)$ in* (3.1)*, there exists a "critical" heteroclinic connection between $\mathcal{Q}^-$ and $\mathcal{Q}^+$. Moreover, there holds $C_{\mathrm{crit}}(\varepsilon) < C_{\mathrm{crit}}$.*

Proof. Recall the definition of the section Σ_2^{in}, cf. (3.12), as well as of the point $\mathcal{P}_2^{\mathrm{in}} = (-1 - \sqrt{p}, 1, 0)$. For $r_2(= \varepsilon)$ sufficiently small, the intersection of the stable manifold $\mathcal{W}_2^{\mathrm{s}}(\ell_2^+)$ with Σ_2^{in} can be written as the graph of a smooth function $P_2^{\mathrm{in}}(C, \varepsilon) = -C$, with

$$P_2^{\mathrm{in}}(C_{\mathrm{crit}}, 0) = -C_{\mathrm{crit}} \qquad \text{and} \qquad \frac{\partial P_2^{\mathrm{in}}}{\partial C}(C_{\mathrm{crit}}, 0) = -1,$$

where $C_{\mathrm{crit}} = 1 + \sqrt{p}$; hence, $\frac{\partial P_2^{\mathrm{in}}}{\partial C}(C, \varepsilon) = -1$ for $C \sim C_{\mathrm{crit}}$ and $\varepsilon > 0$ sufficiently small. The intersection of $\mathcal{W}^{\mathrm{s}}(\mathcal{Q}^+)$ with $\{Q = \varepsilon\}$, which is given by $P^{\mathrm{in}}(C, \varepsilon) \equiv \varepsilon P_2^{\mathrm{in}}(C, \varepsilon)$ after blow-down, therefore satisfies

$$P^{\mathrm{in}}(C, \varepsilon) = -C\varepsilon \qquad \text{and} \qquad \frac{\partial P^{\mathrm{in}}}{\partial C}(C, \varepsilon) = -\varepsilon$$

for $\varepsilon > 0$ sufficiently small.

The intersection of the unstable manifold $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ of $\mathcal{Q}^-$ with the line $\{Q = \varepsilon\}$ in

$$P' = -CP - pQ(1 - Q), \tag{4.1a}$$

$$Q' = P + \frac{p-1}{C}Q(1 - Q), \tag{4.1b}$$

i.e., in (3.1) with $H(Q, \varepsilon) \equiv 1$, is a smooth function $P^{\mathrm{out}}(C, \varepsilon)$, with $\frac{\partial P^{\mathrm{out}}}{\partial C} > 0$. Since $P^{\mathrm{out}}(C, 0) < 0$ is well-defined for $C \lesssim C_{\mathrm{crit}}$ fixed, $P^{\mathrm{out}}(C, \varepsilon)$ must also be $\mathcal{O}(1)$ and negative for $\varepsilon > 0$ sufficiently small which, by regular perturbation theory, implies $P^{\mathrm{in}} > P^{\mathrm{out}}$ for $C \lesssim C_{\mathrm{crit}}$.

Finally, for $C = C_{\mathrm{crit}}$ and $\varepsilon > 0$ small, the flow of (4.1) is trapped in the wedge $\mathcal{T}$ bounded by $\{P = 0\}$ and $\{P = -\sqrt{p}Q\}$; recall the proof of Lemma 3.7. Hence, in $\{Q = \varepsilon\}$, $P^{\mathrm{out}}(C_{\mathrm{crit}}, \varepsilon) \geq -\sqrt{p}\varepsilon$, while $P^{\mathrm{in}}(C_{\mathrm{crit}}, \varepsilon) = -(1 + \sqrt{p})\varepsilon$, by the above, and it follows that $P^{\mathrm{in}} < P^{\mathrm{out}}$ for $C = C_{\mathrm{crit}}$.

In sum, $\mathcal{W}^{\mathrm{s}}(\mathcal{Q}^+)$ and $\mathcal{W}^{\mathrm{u}}(\mathcal{Q}^-)$ must connect in $\{Q = \varepsilon\}$ for some C-value $C_{\mathrm{crit}}(\varepsilon) < C_{\mathrm{crit}}$. Uniqueness of $C_{\mathrm{crit}}(\varepsilon)$ follows from $\frac{\partial P^{\mathrm{in}}}{\partial C} < 0$ and $\frac{\partial P^{\mathrm{out}}}{\partial C} > 0$ for $C \sim C_{\mathrm{crit}}$ and $\varepsilon > 0$ small, which completes the proof. □

4.2. Transition through chart K_1. We now describe the transition through chart K_1 under the flow of Equation (3.15); more specifically, we aim to approximate the transition map between the sections Σ_1^{in} and Σ_1^{out}, as defined in (3.16), which will yield a necessary condition on the speed $C_{\mathrm{crit}}(\varepsilon)$ in Proposition 4.1. We have the following result:

Proposition 4.2. *For a heteroclinic connection to exist between the steady states $\mathcal{Q}^-$ and $\mathcal{Q}^+$ in Equation* (3.1)*, there must necessarily hold*

$$C = C_{\mathrm{crit}}(\varepsilon) = 1 + \sqrt{p} - \frac{1 + \sqrt{p}}{2\sqrt{p}} \frac{\pi^2}{(\ln \varepsilon)^2} + o\big[(\ln \varepsilon)^{-2}\big], \tag{4.2}$$

where $C_{\mathrm{crit}}(\varepsilon)$ is as defined in Proposition 4.1.

Proof. We begin by shifting the point $\mathcal{Q}_1$ to the origin; to that end, we introduce the new variable W via $P_1 = -\sqrt{p} + W$. Moreover, and in reflection of the fact that, for $\varepsilon = 0$ in (3.15), $C_{\mathrm{crit}}(0) = C_{\mathrm{crit}}(= 1 + \sqrt{p})$, we write $C_{\mathrm{crit}}(\varepsilon) = 1 + \sqrt{p} + \Delta C(\varepsilon)$, where $\Delta C = o(1)$ is negative for $\varepsilon > 0$, by Proposition 4.1, with

$\Delta C(0)=0$. In these new variables, Equation (3.15) becomes

$$W' = -(1+\sqrt{p}+\Delta C)(-\sqrt{p}+W) - p(1-r_1) + F_1(W,r_1)(-\sqrt{p}+W), \tag{4.3a}$$

$$r_1' = -F_1(W,r_1)r_1, \tag{4.3b}$$

$$\varepsilon_1' = F_1(W,r_1)\varepsilon_1, \tag{4.3c}$$

with $F_1(W,r_1) = \frac{1-p}{1+\sqrt{p}+\Delta C}(1-r_1) + \sqrt{p} - W$.

Next, we observe that Equation (4.3) is not in standard form, as the Jacobian thereof about the origin is not diagonal. Hence, we introduce the new variable Y via

$$W = Y - \sqrt{p}\frac{1+\sqrt{p}+\sqrt{p}\Delta C}{1+\sqrt{p}-(2-\sqrt{p})\Delta C - \Delta C^2} r_1,$$

which yields

$$Y' = \frac{\sqrt{p}\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C} - \frac{\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C}Y - Y^2 + \mathcal{O}(r_1 Y, r_1^2), \tag{4.4a}$$

$$r_1' = -\widetilde{F}_1(Y,r_1)r_1, \tag{4.4b}$$

$$\varepsilon_1' = \widetilde{F}_1(Y,r_1)\varepsilon_1, \tag{4.4c}$$

with

$$\widetilde{F}_1(Y,r_1) = \frac{1+\sqrt{p}(1+\Delta C)}{1+\sqrt{p}+\Delta C} - \frac{1-2p\sqrt{p}-3p-2(1+p\sqrt{p})\Delta C - \Delta C^2}{(1+\sqrt{p}+\Delta C)[1+\sqrt{p}-(2-\sqrt{p})\Delta C-\Delta C^2]} r_1 - Y. \tag{4.5}$$

(We note that the $\mathcal{O}(r_1 Y, r_1^2)$-terms in Equation (4.4a) are known explicitly, but that they are insignificant to the order considered here.)

Now, we divide the right-hand sides in (4.4) by a factor of $\widetilde{F}_1$, which corresponds to a transformation of the independent variable that leaves the direction of the flow unchanged, as $\widetilde{F}_1(Y,r_1)$ is positive for $\|(Y,r_1)\|$ sufficiently small and $\Delta C = o(1)$:

$$\dot{Y} = \frac{\frac{\sqrt{p}\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C} - \frac{\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C}Y - Y^2}{\widetilde{F}_1(Y,r_1)} + \mathcal{O}(r_1 Y, r_1^2), \tag{4.6a}$$

$$\dot{r}_1 = -r_1, \tag{4.6b}$$

$$\dot{\varepsilon}_1 = \varepsilon_1. \tag{4.6c}$$

Here, the overdot denotes differentiation with respect to a new independent variable ξ. As in [**4**, Section 3], we can now perform a (near-identity) normal form transformation $(Y,r_1)\mapsto(y,R)$ that removes all non-resonant terms – and, in particular, the $\mathcal{O}(r_1 Y, r_1^2)$-terms – in (4.6a), reducing Equation (4.6) to

$$\dot{y} = \frac{\frac{\sqrt{p}\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C} - \frac{\Delta C(2+\Delta C)}{1+\sqrt{p}+\Delta C}y - y^2}{\frac{1+\sqrt{p}(1+\Delta C)}{1+\sqrt{p}+\Delta C} - y}, \tag{4.7a}$$

$$\dot{R} = -R, \tag{4.7b}$$

$$\dot{\varepsilon}_1 = \varepsilon_1. \tag{4.7c}$$

Finally, we translate the variable y via $y = z - \frac{\Delta C(2+\Delta C)}{2(1+\sqrt{p}+\Delta C)}$, which transforms Equation (4.7a) to

$$\dot{z} = \frac{z^2 - \frac{\Delta C(2+\Delta C)}{4}\left[1+\frac{2\sqrt{p}(1+\Delta C)+3p-1}{(1+\sqrt{p}+\Delta C)^2}\right]}{z-1-\frac{\Delta C(2\sqrt{p}+\Delta C)}{2(1+\sqrt{p}+\Delta C)}}. \tag{4.8}$$

Since (4.8) is separable, and of the form $-d\xi = dz\frac{-z+1+\beta}{z^2-\alpha}$ with

$$\alpha = \frac{\Delta C(2+\Delta C)}{4}\Big[1+\frac{2\sqrt{p}(1+\Delta C)+3p-1}{(1+\sqrt{p}+\Delta C)^2}\Big] \quad \text{and} \quad \beta = \frac{\Delta C(2\sqrt{p}+\Delta C)}{2(1+\sqrt{p}+\Delta C)},$$

see again [**4**, Section 3], we may integrate to find

$$-\xi(z) = -\frac{1}{2}\ln(z^2-\alpha) + \frac{1+\beta}{\sqrt{-\alpha}}\arctan\Big(\frac{z}{\sqrt{-\alpha}}\Big) + K, \tag{4.9}$$

where K is a constant of integration; here, we note that α and β are both negative due to $\Delta C = o(1)$ being negative.

Reverting to y in (4.9) and expanding the result for ΔC small, we find

$$\begin{aligned} -\xi(z) = -\frac{1}{2}\ln\big[y^2 + \mathcal{O}(\Delta C)\big] + \bigg(\sqrt{\tfrac{1+\sqrt{p}}{2\sqrt{p}}}\frac{1}{\sqrt{-\Delta C}} + \mathcal{O}(\sqrt{-\Delta C})\bigg) \\ \times \arctan\bigg(\sqrt{\tfrac{1+\sqrt{p}}{2\sqrt{p}}}\frac{y}{\sqrt{-\Delta C}} + \mathcal{O}(\sqrt{-\Delta C})\bigg) + K. \end{aligned} \tag{4.10}$$

To approximate the "time" Ξ taken by solutions in their transition between the two sections corresponding to $\Sigma_1^{\rm in}$ and $\Sigma_1^{\rm out}$ after the above sequence of transformations from P_1 to y, we note that $P_1^{\rm in} \in (-\sqrt{p}, 0)$, whereas $P_1^{\rm out} = -(1+\sqrt{p})$. Hence, it is easy to see that both $y^{\rm in} > 0$ and $y^{\rm out} < 0$ are $\mathcal{O}(1)$ as $\varepsilon \to 0$, and independent of ΔC to leading order. Furthermore, we make use of the fact that, for $|x|$ large, $\arctan x = \pm\frac{\pi}{2} - \big(\frac{1}{x} + \mathcal{O}(x^{-3})\big)$, where the sign equals that of x. In sum, (4.10) yields

$$\Xi = \sqrt{\tfrac{1+\sqrt{p}}{2\sqrt{p}}}\frac{1}{\sqrt{-\Delta C}}\bigg\{\pi + \mathcal{O}(\sqrt{-\Delta C})\bigg[\sqrt{\tfrac{2\sqrt{p}}{1+\sqrt{p}}}\Big(\frac{1}{y^{\rm out}} - \frac{1}{y^{\rm in}}\Big) + \ln\Big|\frac{y^{\rm out}}{y^{\rm in}}\Big|\bigg] + \mathcal{O}\big[(-\Delta C)^{3/2}\big]\bigg\}. \tag{4.11}$$

Finally, since (4.7b) implies $R(\xi) = R^{\rm in}{\rm e}^{-\xi}$, with $R^{\rm in} = R(0) > 0$, it follows that $\xi = -\ln\frac{R}{R^{\rm in}} = -\ln r_1 + \mathcal{O}(1)$. Recalling that $r_1^{\rm out} = \varepsilon$, we conclude that $\Xi = -\ln\varepsilon + \mathcal{O}(1)$ which, together with (4.11), shows Equation (4.2), completing the proof. □

The necessary condition on $C_{\rm crit}(\varepsilon)$ found in Proposition 4.2, in combination with the existence argument in Proposition 4.1, concludes the proof of Theorem 3.3.

Remark 4.3. The correction ΔC in Equation (4.2) reduces to the expression obtained for the classical FKPP equation [**4**] in the limit as $p \to 1$; similarly, one can verify that the various transformations introduced in the proof of Proposition 2.5 reduce to their analogues in [**4**, Proposition 3.2] in that limit.

5. Conclusions

In this paper, we have studied the effects of a Heaviside cut-off on front propagation in a modification of the classical Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation that incorporates non-dispersal of a subpopulation. While the underlying two-component reaction-diffusion model without a cut-off, which was first proposed by Cook [**13**, Section 13.7], has been discussed previously, the inclusion of a cut-off has not been considered before, to the best of our knowledge. Applying geometric singular perturbation theory (GSPT) and the blow-up technique, we have shown existence and (local) uniqueness of a "critical" front solution between the two homogeneous steady states in the model, and we have derived the leading-order correction to the corresponding front propagation speed in terms of the cut-off parameter (ε). In sum, we have hence elucidated the dependence of the speed on both ε and the probability of dispersal (p), showing, in particular, that our results reduce to those found for the FKPP equation with a cut-off when the entire population disperses, *i.e.*, as $p \to 1$. Correspondingly, we have found that the leading-order correction to the propagation speed depends on the inverse of the square of the logarithm of the cut-off parameter. It is interesting to note that this logarithmic asymptotics is due to resonance in the phase-directional chart after blow-up, as was also the case in [**4**]; as is to be expected, the associated normal form is equivalent to the one found there.

As is common practice, we have constructed front solutions as heteroclinic connections between equilibria in the system of differential equations that is obtained from the original reaction-diffusion model by transformation to a co-moving frame. An important preliminary step in our analysis involves the reduction of the resulting, three-dimensional first-order system to an invariant plane; that step not only allows us to adapt the study of the cut-off FKPP equation from [**4**] to the present context, but also to give an elementary proof for the existence of monotonic propagating fronts in Cook's model without a cut-off. Our findings are

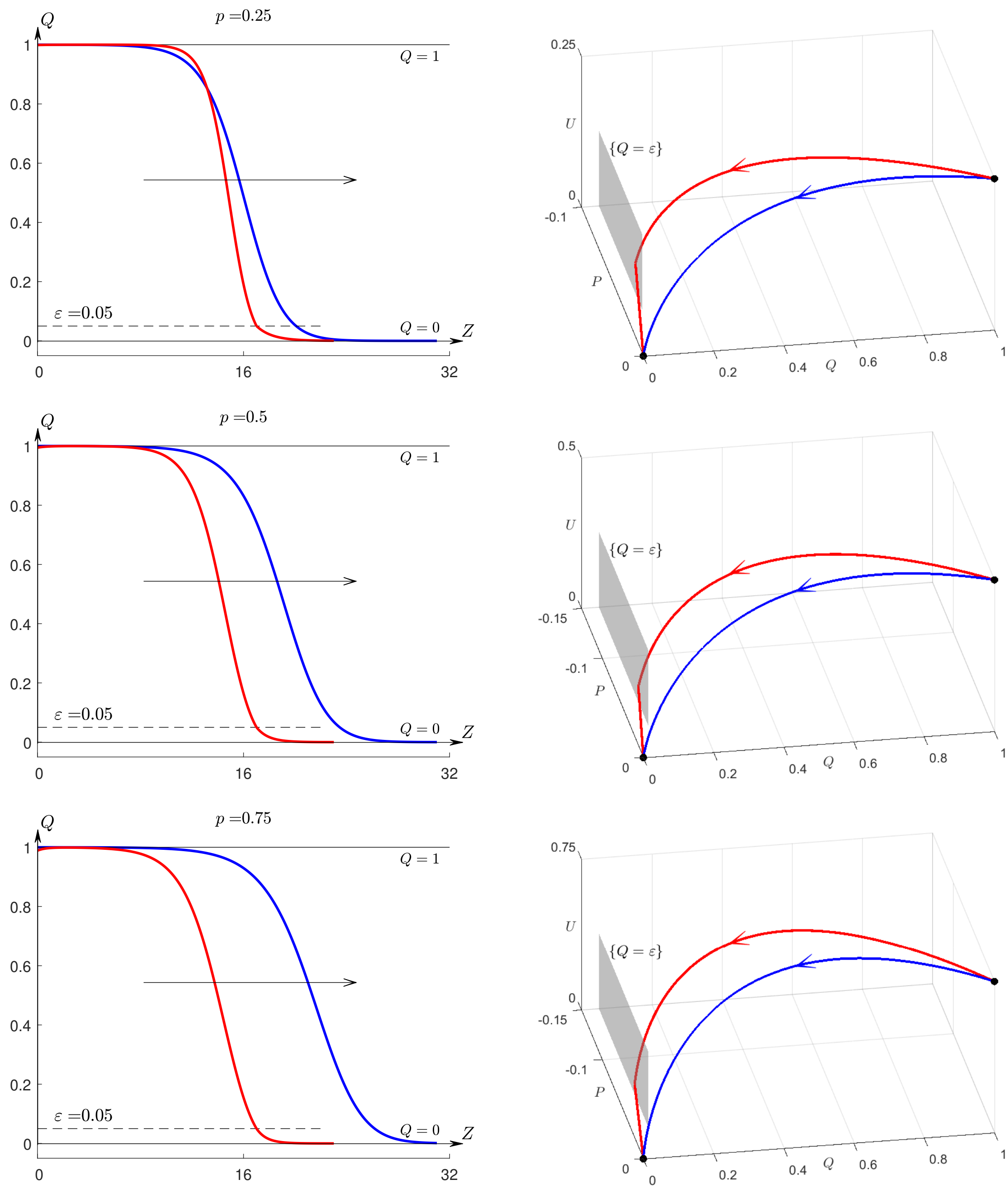

FIGURE 5. Numerical front solutions to Equation (1.5) for varying values of $p \in (0, 1)$, with $Q(Z) = (u + v)(X - CT)$ (left column) and corresponding orbits in (P, Q, U)-space for Equation (2.5) (right column): no cut-off (blue) and Heaviside cut-off with $\varepsilon = 0.05$ (red).

illustrated numerically in Figure 5, where we show front solutions and the corresponding heteroclinic orbits, both without cut-off and with a Heaviside cut-off, for a range of dispersal probabilities. We remark that, while the shape of either of those is not dramatically affected by the cut-off, the front propagation speed is reduced significantly, which is consistent with the asymptotics stated in Theorem 3.3; see also [**4**].

While we have hence given a rather complete geometric analysis of front propagation in Cook's model with a cut-off, several open questions remain. The most obvious of those is the (spectral) stability of the persistent front solution which we have constructed; presumably, that question can be answered in the affirmative via an adaptation of a recently performed stability analysis for the cut-off FKPP equation [**8**]. Another natural question concerns the universality of the correction to the critical front propagation speed stated in Theorem 3.3: while the latter was obtained for a Heaviside cut-off, it can be expected that it is universal within a broad family of cut-off functions, as was also the case in [**4**]. Finally, we are aware of at least two generalisations of Cook's model that could be studied along the lines of the present paper; one [**11**] allows for switching between dispersive and non-dispersive states, while the other [**7**] incorporates a time delay.

Acknowledgements

This work is inspired by a vacation scholarship project which the first author conducted together with Martin Brolly in the School of Mathematics at the University of Edinburgh over the summer of 2018.

References

1. E. Brunet and B. Derrida, *Shift in the velocity of a front due to a cutoff*, Phys. Rev. E **56** (1997), no. 3, 2597–2604 (English).
2. F. Dumortier, *Techniques in the theory of local bifurcations: Blow-up, normal forms, nilpotent bifurcations, singular perturbations*, Bifurcations and periodic orbits of vector fields. Proceedings of the NATO Advanced Study Institute and Séminaire de Mathématiques Supérieures, Montréal, Canada, July 13-24, 1992, Dordrecht: Kluwer Academic Publishers, 1993, pp. 19–73 (English).
3. F. Dumortier and T. J. Kaper, *Wave speeds for the FKPP equation with enhancements of the reaction function*, Z. Angew. Math. Phys. **66** (2015), no. 3, 607–629 (English).
4. F. Dumortier, N. Popović, and T. J. Kaper, *The critical wave speed for the Fisher-Kolmogorov-Petrowskii-Piscounov equation with cut-off*, Nonlinearity **20** (2007), no. 4, 855–877 (English).
5. F. Dumortier and R. Roussarie, *Canard cycles and center manifolds*, Mem. Am. Math. Soc., vol. 577, Providence, RI: American Mathematical Society (AMS), 1996 (English).
6. R. A. Fisher, *The wave of advance of advantageous genes.*, Ann. Eugenics **7** (1937), no. 4, 355–369 (English).
7. S. Harris, *Traveling waves with dispersive variability and time delay*, Phys. Rev. E **68** (2003), no. 3, 031912 (English).
8. P. Kaklamanos, *Personal communication*, 2023.
9. A. Kolmogoroff, I. Petrovsky, and N. Piscounoff, *Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique*, Bull. Univ. État Moscou, Sér. Int., Sect. A: Math. et Mécan. 1, Fasc. 6, 1-25, 1937.
10. M. Krupa and P. Szmolyan, *Extending geometric singular perturbation theory to nonhyperbolic points – fold and canard points in two dimensions*, SIAM J. Math. Anal. **33** (2001), no. 2, 286–314 (English).
11. M. A. Lewis and G. Schmitz, *Biological invasion of an organism with separate mobile and stationary states: Modelling and analysis*, Forma **11** (1996), no. 1, 1–25 (English).
12. J. Mai, I. M. Sokolov, and A. Blumen, *Front propagation in one-dimensional autocatalytic reactions: The breakdown of the classical picture at small particle concentrations*, Phys. Rev. E **62** (2000), no. 1, 141–145 (English).
13. J. D. Murray, *Mathematical biology. Vol. 1: An introduction.*, 3rd ed., Interdiscip. Appl. Math., vol. 17, New York, NY: Springer, 2002 (English).
14. National Institute of Standards and Technology, *NIST Digital Library of Mathematical Functions*, 2010, accessed on April 27, 2023.

University of Edinburgh, School of Mathematics and Maxwell Institute for Mathematical Sciences, James Clerk Maxwell Building, King's Buildings, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom
Email address: `nikola.popovic@ed.ac.uk`

Department of Mathematics, College of Science, China Jiliang University, 258 Xueyuan Street, Qiantang District, 310018 Hangzhou, Zhejiang, China
Email address: `zhouqian.miao@cjlu.edu.cn`