RIGOROUS ASYMPTOTIC EXPANSIONS FOR CRITICAL WAVE SPEEDS IN A FAMILY OF SCALAR REACTION-DIFFUSION EQUATIONS

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Abstract. We investigate traveling wave solutions in a family of reaction-diffusion equations which includes the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with quadratic nonlinearity and a bistable equation with degenerate cubic nonlinearity. It is known that, for each equation in this family, there is a critical wave speed which separates waves of exponential decay from those of algebraic decay at one of the end states. We derive rigorous asymptotic expansions for these critical speeds by perturbing off the classical FKPP and bistable cases. Our approach uses geometric singular perturbation theory and the blow-up technique, as well as a variant of the Melnikov method, and confirms the results previously obtained through asymptotic analysis in [J.H. Merkin and D.J. Needham, J. Appl. Math. Phys. (ZAMP) A, 44(4):707-721, 1993] and [T.P. Witelski, K. Ono, and T.J. Kaper, Appl. Math. Lett., 14(1):65-73, 2001].

1. Introduction

We consider traveling wave solutions for the family of scalar reaction-diffusion equations given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u), \tag{1}$$

where the potential functions $f_m(u)$ are defined by $f_m(u) = 2u^m(1-u)$ for $m \geq 1$. This family of equations includes the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation ($m = 1$) [Fis37, KPP37], as well as a bistable equation ($m = 2$) [CCL75, WOK01], and has arisen in the study of numerous phenomena in biology, optics, combustion, and other disciplines, see e.g. [Bri86, MN93, SM96, NB99].

Traveling waves of velocity c are solutions of (1) that connect the rest states $u = 0$ and $u = 1$ and that are stationary in a frame moving at the constant wave speed c. Let $\xi = x - ct$ and $U(\xi) = u(x,t)$; then, traveling waves are found as solutions of the nonlinear second-order equation

$$U'' + cU' + f_m(U) = 0 \tag{2}$$

with $0 \leq U \leq 1$ for $\xi \in \mathbb{R}$ that satisfy

$$\lim_{\xi\to-\infty} U(\xi) = 1 \quad \text{and} \quad \lim_{\xi\to\infty} U(\xi) = 0, \tag{3}$$

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where the prime denotes differentiation with respect to ξ. It is well-known that for each $m \geq 1$, there is a critical wave speed $c_{\text{crit}}(m) > 0$ such that traveling wave solutions exist for $c \geq c_{\text{crit}}(m)$ in (1) [Bri86, BN91, MN93]. The speed $c_{\text{crit}}(m)$ is critical in the sense that the wave decays exponentially ahead of the wave front (i.e. for $\xi \to \infty$) when $c = c_{\text{crit}}(m)$, whereas it decays at an algebraic rate for $c > c_{\text{crit}}(m)$.

In the (U, U')-phase plane, traveling wave solutions of (1) correspond to heteroclinic trajectories of the system

$$\begin{aligned} U' &= V, \\ V' &= -f_m(U) - cV, \end{aligned} \tag{4}$$

connecting the two equilibria $(1, 0)$ and $(0, 0)$. Geometrically, the dependence of these solutions on c can be understood as follows. The point $(1, 0)$ is a hyperbolic saddle regardless of the value of c, with eigenvalues $\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 2}$. Traveling waves exist when the unstable manifold of this saddle point connects to the equilibrium at the origin. Moreover, the nature of this connection determines the type of the traveling wave. To be precise, for $m > 1$, the origin is a semi-hyperbolic fixed point. The eigenvalues of (4) at $(0, 0)$ are $-c$ and 0, with eigenvectors $(1, -c)^T$ and $(1, 0)^T$, respectively. Whenever $c > c_{\text{crit}}(m)$, the lower left branch of the unstable manifold of $(1, 0)$ approaches the origin on a one-dimensional center manifold, which is locally tangent to the span of the eigenvector $(1, 0)^T$. Hence, solutions decay merely algebraically as $\xi \to \infty$. Precisely for $c = c_{\text{crit}}(m)$, however, this same branch coincides with the one-dimensional strong stable manifold of $(0, 0)$; hence, solutions approach the origin tangent to the span of $(1, -c)^T$ and decay exponentially for $\xi \to \infty$. Therefore, for each $m > 1$, a global bifurcation occurs at $c = c_{\text{crit}}(m)$ due to the switchover from one type of connection to another in (4).

For $m > 1$ and $c < c_{\text{crit}}(m)$, no heteroclinic solutions to (4) exist, as the unstable manifold of $(1, 0)$ does not enter the basin of attraction of the origin, but veers off to become unbounded for $\xi \to \infty$.

Finally, for $m = 1$, the critical speed $c_{\text{crit}}(1) = 2\sqrt{2}$ is determined by a local transition condition, with the origin changing from being a stable node for $c > c_{\text{crit}}(1)$ via a degenerate node at $c = c_{\text{crit}}(1)$ to a stable spiral for $c < c_{\text{crit}}(1)$. In terms of U, this implies $U \sim C\xi e^{-\sqrt{2}\xi}$ for $c = 2\sqrt{2}$ as $\xi \to \infty$, whereas the decay is strictly exponential for $c > 2\sqrt{2}$.

We study (4) for $m = n + \varepsilon$, where $n = 1, 2$, with $0 < \varepsilon \ll 1$ for $n = 1$ and $0 < |\varepsilon| \ll 1$ for $n = 2$, respectively. In both cases, we derive rigorous asymptotic expansions for the critical wave speed $c_{\text{crit}}(m) \equiv c^n_{\text{crit}}(\varepsilon)$ that separates algebraic solutions from those of exponential structure. More specifically, we will prove

$$c^1_{\text{crit}}(\varepsilon) = 2\sqrt{2} - \sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon) \qquad \text{for } \varepsilon \in (0, \varepsilon_0) \tag{5}$$

and

$$c^2_{\text{crit}}(\varepsilon) = 1 - \frac{13}{24}\varepsilon + \mathcal{O}(\varepsilon^2) \qquad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{6}$$

respectively, where $\varepsilon_0 > 0$ has to be chosen sufficiently small and $-\Omega_0 \approx -2.338107$ is the first real zero of the Airy function. Hence, our findings confirm the more formal results previously obtained in [MN93] ($n = 1$) and [WOK01] ($n = 2$) by means of asymptotic analysis. The restriction to ε positive for $n = 1$ is necessary, as it has been shown [NB99] that no traveling wave solutions for (4) can exist when $m < 1$. This fact will also become apparent through our analysis: We will confirm that $m = 1$ is a borderline case and will provide a geometric justification.

Given the above assumptions, it is useful to write (4) as

$$\begin{aligned} U' &= V, \\ V' &= -2U^{n+\varepsilon}(1-U) - cV. \end{aligned} \tag{7}$$

Note that the vector field in (7) is generally only $\mathcal{C}^n$ in U and even just $\mathcal{C}^{n-1}$ when ε is negative. To circumvent this lack of smoothness, as well as the inherent non-hyperbolic character of the problem, and to facilitate an analysis of (7) by geometric singular perturbation techniques, we propose an alternative formulation for (7). Our approach is based on introducing projectivized coordinates in (7) by setting $Z = \frac{V}{U}$. In terms of U and the new variable Z, the equations in (7) become

$$\begin{aligned} U' &= UZ, \\ Z' &= -2U^{n-1+\varepsilon}(1-U) - cZ - Z^2. \end{aligned} \tag{8}$$

The introduction of Z is motivated by the observation that $\frac{U'}{U} \sim -c$ as $\xi \to \infty$ for solutions that approach the origin in (7) along its strong stable manifold, whereas $\frac{U'}{U} \sim 0$ when the approach is along a center manifold. Therefore, the projectivization via Z is a first step to teasing apart the various possible asymptotics in (7), as these correspond to different regimes in the projectivized equations in (8). As will become clear in the following, it also eliminates some – though not all – of the non-hyperbolicity from the problem and, hence, enables us to apply techniques from dynamical systems theory which might otherwise not be applicable.

Next, we define $Y = -\varepsilon \ln U$ to eliminate the U^ε-term in (8):

$$\begin{aligned} U' &= UZ, \\ Z' &= -2e^{-Y}U^{n-1}(1-U) - cZ - Z^2, \\ Y' &= -\varepsilon Z. \end{aligned} \tag{9}$$

System (9) is a fast-slow system, with U, Z fast and Y slow. Moreover, as opposed to (7), (9) is $\mathcal{C}^k$-smooth in all three variables U, Z, and Y, as well as in the parameters ε and c, for any $k \in \mathbb{N}$. In the following, we will consider (9) as a system in which the three variables U, Z, and Y are treated equally. Since (7) and (9) are equivalent by the definition of Z and Y (at least for $U \in (0,1)$), the subsequent analysis (and in particular the expansions for $c_{\text{crit}}^n(\varepsilon)$ derived below) will then carry over from (9) to (7).

As it turns out, the additional complexity introduced by embedding the planar system (8) into a three-dimensional system (9) is compensated by the resulting gain in smoothness. In fact, for $n = 2$, the equations in (9) can be treated directly using a variant of the Melnikov method for fast-slow systems which was first introduced in [Rob83]. In addition to giving us explicitly the first-order correction in the expansion for c_{crit}^2, this approach also proves the regularity of $c_{\text{crit}}^2(\varepsilon)$ as a function of ε, see (6).

The situation is more complicated for $n = 1$. There, one finds that, after a center manifold reduction, the equations are locally equivalent to the singularly perturbed planar fold problem which was analyzed in full detail via the so-called blow-up technique [DR91, Dum93] in a series of articles by Krupa and Szmolyan, see e.g. [KS01] and the references therein.

Our analysis shows that the complicated structure of the problem for $n = 1$ is determined by the characteristics of the underlying singularly perturbed planar fold, and it provides a non-trivial application of the more general results presented in [KS01]. This correspondence also explains the a priori unexpected expansion for $c^1_{\text{crit}}(\varepsilon)$ in fractional powers of ε, cf. (5), as these powers arise naturally through the blow-up transformation. We retrace the analysis of [KS01] here, taking into account the inclusion of the parameter c. It turns out that the additional degree of freedom introduced into the problem by c is essential, as it is this freedom which allows us to identify the one solution of (9) corresponding to the critical speed c^1_{crit} out of an entire family of possible solutions. In particular, c^1_{crit} is determined by the geometric condition that this solution stay close to the repelling branch of a critical manifold for (9) for an infinite amount of time. In that sense, the criticality of c^1_{crit} can be interpreted as a canard phenomenon [Die84], see also [GHW00], where such solutions are referred to as "fold initiated canards".

This article is organized as follows. In Section 2, we show how the equations in (9) with $n = 1$ can be reduced to the singularly perturbed planar fold problem. We then adapt the results from [KS01] to obtain the desired expansion for $c^1_{\text{crit}}(\varepsilon)$. In Section 3, we present a modification of the Melnikov method for fast-slow systems from [Rob83], which we then apply to (9) with $n = 2$ to derive the expansion for $c^2_{\text{crit}}(\varepsilon)$.

Remark 1. *The equations in* (8) *may also be derived from* (2) *via the well-known transformation*

$$U(\xi) = C \exp\Big[\int^{\xi} Z(s)\, ds\Big], \tag{10}$$

which (for $U \neq 0$) shows (2) *to be equivalent to* (8).

Remark 2. *Alternatively,* (8) *can be obtained from* (4) *through blow-up, i.e. the degenerate origin in* (4) *can be desingularized directly by means of a homogeneous blow-up transformation. Then,* (8) *is retrieved in one of the charts which are commonly introduced to describe the blown-up vector field. This is to be expected, since blow-up can also be viewed as a kind of projectivization. However, we do not pursue this approach here.*

2. Traveling waves for $m = 1 + \varepsilon$

When $m = 1 + \varepsilon$, the equations in (4) are given by

$$\begin{aligned} U' &= V, \\ V' &= -2U^{1+\varepsilon}(1-U) - cV. \end{aligned} \tag{11}$$

It is well-known that $c^1_{\text{crit}} = 2\sqrt{2}$ when $\varepsilon = 0$ in (11). The main result of this section is the following theorem on the asymptotics of c^1_{crit} for ε positive and small:

Theorem 1. *There exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the critical wave speed c^1_{crit} for* (11) *can be represented as a function of ε. Moreover, the expansion of c^1_{crit} is regular in powers of $\varepsilon^{\frac{1}{3}}$ and is given by*

$$c^1_{\text{crit}}(\varepsilon) = 2\sqrt{2} - \sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon). \tag{12}$$

Here, $-\Omega_0 \approx -2.338107$ is the first real zero of the Airy function.

Up to a factor of $\sqrt{2}$ (which is due to an additional factor of 2 in our definition of f_m), this is exactly the expansion found in [MN93] by the method of matched asymptotics. In particular, note that $c^1_{\rm crit}$ is non-smooth (not even $\mathcal{C}^1$) in ε as $\varepsilon \to 0$.

Remark 3. *By proving Theorem 1 using geometric singular perturbation theory, we will obtain an alternative proof of the well-known existence and uniqueness of traveling waves for $c \geq c^1_{\rm crit}$ in (11).*

2.1. The projectivized equations for $\varepsilon = 0$. For $n = 1$, one obtains from (9) the projectivized system

$$
\begin{aligned}
U' &= UZ,\\
Z' &= -2(1-U)e^{-Y} - cZ - Z^2,\\
Y' &= -\varepsilon Z.
\end{aligned}
\tag{13}
$$

Recall that for $\varepsilon = 0$ in (13), the critical wave speed is given by $c^1_{\rm crit} = 2\sqrt{2}$, which will be used repeatedly throughout the following analysis. Moreover, there holds $Y = 0$ by definition. Then, a straightforward computation shows that there are two equilibria for (13), with $U = 1$ and $Z = 0$ or $U = 0$ and $Z = -\sqrt{2}$. More precisely, one has the following simple lemma:

Lemma 1. *When $\varepsilon = 0$, the point $Q^- : (1, 0)$ is a hyperbolic saddle fixed point of*

$$
\begin{aligned}
U' &= UZ,\\
Z' &= -2(1-U) - 2\sqrt{2}Z - Z^2,
\end{aligned}
$$

with the eigenvalues given by $\lambda_1 = 2-\sqrt{2}$ and $\lambda_2 = -2-\sqrt{2}$. The corresponding eigenspaces are spanned by $(1, 2-\sqrt{2})^T$ and $(1, -2-\sqrt{2})^T$. The point $Q^+ : (0, -\sqrt{2})$ is a semi-hyperbolic fixed point, with eigenvalues 0 and $-\sqrt{2}$ and corresponding eigendirections $(0, 1)^T$ and $(1, -\sqrt{2})^T$.

Additionally, from the existence of a singular heteroclinic solution to (11) it follows that there exists an orbit Γ connecting Q^- and Q^+; note that no closed-form expression for Γ seems to be known, see e.g. [Bri86] and the references therein.

2.2. Strategy of the proof via blow-up. In this section, we outline the proof of Theorem 1, which is based on the blow-up method. We show that, after a center manifold reduction, equations (13) locally reduce to the singularly perturbed planar fold problem, which has been analyzed in detail in [KS01]. In particular, this correspondence will explain the a priori unexpected expansion for $c^1_{\rm crit}(\varepsilon)$ in fractional powers of ε. Standard center manifold theory will then enable us to extend our results to the full system (13).

We set out by rewriting (13) in terms of the new variables $\widetilde{Z} := \sqrt{2}+Z$ and $\tilde{c} := -2\sqrt{2}+c$, whereby for $\varepsilon = 0$, the equilibrium at Q^+ is shifted to the origin, which we label $\widetilde{Q}^+$.

Moreover, we extend (13) by appending the trivial equations $\tilde{c}' = 0$ and $\varepsilon' = 0$:

$$
\begin{aligned}
U' &= -U(\sqrt{2} - \widetilde{Z}), \\
\widetilde{Z}' &= 2 - 2(1-U)e^{-Y} + \tilde{c}(\sqrt{2} - \widetilde{Z}) - \widetilde{Z}^2, \\
Y' &= \varepsilon(\sqrt{2} - \widetilde{Z}), \\
\tilde{c}' &= 0, \\
\varepsilon' &= 0.
\end{aligned} \tag{14}
$$

The following result follows immediately from Lemma 1 and standard invariant manifold theory [Car81, CLW94].

Proposition 1. *For any $k \in \mathbb{N}$, the equations in* (14) *possess an attracting $\mathcal{C}^k$-smooth center manifold $\mathcal{W}^c$ at $\widetilde{Q}^+$. $\mathcal{W}^c$ is four-dimensional and is, to all orders, given by $\{U = 0\}$. Moreover, in some neighborhood of $\widetilde{Q}^+$, there exists a stable invariant foliation $\mathcal{F}^s$ of $\mathcal{W}^c$ with one-dimensional fibers, where the contraction along $\mathcal{F}^s$ is stronger than $e^{-\lambda\xi}$ for some $0 < \lambda < \sqrt{2}$.*

The restriction of (14) to $\mathcal{W}^c$ is given by

$$
\begin{aligned}
Z' &= 2(1 - e^{-Y}) + c(\sqrt{2} - Z) - Z^2, \\
Y' &= \varepsilon(\sqrt{2} - Z), \\
c' &= 0, \\
\varepsilon' &= 0.
\end{aligned} \tag{15}
$$

Here and in the following, we omit the tildes for convenience of notation. Since by Proposition 1, $\mathcal{W}^c$ is exponentially attracting, the fate of solutions starting in the unstable manifold $\mathcal{W}^u(Q^-)$ of the point Q^- in the original system (13) for small ε and ξ sufficiently large will be determined by the dynamics of (15) on $\mathcal{W}^c$. When referring to solutions of (13), we will henceforth always mean solutions on $\mathcal{W}^u(Q^-)$ which converge to Q^- as $\xi \to -\infty$.

Clearly, (15) is a fast-slow system in (extended) standard form, with Z fast and Y slow. By setting $\varepsilon = 0$ in (15) and taking into account that $c = o(1)$ by definition, one finds that the critical manifold $\mathcal{S}$ for (15) is given by

$$
\begin{aligned}
\mathcal{S} &= \mathcal{S}^r \cup \mathcal{S}^a \\
&= \left\{(Z, Y) \,\middle|\, Z = -\sqrt{2(1 - e^{-Y})}, Y \geq 0\right\} \cup \left\{(Z, Y) \,\middle|\, Z = \sqrt{2(1 - e^{-Y})}, Y \geq 0\right\}.
\end{aligned} \tag{16}
$$

Moreover, one sees that $\mathcal{S}$ is normally hyperbolic except at $(Z, Y) = (0, 0)$, with $\mathcal{S}^r$ being normally repelling and $\mathcal{S}^a$ normally attracting for the corresponding layer problem,

$$
\begin{aligned}
Z' &= 2(1 - e^{-Y}) - Z^2, \\
Y' &= 0.
\end{aligned}
$$

The geometry of (15) for $\varepsilon = 0$ is illustrated in Figure 1.

By standard geometric singular perturbation theory [Fen79, Jon95], we conclude that outside any small neighborhood of the origin, $\mathcal{S}^r$ and $\mathcal{S}^a$ perturb smoothly to locally invariant manifolds $\mathcal{S}^r_\varepsilon$ and $\mathcal{S}^a_\varepsilon$, respectively, for $\varepsilon \neq 0$ sufficiently small.

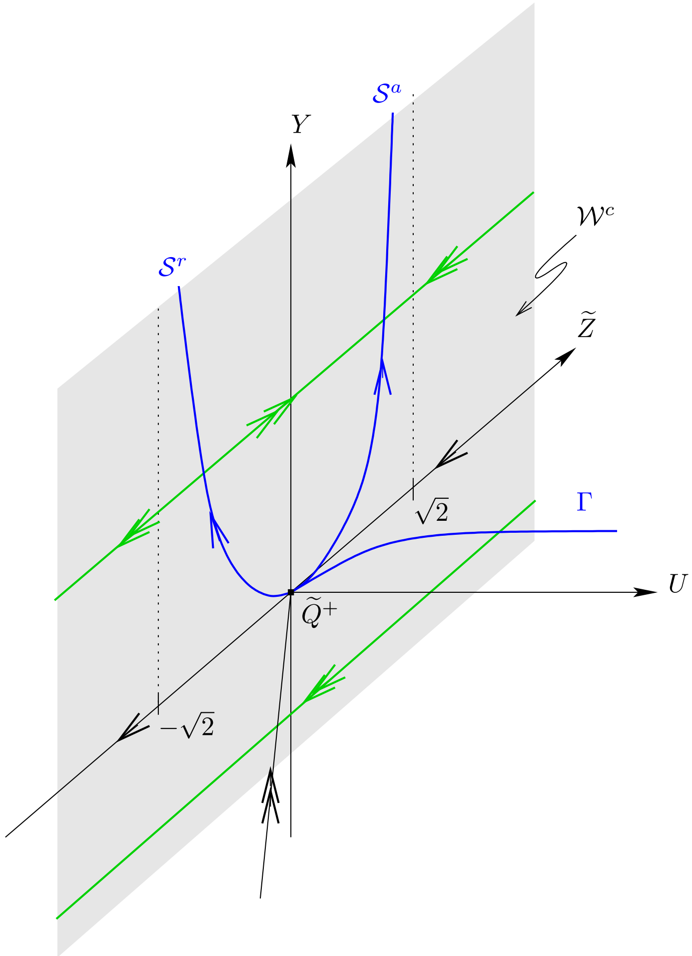
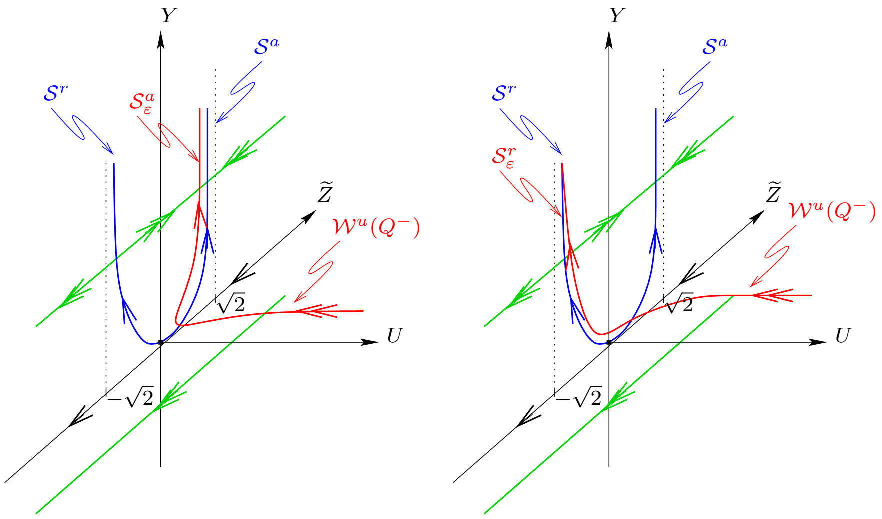


FIGURE 1. The dynamics of (15) on $\mathcal{W}^c$ for $\varepsilon = 0$.

Note that the repelling branch $\mathcal{S}^r_\varepsilon$ of $\mathcal{S}_\varepsilon$ is the one we are interested in. For $\varepsilon > 0$ sufficiently small, it corresponds to the unique heteroclinic solution of (13) which exhibits exponential decay for $\xi \to \infty$; therefore, c equals $c^1_{\text{crit}}(\varepsilon)$, the critical wave speed (Figure 2(b)). This fact will become more apparent through the analysis in the subsequent subsections. Generically, however, $\mathcal{W}^u(Q^-)$ will miss $\mathcal{S}^r_\varepsilon$: For $c > c^1_{\text{crit}}(\varepsilon)$, any solution on $\mathcal{W}^u(Q^-)$ will asymptote to $\mathcal{S}^a_\varepsilon$ (Figure 2(a)), whereas for $c < c^1_{\text{crit}}(\varepsilon)$, there are no bounded solutions on $\mathcal{W}^u(Q^-)$, as any solution which enters $\mathcal{W}^c$ will be repelled away from $\mathcal{S}_\varepsilon$ altogether and go off to $Z = -\infty$ (Figure 2(c)).

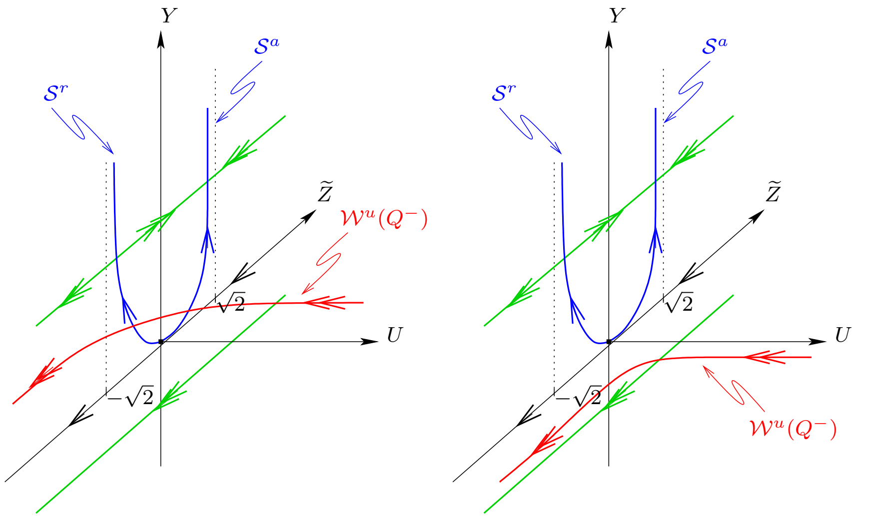
Finally, the equations in (15) indicate why there can be no bounded solution to (13) (or, equivalently, to (11)) for $\varepsilon < 0$ regardless of the value of c, as pointed out in [MN93]: Since $Y = -\varepsilon \ln U$, it would then follow that $Y < 0$. For any such Y, however, the dynamics on $\mathcal{W}^c$ are governed exclusively by the fast flow in (15), which implies $Z' < -Z^2$ and hence $Z \to -\infty$ (see Figure 2(d)).

To analyze the passage of solutions past the origin in more detail, we have to consider the regime where Y and Z are small. Then, one can expand the right-hand side in (15) to



(a) $\varepsilon > 0$: For $c > c^1_{\text{crit}}$, solutions on $\mathcal{W}^u(Q^-)$ asymptote to $\mathcal{S}^a_\varepsilon$.

(b) $\varepsilon > 0$: When $c = c^1_{\text{crit}}$, $\mathcal{W}^u(Q^-)$ coincides with $\mathcal{S}^r_\varepsilon$.



(c) $\varepsilon > 0$: For $c < c^1_{\text{crit}}$, all solutions on $\mathcal{W}^u(Q^-)$ are unbounded.

(d) $\varepsilon < 0$: No bounded solution exists for any value of c.

FIGURE 2. The geometry of (13) for $\varepsilon \neq 0$ small.

obtain

$$\begin{aligned} Z' &= 2Y - Y^2 + \mathcal{O}(Y^3) + \sqrt{2}c - cZ - Z^2, \\ Y' &= \varepsilon(\sqrt{2} - Z), \\ c' &= 0, \\ \varepsilon' &= 0. \end{aligned} \tag{17}$$

For Y, Z, and c small, system (17) is, to leading order, precisely the singularly perturbed fold problem treated in full detail in [KS01]. However, our equations are different insofar as the additional parameter c has to be accounted for. Moreover, the structure of the higher-order terms in (17) is specific, as

$$
\begin{aligned}
Z' &= 2Y - Z^2 + \mathcal{O}(c, cZ, Y^2),\\
Y' &= \varepsilon\big(\sqrt{2} + \mathcal{O}(Z)\big),
\end{aligned}
$$

whereas the analysis in [KS01] is concerned with systems of the more general form

$$
\begin{aligned}
x' &= -y + x^2 + \mathcal{O}(\varepsilon, xy, y^2, x^3),\\
y' &= \varepsilon\big(-1 + \mathcal{O}(x, y, \varepsilon)\big).
\end{aligned}
$$

Hence, after a rescaling of the variables in (17) via $\widehat{Z} = \frac{Z}{\sqrt{2}}$, $\widehat{Y} = Y + \frac{c}{\sqrt{2}}$, and $\hat{\xi} = -\sqrt{2}\xi$, our equations do fit into the framework of [KS01], as our assumptions actually imply $c = o(1)$.

2.3. **Desingularization of the origin in** (17)**.** We will in the following briefly recall the results of [KS01] adapted to the present setting and indicate the necessary modifications. To desingularize the non-hyperbolic origin in (17), we introduce the blow-up transformation

$$
Z = \bar{r}\bar{z}, \quad Y = \bar{r}^2\bar{y}, \quad c = \bar{r}^2\bar{c}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon} \tag{18}
$$

which maps the manifold $B := \mathbb{S}^2 \times [-c_0, c_0] \times [0, r_0]$ to $\mathbb{R}^4$. Here, $\mathbb{S}^2 = \big\{(\bar{z}, \bar{y}, \bar{\varepsilon}) \,\big|\, \bar{z}^2 + \bar{y}^2 + \bar{\varepsilon}^2 = 1\big\}$ denotes the two-sphere in $\mathbb{R}^4$, and c_0 and r_0 are positive constants which have to be chosen sufficiently small; in fact, given r_0, ε_0 in Theorem 1 is fixed via $\varepsilon_0 = r_0^3$. Apart from the rescaling of c which has to be included here, this quasi-homogeneous blow-up has been established to give the correct scaling for the treatment of the planar fold in [KS01].

The blown-up vector field induced by (17) is most conveniently studied by introducing different charts for the manifold B. As in [KS01], we require the rescaling chart K_2 which covers the upper half-sphere defined by $\bar{\varepsilon} > 0$, as well as two other charts (K_1 and K_3) which describe the portions of the equator of $\mathbb{S}^2$ corresponding to $\bar{y} > 0$ and $\bar{z} > 0$, respectively. We will first analyze the dynamics in each of these charts separately and then, in Section **2.4**, combine the results into the proof of Theorem 1.

Remark 4. *Given any object* $\Box$ *in the original setting, we will in the following denote the corresponding object in the blown-up coordinates by* $\overline{\Box}$*; in charts* K_i*,* $i = 1, 2, 3$*, the same object will appear as* $\Box_i$ *when necessary.*

2.3.1. *Dynamics in chart* K_1. Chart K_1, which is defined by $\bar{y} = 1$, is introduced to analyze the vector field in (17) for $Y > 0$, but small. From (18), we have

$$
Z = r_1 z_1, \quad Y = r_1^2, \quad c = r_1^2 c_1, \quad \varepsilon = r_1^3 \varepsilon_1 \tag{19}
$$

for the blow-up transformation in K_1. Substituting (19) into (17) and desingularizing (i.e. dividing out a factor r_1 from the resulting equations), we obtain

$$
\begin{aligned}
z_1' &= 2 - z_1^2 + c_1(\sqrt{2} - r_1 z_1) - \frac{z_1\varepsilon_1}{2}(\sqrt{2} - r_1 z_1) + \mathcal{O}(r_1^2),\\
r_1' &= \frac{r_1\varepsilon_1}{2}(\sqrt{2} - r_1 z_1),\\
c_1' &= -c_1\varepsilon_1(\sqrt{2} - r_1 z_1),\\
\varepsilon_1' &= -\frac{3\varepsilon_1^2}{2}(\sqrt{2} - r_1 z_1),
\end{aligned} \tag{20}
$$

where the prime now denotes differentiation with respect to a new (rescaled) independent variable ξ_1.

The invariant sets of this system are given by $\{r_1 = 0\}$, $\{c_1 = 0\}$, and $\{\varepsilon_1 = 0\}$; moreover, in the intersection of these three planes lies the invariant line $\ell_1 := \{(z_1, 0, 0, 0) \,|\, z_1 \in \mathbb{R}\}$. The dynamics on ℓ_1 are governed by $z_1' = 2 - z_1^2$. As in [KS01, Section 2.5], one can conclude the existence of two hyperbolic equilibria $P_1^a = (\sqrt{2}, 0, 0, 0)$ and $P_1^r = (-\sqrt{2}, 0, 0, 0)$ on ℓ_1, with P_1^a attracting and P_1^r repelling, and the relevant eigenvalues given by $\mp 2\sqrt{2}$. Moreover, in $\{c_1 = 0\} \cap \{\varepsilon_1 = 0\}$, (20) reduces to

$$
\begin{aligned}
z_1' &= 2 - z_1^2 + \mathcal{O}(r_1^2), \\
r_1' &= 0,
\end{aligned}
\tag{21}
$$

which, by the Implicit Function Theorem, implies that there exist two normally hyperbolic curves of equilibria for r_1 small, say $\mathcal{S}_1^a$ and $\mathcal{S}_1^r$. As in [KS01], these two curves correspond to the two branches $\mathcal{S}^a$ and $\mathcal{S}^r$ of the critical manifold $\mathcal{S}$ after transformation to chart K_1.

Similarly, in $\{r_1 = 0\}$, (20) becomes

$$
\begin{aligned}
z_1' &= 2 - z_1^2 + \sqrt{2}c_1 - \frac{z_1\varepsilon_1}{\sqrt{2}}, \\
c_1' &= -\sqrt{2}c_1\varepsilon_1, \\
\varepsilon_1' &= -\frac{3\varepsilon_1^2}{\sqrt{2}}.
\end{aligned}
\tag{22}
$$

One recovers P_1^a and P_1^r. Also, due to the second and third equation, there is a double zero eigenvalue now, with corresponding eigenvectors $(1, 0, -2\sqrt{2})^T$ and $(0, 1, \pm\sqrt{2})^T$, respectively. Thus, there exists an attracting (repelling) two-dimensional center manifold $\mathcal{N}_1^a$ ($\mathcal{N}_1^r$) at P_1^a (P_1^r) for $|c_1|$ and $|\varepsilon_1|$ small (in fact, it suffices to consider $\varepsilon_1 > 0$ here, as we assume $\varepsilon > 0$). In sum, by having followed [KS01], we have

Proposition 2. *For r_1, $|c_1|$, and ε_1 in* (20) *sufficiently small and $k \in \mathbb{N}$, there holds:*

(1) *There exists an attracting three-dimensional $\mathcal{C}^k$-smooth center manifold $\mathcal{M}_1^a$ at P_1^a containing both $\mathcal{S}_1^a$ and $\mathcal{N}_1^a$. The branch of $\mathcal{N}_1^a$ contained in $\{\varepsilon_1 > 0\}$ is not unique.*
(2) *There exists a repelling three-dimensional $\mathcal{C}^k$-smooth center manifold $\mathcal{M}_1^r$ at P_1^r containing both $\mathcal{S}_1^r$ and $\mathcal{N}_1^r$. The branch of $\mathcal{N}_1^r$ contained in $\{\varepsilon_1 > 0\}$ is unique.*
(3) *There exists a stable invariant foliation $\mathcal{F}_1^s$ of $\mathcal{M}_1^a$ with one-dimensional fibers, where the contraction along $\mathcal{F}_1^s$ is stronger than $e^{-\lambda\xi_1}$ for some $0 < \lambda < 2\sqrt{2}$.*
(4) *There exists an unstable invariant foliation $\mathcal{F}_1^u$ of $\mathcal{M}_1^r$ with one-dimensional fibers, where the expansion along $\mathcal{F}_1^u$ is stronger than $e^{\lambda\xi_1}$ for some $0 < \lambda < 2\sqrt{2}$.*

2.3.2. *Dynamics in chart K_2.* In chart K_2, which corresponds to $\bar{\varepsilon} = 1$ in (18), the blow-up transformation is given by

$$
Z = r_2 z_2, \quad Y = r_2^2 y_2, \quad c = r_2^2 c_2, \quad \varepsilon = r_2^3.
$$

After desingularization, the equations in (17) become

$$
\begin{aligned}
z_2' &= 2y_2 + \sqrt{2}c_2 - z_2^2 - r_2 c_2 z_2 + \mathcal{O}(r_2^2), \\
y_2' &= \sqrt{2} - r_2 z_2, \\
c_2' &= 0, \\
r_2' &= 0.
\end{aligned} \tag{23}
$$

System (23) can be regarded as a two-dimensional system for (y_2, z_2) parametrized by c_2 and r_2. We will first study the simplified equations obtained from (23) by neglecting the perturbative terms in r_2 and will consider the effect of the perturbation afterwards.

Setting $r_2 = 0$ in (23), we find

$$
\begin{aligned}
z_2' &= 2y_2 + \sqrt{2}c_2 - z_2^2, \\
y_2' &= \sqrt{2},
\end{aligned} \tag{24}
$$

which is a Riccati equation that can be solved in terms of special functions. In analogy to [KS01, Proposition 2.3], we have

Proposition 3. *Equations* (24) *have the following properties:*

(1) *Each orbit of* (24) *has a horizontal asymptote* $\{y_2 = \alpha_+\}$, *where* α_+ *depends on the orbit, which it approaches from above as* $z_2 \to \infty$.

(2) *There exists a unique orbit* $\gamma_2 = \gamma_2(c_2)$ *which is asymptotic to the left branch of* $\{2y_2 + \sqrt{2}c_2 - z_2^2 = 0\}$ *for* $z_2 \to -\infty$ *and which asymptotes to* $\{y_2 = -\Omega_0 - \frac{c_2}{\sqrt{2}}\}$, *with* $\Omega_0 > 0$, *as* $z_2 \to \infty$.

(3) *The orbit* γ_2 *can be parametrized as* $(z_2, s(z_2, c_2))$ $(z_2 \in \mathbb{R})$, *where*

$$
s(z_2, c_2) = -\frac{c_2}{\sqrt{2}} + \frac{z_2^2}{2} + \frac{1}{\sqrt{2}z_2} + \mathcal{O}(z_2^{-4}) \quad \textit{for } z_2 \to -\infty
$$

and

$$
s(z_2, c_2) = -\Omega_0 - \frac{c_2}{\sqrt{2}} + \frac{\sqrt{2}}{z_2} + \mathcal{O}(z_2^{-3}) \quad \textit{for } z_2 \to \infty.
$$

(4) *All orbits to the right of* γ_2 *are forward asymptotic to the right branch of* $\{2y_2 + \sqrt{2}c_2 - z_2^2 = 0\}$.

(5) *All orbits to the left of* γ_2 *have a horizontal asymptote* $\{y_2 = \alpha_-\}$, *where* $\alpha_- > \alpha_+$ *depends on the orbit, such that* y_2 *approaches* α_- *from below as* $z_2 \to -\infty$.

Proof. After introducing the new coordinates $\tilde{y}_2 = y_2 + \frac{c_2}{\sqrt{2}}$ and $\tilde{z}_2 = \frac{z_2}{\sqrt{2}}$ in (24), we obtain

$$
\begin{aligned}
\tilde{z}_2' &= -\sqrt{2}(-\tilde{y}_2 + \tilde{z}_2^2), \\
\tilde{y}_2' &= \sqrt{2}.
\end{aligned}
$$

Rescaling time to divide out a factor $-\sqrt{2}$ from the right-hand sides above, we find precisely Equations (2.16) from [KS01]. The result then follows immediately from their Proposition 2.3. ∎

The situation is illustrated in Figure 3.

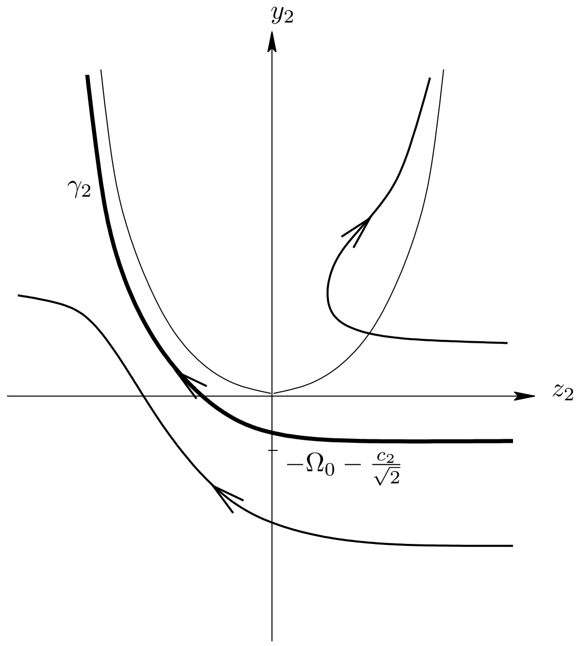


FIGURE 3. Solutions of the Riccati equation (24).

Remark 5. *As pointed out in* [KS01]*, the constant* $\Omega_0 > 0$ *is the smallest positive zero of*

$$\mathrm{J}_{-\frac{1}{3}}\Big(\frac{2}{3}z^{\frac{3}{2}}\Big) + \mathrm{J}_{\frac{1}{3}}\Big(\frac{2}{3}z^{\frac{3}{2}}\Big),$$

where $\mathrm{J}_{\pm\frac{1}{3}}$ *are Bessel functions of the first kind. Moreover, using the identity*

$$\frac{3}{\sqrt{z}}\mathrm{Ai}(-z) = \mathrm{J}_{-\frac{1}{3}}\Big(\frac{2}{3}z^{\frac{3}{2}}\Big) + \mathrm{J}_{\frac{1}{3}}\Big(\frac{2}{3}z^{\frac{3}{2}}\Big)$$

from [AS74]*, we see that* $-\Omega_0$ *can also be found as the smallest negative zero of the Airy function* Ai*.*

Remark 6. *Setting* $z_2 = \frac{w_2'}{w_2}$ *in* (24) *and making use of* $y_2 = \sqrt{2}\xi_2$ *leads to*

$$w_2'' - \sqrt{2}(2\xi_2 + c_2)w_2 = 0,$$

which is the analog of the Airy equation derived in [MN93] *in what is called the "middle region" there. Incidentally, note that Airy equations arise in the analysis of turning point problems by means of JWKB techniques and asymptotic matching, cf. also Section* **2.6** *below. To be precise, they (as well as the corresponding Riccati equations obtained via the above transformation) occur in the derivation of the so-called connection formulae for these problems* [BO78]*.*

2.3.3. *Dynamics in chart* K_3*.* In K_3, one sets $\bar{z} = 1$ in (18) to obtain

$$Z = r_3, \quad Y = r_3^2 y_3, \quad c = r_3^2 c_3, \quad \varepsilon = r_3^3 \varepsilon_3. \tag{25}$$

Chart K_3 covers the neighborhood of the equator of $\mathbb{S}^2$ corresponding to $Z > 0$ and is used to analyze the dynamics of the vector field in (17) close to the Y-axis.

After desingularizing, we have the following equations:

$$
\begin{aligned}
r_3' &= -r_3 F(r_3, y_3, c_3), \\
y_3' &= 2y_3 F(r_3, y_3, c_3) + \varepsilon_3(\sqrt{2} - r_3), \\
c_3' &= 2c_3 F(r_3, y_3, c_3), \\
\varepsilon_3' &= 3\varepsilon_3 F(r_3, y_3, c_3),
\end{aligned} \tag{26}
$$

where $F(r_3, y_3, c_3) := 1 - 2y_3 - \sqrt{2}c_3 + r_3 c_3 + \mathcal{O}(r_3^2)$. The hyperplanes $\{r_3 = 0\}$, $\{c_3 = 0\}$, and $\{\varepsilon_3 = 0\}$ as well as the (r_3, c_3)-plane are invariant under the resulting flow. Due to the fact that $F \approx 1$ for y_3 small, we may divide out a factor F from (26) to obtain

$$
\begin{aligned}
r_3' &= -r_3, \\
y_3' &= 2y_3 + \sqrt{2}\varepsilon_3 - r_3\varepsilon_3 + 2\sqrt{2}y_3\varepsilon_3 + 2c_3\varepsilon_3 + \mathcal{O}(3), \\
c_3' &= 2c_3, \\
\varepsilon_3' &= 3\varepsilon_3.
\end{aligned} \tag{27}
$$

Lemma 2. *The origin is a hyperbolic equilibrium of* (27), *with the eigenvalues given by* $\lambda_1 = -1$, $\lambda_2 = 2$ *(double), and* $\lambda_3 = 3$. *The corresponding eigenspaces are spanned by* $(1, 0, 0, 0)^T$, $\{(0, 1, 0, 0)^T, (0, 0, 1, 0)^T\}$, *and* $(0, \sqrt{2}, 0, 1)^T$, *respectively.*

The point $Q_3 := (0, 0, 0, 0)$ is the entry point of the fast flow and is the analog of the equilibrium termed q_{out} in [KS01].

Remark 7. *For* $y_3 = \frac{1}{2}$ *in* (26), *one retrieves the point* P_1^a *(corresponding to the attracting branch* $\mathcal{S}^a$ *of* $\mathcal{S}$*) from chart* K_1.

2.4. Proof of Theorem 1. To complete our picture of the dynamics of (17) in a neighborhood of the fold point at the origin in $\mathcal{W}^c$, we have to combine the dynamics in charts K_1, K_2, and K_3. The following lemma relates the coordinates in these three charts on their respective domains of overlap:

Lemma 3. *The change of coordinates* $\kappa_{12} : K_1 \to K_2$ *from* K_1 *to* K_2 *reads*

$$
z_2 = z_1\varepsilon_1^{-\frac{1}{3}}, \quad y_2 = \varepsilon_1^{-\frac{2}{3}}, \quad c_2 = c_1\varepsilon_1^{-\frac{2}{3}}, \quad r_2 = r_1\varepsilon_1^{\frac{1}{3}}, \tag{28}
$$

with the inverse $\kappa_{21} = \kappa_{12}^{-1}$ *given by*

$$
z_1 = z_2 y_2^{-\frac{1}{2}}, \quad r_1 = r_2 y_2^{\frac{1}{2}}, \quad c_1 = c_2 y_2^{-1}, \quad \varepsilon_1 = y_2^{-\frac{3}{2}}. \tag{29}
$$

Similarly, for $\kappa_{23} : K_2 \to K_3$ *and* $\kappa_{32} = \kappa_{23}^{-1}$, *one finds*

$$
r_3 = r_2 z_2, \quad y_3 = y_2 z_2^{-2}, \quad c_3 = c_2 z_2^{-2}, \quad \varepsilon_3 = z_2^{-3} \tag{30}
$$

and

$$
z_2 = \varepsilon_3^{-\frac{1}{3}}, \quad y_2 = y_3\varepsilon_3^{-\frac{2}{3}}, \quad c_2 = c_3\varepsilon_3^{-\frac{2}{3}}, \quad r_2 = r_3\varepsilon_3^{\frac{1}{3}}, \tag{31}
$$

respectively.

Next, we study the family of orbits $\{\gamma_2(c_2)\}$ retrieved in K_2 for $r_2 = 0$, investigating what this family corresponds to in the two remaining charts.

Lemma 4. *For $z_2 > 0$, let $\gamma_1 := \kappa_{21}(\gamma_2)$, where κ_{21} is defined as in Lemma 3; similarly, for $y_2 > 0$, let $\gamma_3 := \kappa_{23}(\gamma_2)$. Then, the following holds:*

(1) *In K_1, γ_1 is contained in the unique branch of $\mathcal{N}_1^r$ from Proposition 2, and converges to P_1^r as $\varepsilon_1 \to 0$ irrespective of the choice of c_2.*
(2) *The orbit γ_3 lies in $\{r_3 = 0\}$, and converges to Q_3 as $\varepsilon_3 \to 0$ in K_3. Moreover, the convergence is along the c_3-axis for $c_2 = -\sqrt{2}\Omega_0$. For any $c_2 \neq -\sqrt{2}\Omega_0$, γ_3 is tangent to* $\operatorname{span}\{(0,1,0,0)^T, (0,0,1,0)^T\}$ *as $\varepsilon_3 \to 0$.*

Proof. Given Proposition 3 and Lemma 3, one finds the expansion

$$\Bigg(z_2\Big(-\frac{c_2}{\sqrt{2}} + \frac{z_2^2}{2} + \frac{1}{\sqrt{2}z_2} + \mathcal{O}(z_2^{-4})\Big)^{-\frac{1}{2}}, 0, c_2\Big(-\frac{c_2}{\sqrt{2}} + \frac{z_2^2}{2} + \frac{1}{\sqrt{2}z_2} + \mathcal{O}(z_2^{-4})\Big)^{-1},$$
$$\Big(-\frac{c_2}{\sqrt{2}} + \frac{z_2^2}{2} + \frac{1}{\sqrt{2}z_2} + \mathcal{O}(z_2^{-4})\Big)^{-\frac{3}{2}}\Bigg)$$

for γ_1 as $z_2 \to -\infty$. Expanding the above in powers of z_2 shows that γ_1 converges to $P_1^r = (-\sqrt{2}, 0, 0, 0)$ tangent to the center direction $(1, 0, -2, 0)^T$ for $z_2 \to -\infty$ and $y_2 \to \infty$ in K_2, which implies $\varepsilon_1 \to 0$ in K_1, see also [KS01, Proposition 2.6].

Similarly, by applying the expansion given in Proposition 3 as $z_2 \to \infty$, one finds

$$\Bigg(0, \frac{-\Omega_0 - \frac{c_2}{\sqrt{2}}}{z_2^2} + \frac{\sqrt{2}}{z_2^3} + \mathcal{O}(z_2^{-5}), \frac{c_2}{z_2^2}, \frac{1}{z_2^3}\Bigg)$$

for γ_3. Since the limit as $z_2 \to \infty$ corresponds to letting $\varepsilon_3 \to 0$ in K_3, this completes the proof. ∎

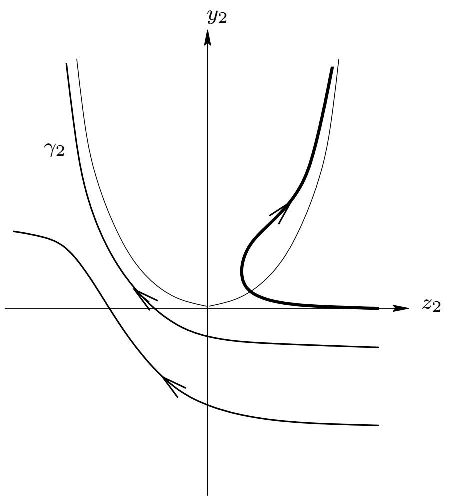
Remark 8. *An argument similar to the above shows that the orbits in K_2 specified in Proposition 3(4) correspond to one of the non-unique branches of $\mathcal{N}_1^a$ each and converge to P_1^a in K_1. In K_3, these orbits are still asymptotic to Q_3 as $\varepsilon_3 \to 0$ and approach Q_3 tangent to* $\operatorname{span}\{(0,1,0,0)^T, (0,0,1,0)^T\}$.

We are now ready to state the proof of our main result:

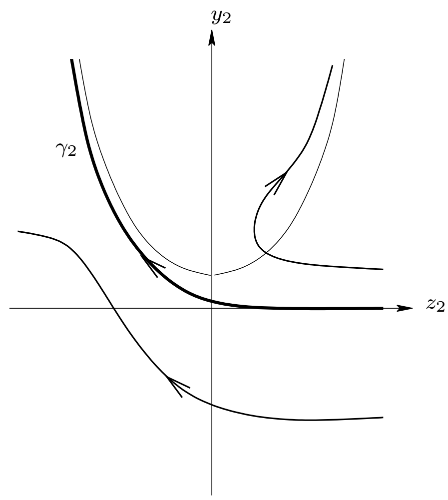
Proof of Theorem 1. First, we show that in the singular limit of $\varepsilon = 0$, the projection of $\mathcal{W}^u(Q^-) \times [-\tilde{c}_0, \tilde{c}_0]$ (for any $\tilde{c}_0 > 0$ small) onto $\mathcal{W}^c$ must lie in the $(\widetilde{Z}, \tilde{c})$-plane. Indeed, for $\varepsilon = 0$, there holds $Y = 0$, and there is clearly no variation in Y under (14) along the singular orbit Γ. In K_3, it then follows from the invariance of the (r_3, c_3)-plane under (26) that the corresponding singular orbit Γ_3 can have no component in the y_3-direction. Hence, it must lie in $\{y_3 = 0\} \cap \{\varepsilon_3 = 0\}$.

Second, recall that the special orbit γ_2 in K_2 is the one we are interested in, as it is asymptotic to the left branch of $\{2y_2 + \sqrt{2}c_2 - z_2^2 = 0\}$ for $z_2 \to -\infty$ by Proposition 3(2) and hence corresponds to the one solution of (17) which connects to the repelling branch $\mathcal{S}^r$ in the original coordinates. Since $Y \geq 0$ by definition, it follows from Lemma 3 that necessarily $y_2 \geq 0$, as well. Hence, Proposition 3(3) implies that $c_2 \leq -\sqrt{2}\Omega_0$ must hold for the orbit $\gamma_2(c_2)$ to be admissible.

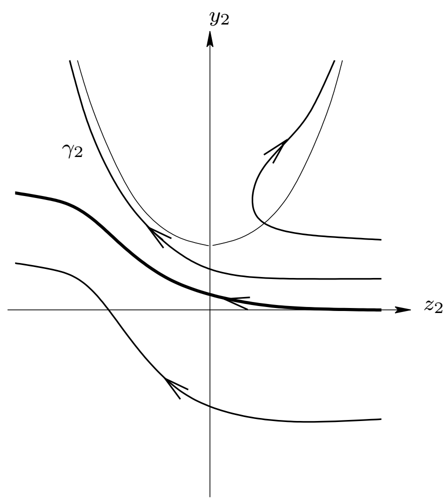
Now, by combining these two observations, one can see directly that $c_2 = -\sqrt{2}\Omega_0$. Namely, of all the admissible orbits (as found in our second observation), the only such orbit which has zero tangential y_3-component – as required by our first observation – is obtained precisely



(a) For $c_2 > -\sqrt{2}\Omega_0$, the orbit is asymptotic to the right branch of $\{2y_2+\sqrt{2}c_2 - z_2^2 = 0\}$.



(b) When $c_2 = -\sqrt{2}\Omega_0$, the orbit equals γ_2 and is asymptotic to the left branch of $\{2y_2 - 2\Omega_0 - z_2^2 = 0\}$.



(c) For $c_2 < -\sqrt{2}\Omega_0$, the orbit becomes unbounded as $z_2 \to -\infty$.

FIGURE 4. The phase portrait of the unperturbed equations (24) in K_2 for different choices of c_2. The bold curve represents the unique orbit for which $y_2 \to 0$ as $z_2 \to \infty$.

for $c_2 = -\sqrt{2}\Omega_0$, see Lemma 4. Here, $-\Omega_0 \approx -2.338107$ is the smallest negative zero of the Airy function, cf. Remark 5.

By Proposition 3(3), it follows that this choice of c_2 implies $y_2 \to 0$ as $z_2 \to \infty$ (Figure 4(b)). By Proposition 3(1), other choices of c_2 can yield solutions to (24) for which $y_2 \to 0$ as $z_2 \to \infty$. However, such solutions are either forward asymptotic to the right branch of $\{2y_2 + \sqrt{2}c_2 - z_2^2 = 0\}$ (Figure 4(a)), or become unbounded for $z_2 \to -\infty$ (Figure 4(c)). Consequently, they correspond to solutions of (17) which are forward asymptotic to $\mathcal{S}^a$ or which become unbounded for $Z \to -\infty$, respectively. Hence, by fixing $c_2 = -\sqrt{2}\Omega_0$

as required by the asymptotics of γ_2 in K_2, we have determined the leading-order behavior of $\tilde{c}(\varepsilon)$, which will give the desired expansion for $\tilde{c}$ after blow-down, since $\tilde{c} = r_2^2 c_2$.

It remains to show that the next-order correction in $\tilde{c}$ will be $\mathcal{O}(\varepsilon)$. Let us define sections Σ_3^{in} and Σ_3^{out} in K_3 via

$$\Sigma_3^{out} = \left\{(r_3, y_3, c_3, \varepsilon_3) \in \mathcal{U}_3 \,\middle|\, \varepsilon_3 = \delta\right\} \quad \text{and} \quad \Sigma_3^{in} = \left\{(r_3, y_3, c_3, \varepsilon_3) \in \mathcal{U}_3 \,\middle|\, r_3 = \rho\right\},$$

where $\mathcal{U}_3$ is a sufficiently small neighborhood of the origin in K_3 and $0 < \rho, \delta \ll 1$ are fixed. Let Σ_2^{in} denote the section in K_2 obtained from Σ_3^{out} via the coordinate change κ_{32} specified in Lemma 3. Moreover, let $\Sigma^{in} = \left\{(Z, Y, c, \varepsilon) \in \mathcal{U} \,\middle|\, Z = \rho\right\}$ be the section corresponding to Σ_3^{in} in the original coordinates, with $\mathcal{U}$ an appropriately defined neighborhood of the origin.

Lemma 5. *In Σ^{in}, there holds*

$$\tilde{c}(\varepsilon) = -\sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon).$$

Proof. First, by regular perturbation theory, we know that

$$c_2 = -\sqrt{2}\Omega_0 + \alpha r_2 + \mathcal{O}(r_2^2) \tag{32}$$

on bounded domains in K_2, where α is some constant. Hence, in K_3,

$$c_3 = -\sqrt{2}\Omega_0 \varepsilon_3^{\frac{2}{3}} + \alpha r_3 \varepsilon_3 + \mathcal{O}\big(r_3^2 \varepsilon_3^{\frac{4}{3}}\big),$$

since $c_3 = c_2 \varepsilon_3^{\frac{2}{3}}$ and $r_2 = r_3 \varepsilon_3^{\frac{1}{3}}$, recall Lemma 3. Specifically, it follows that

$$c_3^{out} := -\sqrt{2}\Omega_0 \delta^{\frac{2}{3}} + \alpha \varepsilon^{\frac{1}{3}} \delta^{\frac{2}{3}} + \mathcal{O}\big(\varepsilon^{\frac{2}{3}} \delta^{\frac{2}{3}}\big)$$

in Σ_3^{out}, as $r_3 = \varepsilon^{\frac{1}{3}} \delta^{-\frac{1}{3}}$. Following [KS01], we now consider the transition map $\Pi_3 : \Sigma_3^{in} \to \Sigma_3^{out}$, with $(r_3, y_3, c_3, \varepsilon_3) = (\varepsilon^{\frac{1}{3}} \delta^{-\frac{1}{3}}, y_3^{out}, c_3^{out}, \delta) \in \Sigma_3^{out}$ fixed. In particular, by solving the equation for c_3 in (27) in backward time given c_3^{out}, we find

$$c_3(\xi_3) = c_3^{out} e^{2(\xi_3 - \Xi_3)};$$

here, Ξ_3 denotes the transition "time" from Σ_3^{in} to Σ_3^{out}. Then, recalling $r_3^{in} = \rho$ and $r_3^{out} = \varepsilon^{\frac{1}{3}} \delta^{-\frac{1}{3}}$, we deduce that $\Xi_3 = \frac{1}{3} \ln \frac{\delta \rho^3}{\varepsilon}$ and hence

$$c_3(\xi_3) = \Big(-\sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \alpha \varepsilon + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Big) \frac{e^{2\xi_3}}{\rho^2}. \tag{33}$$

The assertion now follows, since $\xi_3 = 0$ and $\tilde{c} = \rho^2 c_3^{in}$ in Σ_3^{in}. ■

Reverting to our original notation, we see from Lemma 5 that

$$c_{\text{crit}}^1(\varepsilon) = 2\sqrt{2} + \tilde{c}(\varepsilon) = 2\sqrt{2} - \sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon)$$

on $\mathcal{W}^c$. Finally, by standard center manifold theory, this expansion for c_{crit}^1 can be extended to the full equations in (13) (at least for $U > 0$ small), which concludes the proof of Theorem 1. ■

To summarize, we have shown that to lowest order, the solution corresponding to $c = c_{\text{crit}}^1(\varepsilon)$, the critical wave speed in (15), is given by the special orbit $\bar{\gamma}$ in blown-up phase space. Moreover, we have obtained the first-order correction to $c_{\text{crit}}^1(0) = 2\sqrt{2}$.

Note, however, that in contrast to the classical planar fold problem, we obtain an entire family of special orbits $\gamma_2 = \gamma_2(c_2)$ in K_2, parametrized by the parameter c_2. In fact, the

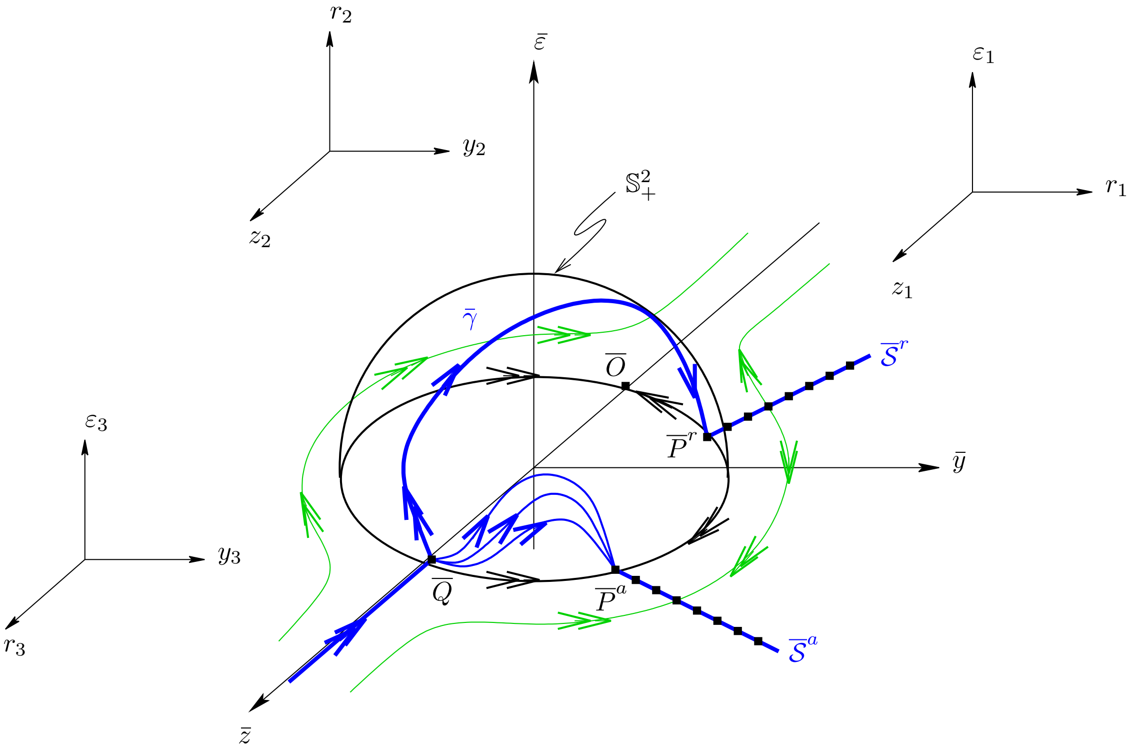


FIGURE 5. The dynamics of the blown-up vector field on the locus $\mathbb{S}^2_+$.

additional degree of freedom provided by c_2 is necessary to identify the one solution of (17) corresponding to the critical speed c^1_{crit}. This identification cannot be achieved in K_2, though; to that end, one has to investigate the asymptotics (and, in particular, the tangency) of γ_3 in K_3. The fact that this tangency is irrelevant in the classical problem implies that one requires no additional parameters there.

The dynamics of the blown-up vector field on the locus $\mathbb{S}^2_+ := \mathbb{S}^2 \cap \{\bar{\varepsilon} \geq 0\}$ (i.e. for $\bar{r} = 0$ and $\bar{c} = -\sqrt{2}\Omega_0$) are roughly illustrated in Figure 5. A more detailed illustration of the geometry of the singularly perturbed planar fold, especially with regard to the dynamics in the individual charts, can be found in [KS01].

Remark 9. *The above analysis shows that $\tilde{c}(\varepsilon) = -\sqrt{2}\Omega_0\varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon)$, see Lemma 5, which in turn gives the first-order correction to the original $c^1_{\text{crit}}(0) = 2\sqrt{2}$. To compute coefficients for the higher-order terms in $\tilde{c}$, one would have to determine the corresponding coefficients in* (32)*.*

2.5. Resonance and logarithmic switchback. The eigenvalues of the linearization of (27) at Q_3 are in resonance, as is the case in [KS01]. Hence, one can apply an analysis similar to theirs to prove the occurrence of logarithmic (switchback) terms in the transition map $\Pi_3 : \Sigma^{in}_3 \to \Sigma^{out}_3$ for (27), where the sections Σ^{in}_3 and Σ^{out}_3, as well as Σ^{in} are defined as above.

Lemma 6. *There holds*

$$Y = -\frac{1}{3}\varepsilon \ln \varepsilon + \mathcal{O}(\varepsilon) \tag{34}$$

in Σ^{in}.

Proof. In analogy to the proof of Lemma 5, we fix $(\varepsilon^{\frac{1}{3}}\delta^{-\frac{1}{3}}, y_3^{out}, c_3^{out}, \delta) \in \Sigma_3^{out}$. Moreover, we replace y_3 in (27) by $\tilde{y}_3 = y_3 - \sqrt{2}\varepsilon_3$ and then define

$$\hat{y}_3 = \tilde{y}_3 - \varepsilon_3^2 - \frac{2\sqrt{2}}{3}\tilde{y}_3\varepsilon_3 - \frac{2}{3}c_3\varepsilon_3$$

to eliminate all quadratic terms except for the resonant term $r_3\varepsilon_3$ from the equation for y_3 in (27), which gives

$$\hat{y}_3' = 2\hat{y}_3 - r_3\varepsilon_3 + \mathcal{O}(3).$$

To solve the resulting equations to leading order, note that $r_3 = \rho e^{-\xi_3}$ and $\varepsilon_3 = \varepsilon\rho^{-3}e^{3\xi_3}$, whence

$$\hat{y}_3(\xi_3) \sim \hat{y}_3^{out} e^{2\xi_3} - \frac{\varepsilon}{\rho^2}\xi_3 e^{2\xi_3}.$$

Therefore, undoing the above transformations and reverting to the original variable y_3, we obtain

$$y_3 \sim \hat{y}_3^{out}\delta^{-\frac{2}{3}}\varepsilon_3^{\frac{2}{3}} + \sqrt{2}\varepsilon_3 + \frac{1}{3}\varepsilon^{\frac{1}{3}}\varepsilon_3^{\frac{2}{3}} \ln\frac{\delta}{\varepsilon_3}; \tag{35}$$

here we have used $r_3\varepsilon_3 = \varepsilon\rho^{-2}e^{2\xi_3}$, $r_3 = \varepsilon^{\frac{1}{3}}\varepsilon_3^{-\frac{1}{3}}$, and $\xi_3 = \frac{1}{3}\ln\frac{\varepsilon_3\rho^3}{\varepsilon}$. Now, Lemma 3 and the proof of Lemma 4 give $y_3 \sim \sqrt{2}\varepsilon_3$ for $\varepsilon_3 \to 0$, and therefore $\hat{y}_3^{out} = 0$. It follows that in Σ_3^{in},

$$y_3^{in} = \frac{\sqrt{2}}{\rho^3}\varepsilon + \frac{1}{3\rho^2}\varepsilon \ln\frac{\delta\rho^3}{\varepsilon} + \mathcal{O}(2),$$

which implies (34) after blow-down. ■

The above result is analogous to the one obtained in [KS01], up to the fact that our Y is logarithmic in ε to leading order, which is due to our choice of c_2 in K_2.

Remark 10. *By the definition of* $Y = -\varepsilon \ln U$, (34) *implies that* $U \sim \varepsilon^{\frac{1}{3}}$ *on* $\mathcal{W}^c$. *On the other hand, we know from Proposition 1 that* $U \sim e^{-\lambda\xi}$ *for some* $0 < \lambda < \sqrt{2}$. *Hence, it follows that for* $c = c_{\mathrm{crit}}^1(\varepsilon)$ *in* (14), ξ *is of the order* $\mathcal{O}(-\ln\varepsilon)$ *on* $\mathcal{W}^c$.

2.6. Concluding remarks. The case $m = 1 + \varepsilon$ has been analyzed in [MN93] by means of asymptotic matching. More specifically, the analysis of [MN93] is concerned with autocatalysis of general order, $A + pB \to (p+1)B$ for two species A and B with $p \geq 1$, in an N-dimensional reactor. It is shown that traveling waves are generated when p lies in the interval $1 \leq p < 1 + \frac{2}{N}$ for any initial input of autocatalyst B, whereas for $p \geq 1 + \frac{2}{N}$ the input must exceed a certain threshold value before such waves can develop. In particular, in the final section of [MN93] traveling waves are studied in the limit as $p \to 1^+$. To relate our results to this aspect of their work, let us collect a few observations.

First, consider Y, Z small; then, $1-e^{-Y} \sim Y$, and $\widetilde{Z} \sim -\sqrt{2Y}$ on $\mathcal{S}^r$. Noting that $\frac{U'}{U} = Z = -\sqrt{2} + \widetilde{Z}$, we find

$$\frac{dU}{U} = -\sqrt{2}(1+\sqrt{Y})d\xi.$$

Next, we make use of $Y' \sim \sqrt{2}\varepsilon$ (see (17)) to obtain

$$U \sim \exp\Big[-\frac{Y+\frac{2}{3}Y^{\frac{3}{2}}}{\varepsilon}\Big], \tag{36}$$

which is exactly the asymptotics found in [MN93, Equation (38b)] (note that our U, Y, and Z correspond to their β, ζ, and ϕ, respectively). Hence, we have retrieved the "third region" from [MN93], which corresponds to the slow dynamics on $\mathcal{S}$ close to the fold point here. Moreover, our analysis rigorously justifies the JWKB-like asymptotics for U stated in [MN93] (recall that by the definition of Z, $U = e^{\int^\xi Z(s)\,ds}$).

Finally, take Y large and note that with $-\sqrt{2(1-e^{-Y})} \sim -\sqrt{2}$, one has

$$\frac{dU}{U} \sim -2\sqrt{2}d\xi$$

on $\mathcal{S}^r$. Hence, $U \sim e^{-2\sqrt{2}\xi}$, which is exactly the leading-order behavior found in [MN93, Equations (24) and (39c)] for ξ large (up to a scaling factor of $\sqrt{2}$ in c^1_{crit}). Similarly, on $\mathcal{S}^a$ we would obtain $Z \sim 0$, which corresponds to a merely algebraically decaying U.

3. Traveling waves for $m = 2 + \varepsilon$

When $m = 2+\varepsilon$, the equations in (7) read

$$\begin{aligned} U' &= V, \\ V' &= -2U^{2+\varepsilon}(1-U) - cV. \end{aligned} \tag{37}$$

It is well-known that for $m = 2$, one has $c^2_{\text{crit}}(0) = 1$. Moreover, in that case there is a closed-form heteroclinic solution to (37) which can be implicitly written as

$$V(U) = U(U-1), \tag{38}$$

see [CCL75]. In analogy to Theorem 1, we have the following result:

Theorem 2. *There exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the critical wave speed c^2_{crit} for (37) can be represented as a function of ε. Moreover, $c^2_{\text{crit}}(\varepsilon)$ is $\mathcal{C}^k$-smooth, for any $k \in \mathbb{N}$, and is given by*

$$c^2_{\text{crit}}(\varepsilon) = 1 - \frac{13}{24}\varepsilon + \mathcal{O}(\varepsilon^2). \tag{39}$$

This theorem establishes rigorously the result derived in [WOK01] by means of asymptotic analysis.

3.1. The projectivized equations for $\varepsilon = 0$. In projectivized coordinates, the equations for $n = 2$ become

$$
\begin{aligned}
U' &= UZ, \\
Z' &= -2e^{-Y}U(1-U) - cZ - Z^2, \\
Y' &= -\varepsilon Z,
\end{aligned}
\tag{40}
$$

see (9). The layer problem obtained by setting $\varepsilon = 0$ in (40) is given by

$$
\begin{aligned}
U' &= UZ, \\
Z' &= -2e^{-Y_0}U(1-U) - cZ - Z^2, \\
Y &= Y_0.
\end{aligned}
\tag{41}
$$

For the following analysis, it is important to consider (40) as a system in which the three variables U, Z, and Y are treated equally. Although $Y_0 = 0$ is the value of Y we are ultimately interested in by definition, we will regard Y as being independent of U until the very end of this section. Therefore, we assume $Y_0 \in \mathcal{Y}$ now, where $\mathcal{Y} \subset \mathbb{R}$ is a bounded set containing $Y_0 = 0$; without loss of generality, we may take $\mathcal{Y} = (-\upsilon, \upsilon)$ with $\upsilon > 0$. For convenience of notation, we will set $\kappa := e^{-Y_0}$ in the following and consequently write $Y_0 = -\ln \kappa$.

Lemma 7. *For each $\kappa > 0$, and with $c = \sqrt{\kappa}$ in (41), the points $Q_\kappa^- : (1, 0, -\ln \kappa)$ and $Q_\kappa^+ : (0, -\sqrt{\kappa}, -\ln \kappa)$ are hyperbolic saddle fixed points, with eigenvalues given by $\lambda_1 = \sqrt{\kappa}$, $\lambda_2 = -2\sqrt{\kappa}$ and $\lambda_1 = -\sqrt{\kappa}$, $\lambda_2 = \sqrt{\kappa}$, respectively. Moreover, these two equilibria are connected by a heteroclinic orbit Γ_κ which is given implicitly by $Z(U) = \sqrt{\kappa}(U-1)$ and explicitly by*

$$
(U_0, Z_0)(\xi) := \left((1 + e^{\sqrt{\kappa}\xi})^{-1}, -\sqrt{\kappa}e^{\sqrt{\kappa}\xi}(1 + e^{\sqrt{\kappa}\xi})^{-1}\right). \tag{42}
$$

The origin is a semi-hyperbolic fixed point of (41), with eigenvalues 0 and $-\sqrt{\kappa}$ and corresponding eigendirections $(1, -2\sqrt{\kappa}, 0)^T$ and $(0, 1, 0)^T$.

Proof. For any $Y_0 \in \mathcal{Y}$ in (41), with $c = e^{-\frac{Y_0}{2}} = \sqrt{\kappa}$ defined correspondingly, a direct calculation yields the three equilibria and their respective stability types, as asserted in the statement of the lemma.

To find explicit formulae for the corresponding heteroclinics, we replace Z by $\widetilde{Z}$, where $Z = \sqrt{\kappa}\widetilde{Z}$, and rescale $\tilde{\xi} = \sqrt{\kappa}\xi$ in (41) to obtain

$$
\begin{aligned}
\frac{dU}{d\tilde{\xi}} &= U\widetilde{Z}, \\
\frac{d\widetilde{Z}}{d\tilde{\xi}} &= -2U(1-U) - \widetilde{Z} - \widetilde{Z}^2.
\end{aligned}
\tag{43}
$$

Now, system (43) has a heteroclinic orbit given implicitly by $\widetilde{Z}(U) = U - 1$, see (38). By separation of variables one finds an explicit expression for this orbit as

$$
(U_0, \widetilde{Z}_0)(\tilde{\xi}) := \left((1 + e^{\tilde{\xi}})^{-1}, -e^{\tilde{\xi}}(1 + e^{\tilde{\xi}})^{-1}\right).
$$

Reverting to the original variables ξ and Z, we have the desired result. ■

For $\kappa = 1$, we will in the following write $Q_1^- \equiv Q^-$, $Q_1^+ \equiv Q^+$, and $\Gamma_1 \equiv \Gamma$, respectively.

Remark 11. *The heteroclinic orbit specified in Lemma 7 is a connection between two hyperbolic equilibria, as opposed to the corresponding orbit* (38) *for* (37) *which involves the degenerate origin. The additional hyperbolicity gained by the projectivization will allow us to treat* (41) *using a standard Melnikov approach.*

Remark 12. *In* [Mur02], *the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(u\frac{\partial u}{\partial x}\Big) + f_1(u)$$

is cited as a non-trivial example of density-dependent diffusion with logistic population growth. A transformation to traveling wave coordinates yields

$$\begin{aligned} U' &= V, \\ UV' &= -2U(1-U) - cV - V^2, \end{aligned}$$

which, after desingularizing time to remove the singularity at $U = 0$*, gives just* (41) *(with* Z *replaced by* V *and* $Y_0 = 0$*).*

3.2. **Melnikov theory for fast-slow systems.** To prove the persistence of the unperturbed heteroclinic orbit (42) for $\varepsilon \neq 0$ sufficiently small, we employ a slight generalization of the Melnikov theory presented in [Rob83]. There, the focus is on fast-slow systems of the form

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}_0(\mathbf{x}, \mathbf{u}) + \varepsilon \mathbf{f}_1(\mathbf{x}, \mathbf{u}) + \mathcal{O}(\varepsilon^2), \\ \mathbf{u}' &= \varepsilon \mathbf{r}(\mathbf{x}, \mathbf{u}) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{44}$$

where $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^d$ with $d \in \mathbb{N}$ arbitrary, and the prime denotes differentiation with respect to t. Additionally, the vector field $\mathbf{f}_0$ is supposed to be divergence free, i.e. $\mathrm{tr} D\mathbf{f}_0 = 0$ holds. For $\varepsilon = 0$ and each $\mathbf{u}_0$ in some bounded set $\mathcal{U}$, the equations in (44) are assumed to have a hyperbolic saddle fixed point $\mathbf{z}(\mathbf{u}_0, 0)$. Moreover, Robinson imposes a so-called "saddle connection assumption", that is, this fixed point is connected to itself by a homoclinic orbit $\mathbf{x}_0(t, \mathbf{u}_0)$.

We generalize the assumptions of [Rob83] in two ways. First, we need to consider systems of the form (44) which, for $\varepsilon = 0$ and for each $\mathbf{u}_0 \in \mathcal{U}$, have two distinct hyperbolic saddle points, $\mathbf{z}^+(\mathbf{u}_0, 0)$ and $\mathbf{z}^-(\mathbf{u}_0, 0)$ say, that are connected by a heteroclinic orbit $\mathbf{x}_0(t, \mathbf{u}_0)$. Here, it turns out that the same theory applies, with only minor modifications. Second, the unperturbed vector field $\mathbf{f}_0$ in which we are interested has non-zero divergence. This necessitates the inclusion of an additional multiplicative factor in the corresponding Melnikov integral. The required calculations follow closely the earlier work of Salam [Sal87] on small-amplitude perturbations of dissipative systems. For the sake of clarity, however, we will keep the notation of [Rob83] wherever possible.

Note that for $\varepsilon = 0$ in (44), the above assumptions imply the existence of two normally hyperbolic invariant sets

$$\mathcal{M}_0^+ = \big\{(\mathbf{x}, \mathbf{u}) \,\big|\, \mathbf{x} = \mathbf{z}^+(\mathbf{u}, 0), \mathbf{u} \in \mathcal{U}\big\} \quad \text{and} \quad \mathcal{M}_0^- = \big\{(\mathbf{x}, \mathbf{u}) \,\big|\, \mathbf{x} = \mathbf{z}^-(\mathbf{u}, 0), \mathbf{u} \in \mathcal{U}\big\}, \tag{45}$$

as well as of the corresponding stable and unstable manifolds $\mathcal{W}^j(\mathcal{M}_0^+)$ and $\mathcal{W}^j(\mathcal{M}_0^-)$, where $j \in \{u, s\}$. Without loss of generality, we assume that one branch of each of $\mathcal{W}^s(\mathcal{M}_0^+)$ and $\mathcal{W}^u(\mathcal{M}_0^-)$ coincide to form the heteroclinic manifold connecting $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$. For any

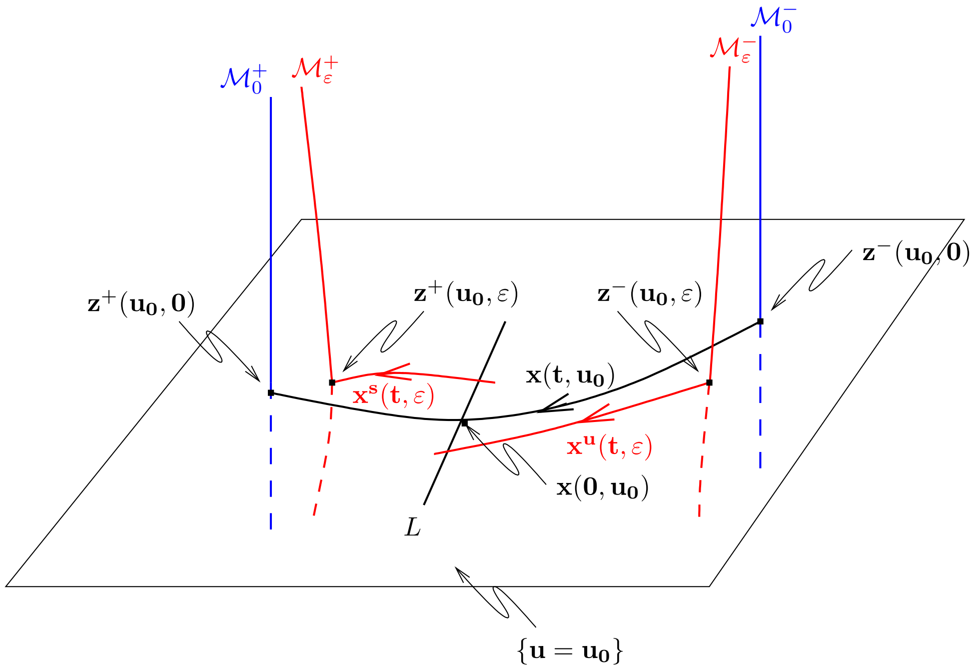


FIGURE 6. Geometry of system (44).

$\mathbf{u}_0 \in \mathcal{U}$ fixed, we then retain the orbit $\mathbf{x}_0$ in the intersection of $\mathcal{W}^s(\mathcal{M}_0^+) = \mathcal{W}^u(\mathcal{M}_0^-)$ with $\{\mathbf{u} = \mathbf{u}_0\}$, see Figure 6.

It follows from standard persistence theory for normally hyperbolic invariant manifolds [Fen71, Fen79] that $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$ will persist for $\varepsilon \neq 0$ sufficiently small as normally hyperbolic invariant sets

$$\mathcal{M}_\varepsilon^+ = \{(\mathbf{x},\mathbf{u}) \,|\, \mathbf{x} = \mathbf{z}^+(\mathbf{u},\varepsilon), \mathbf{u} \in \mathcal{U}\} \quad \text{and} \quad \mathcal{M}_\varepsilon^- = \{(\mathbf{x},\mathbf{u}) \,|\, \mathbf{x} = \mathbf{z}^-(\mathbf{u},\varepsilon), \mathbf{u} \in \mathcal{U}\}. \tag{46}$$

Similarly, $\mathcal{W}^j(\mathcal{M}_0^+)$ and $\mathcal{W}^j(\mathcal{M}_0^-)$ perturb to the corresponding stable and unstable manifolds $\mathcal{W}^j(\mathcal{M}_\varepsilon^+)$ and $\mathcal{W}^j(\mathcal{M}_\varepsilon^-)$, respectively.

As in [Rob83], we define a distance function Δ by

$$\Delta(t,\mathbf{u}_0,\varepsilon) = \left[\mathbf{x}^u(t,\varepsilon) - \mathbf{x}^s(t,\varepsilon)\right] \wedge \mathbf{f}_0(\mathbf{x}_0(t)) \tag{47}$$

and its first derivative with respect to ε by

$$\Delta_1(t,\mathbf{u}_0) = \frac{\partial}{\partial\varepsilon}\Delta(t,\mathbf{u}_0,\varepsilon)\Big|_{\varepsilon=0} = \left[\frac{\partial}{\partial\varepsilon}\mathbf{x}^u(t,\varepsilon) - \frac{\partial}{\partial\varepsilon}\mathbf{x}^s(t,\varepsilon)\right] \wedge \mathbf{f}_0(\mathbf{x}_0(t)). \tag{48}$$

Here $\mathbf{u}_0 \in \mathcal{U}$, and the wedge product is the scalar cross product in the plane. As $\Delta(0,\mathbf{u}_0,0) = 0$, the function Δ_1 measures the infinitesimal separation of the stable and unstable manifolds $\mathcal{W}^s(\mathcal{M}_\varepsilon^+)$ and $\mathcal{W}^u(\mathcal{M}_\varepsilon^-)$ as a function of ε. Note that $\mathbf{x}^s(t,\varepsilon)$ and $\mathbf{x}^u(t,\varepsilon)$ denote the respective sections of these manifolds obtained by fixing $\mathbf{u}_0$.

Proposition 4. *Under the above assumptions, we have*

$$\Delta_1(0,\mathbf{u}_0) = \int_{-\infty}^{\infty} \Big\{ \mathbf{f}_1(\mathbf{x}_0(t),\mathbf{u}_0) + \frac{\partial \mathbf{f}_0}{\partial \mathbf{u}}(\mathbf{x}_0(t),\mathbf{u}_0)\frac{\partial \mathbf{u}}{\partial \varepsilon}\Big\} \wedge \mathbf{f}_0(\mathbf{x}_0(t),\mathbf{u}_0) \times \exp\Big[-\int_0^t \mathrm{tr} D\mathbf{f}_0(\mathbf{x}_0(s),\mathbf{u}_0)\,ds\Big]\,dt, \tag{49}$$

where $\frac{\partial \mathbf{u}}{\partial \varepsilon}$ *satisfies* $\big(\frac{\partial \mathbf{u}}{\partial \varepsilon}\big)' = \mathbf{r}(\mathbf{x}_0(t),\mathbf{u}_0)$, *cf.* [Rob83].

Proof. Let us first define

$$\Delta_1^j(t) := \frac{\partial}{\partial \varepsilon}\mathbf{x}^j(t,\varepsilon) \wedge \mathbf{f}_0(\mathbf{x}_0(t))$$

for $j \in \{u,s\}$, whence $\Delta_1(t,\mathbf{u}_0) = \Delta_1^u(t) - \Delta_1^s(t)$. As in [Rob83], it follows that

$$\frac{d}{dt}\Delta_1^j = \underbrace{\Big\{\mathbf{f}_1 + \frac{\partial \mathbf{f}_0}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{u}}{\partial \varepsilon}\Big\} \wedge \mathbf{f}_0}_{=:b(t)} + \underbrace{\mathrm{tr} D\mathbf{f}_0}_{=:a(t)}\Delta_1^j,$$

with the exception that the contribution from the trace does not vanish now. Using these abbreviations, the above equations for Δ_1^j can be solved by variation of constants, and rearranged to give the following expressions for $\Delta_1^j(0)$:

$$\Delta_1^u(0) = \Delta_1^u(t)e^{\int_t^0 a(s)\,ds} + \int_t^0 b(s)e^{\int_s^0 a(r)\,dr}\,ds \qquad \text{for } t \in (-\infty,0], \tag{50a}$$

$$\Delta_1^s(0) = \Delta_1^s(t)e^{-\int_0^t a(s)\,ds} - \int_0^t b(s)e^{-\int_0^s a(r)\,dr}\,ds \qquad \text{for } t \in [0,\infty). \tag{50b}$$

We now take the limit of $t \to \pm\infty$ in (50). As in [Rob83, Lemma 3.5], we conclude that $\lim_{t\to-\infty}\Delta_1^u(t) = 0 = \lim_{t\to\infty}\Delta_1^s(t)$. In particular, we know that $\frac{\partial \mathbf{x}^j}{\partial \varepsilon}$ is of the order $\mathcal{O}(t)$, whereas $\mathbf{f}_0 \to 0$ exponentially fast as $t \to \pm\infty$, with exponential rates given by the eigenvalues λ^j of $D\mathbf{f}_0$ at the saddle points. As in Lemma 2.1 and the preceding discussion in [Sal87], we see that either $\lambda^s < 0 \le \mathrm{tr}D\mathbf{f}_0 < \lambda^u$ or $\lambda^s < \mathrm{tr}D\mathbf{f}_0 \le 0 < \lambda^u$. Hence, in both cases, the first terms in both equations in (50) vanish exponentially, since $\lambda^s - \mathrm{tr}D\mathbf{f}_0 < 0$ and $\lambda^u - \mathrm{tr}D\mathbf{f}_0 > 0$. As the integrands in the two resulting expressions for $\Delta_1^j(0)$, $j \in \{u,s\}$ are the same, we can recombine (50) in the limit as $t \to \pm\infty$ into the desired form (49), which concludes the proof. ■

3.3. Proof of Theorem 2. We will apply the Melnikov theory (Proposition 4) presented in the previous section to system (40). We begin by showing that the reduced equations in (41) satisfy the hypotheses of the theory.

Note that the fast variables $\mathbf{x}$ in Robinson's notation correspond to our (U,Z) in (40), whereas the slow parameter $\mathbf{u}$ is given by Y in our case. Moreover, we rewrite the second equation from (40) in terms of $\tilde{c} := c - \sqrt{\kappa}$ to obtain

$$Z' = -2\kappa U(1-U) - (\sqrt{\kappa}+\tilde{c})Z - Z^2. \tag{51}$$

Here $\kappa = e^{-Y}$, as before, and we only assume $\tilde{c} = o(1)$ as $\varepsilon \to 0$ for the moment; the exact nature of the ε-dependence of $\tilde{c}$ will be determined through the following analysis. In the notation of [Rob83], we then have

$$\mathbf{f}_0(U,Z,Y) = \big(UZ, -2e^{-Y}U(1-U) - e^{-\frac{Y}{2}}Z - Z^2\big)^T, \tag{52}$$

$$\mathbf{f}_1(U,Z,Y) = (0, -\tilde{c}Z)^T, \tag{53}$$

$$\mathbf{r}(U,Z,Y) = -Z. \tag{54}$$

In contrast to the assumptions made in [Rob83], however, the vector field in (40) is not divergence free, as

$$D\mathbf{f}_0 = \begin{bmatrix} Z & U \\ -2\kappa(1-2U) & -\sqrt{\kappa}-2Z \end{bmatrix} \tag{55}$$

and $\mathrm{tr}D\mathbf{f}_0 = -\sqrt{\kappa} - Z \not\equiv 0$. Hence, we make use of Proposition 4 above to account for the additional terms arising from $\mathrm{tr}D\mathbf{f}_0$.

When $\varepsilon \neq 0$, there is no equilibrium for $Z = -\sqrt{\kappa}$ in (40). Still, reverting to our original notation and recalling that Y_0 varies in $\mathcal{Y} \subset \mathbb{R}$, it follows from standard persistence theory for normally hyperbolic invariant manifolds that the two curves

$$\mathcal{L}_0^+ = \bigcup_{Y_0\in\mathcal{Y}} \big\{\big(0, -e^{-\frac{Y_0}{2}}, Y_0\big)\big\} \quad \text{and} \quad \mathcal{L}_0^- = \bigcup_{Y_0\in\mathcal{Y}} \{(1,0,Y_0)\} \tag{56}$$

persist for ε sufficiently small as curves $\mathcal{L}_\varepsilon^+$ and $\mathcal{L}_\varepsilon^-$, say. Moreover, from Lemma 7 we conclude that there exist two-dimensional stable and unstable manifolds $\mathcal{W}^s(\mathcal{L}_0^+)$ and $\mathcal{W}^u(\mathcal{L}_0^-)$, respectively, which will perturb to the corresponding manifolds $\mathcal{W}^s(\mathcal{L}_\varepsilon^+)$ and $\mathcal{W}^u(\mathcal{L}_\varepsilon^-)$ by Fenichel theory [Fen79]. Note that for $\varepsilon = 0$, the two branches of $\mathcal{W}^s(\mathcal{L}_0^+)$ and $\mathcal{W}^u(\mathcal{L}_0^-)$ restricted to $U \in [0,1]$ coincide, which will be of importance later on, cf. Figure 7.

To prove Theorem 2, i.e. to establish the smoothness of $c_{\mathrm{crit}}^2(\varepsilon)$ and to derive its leading-order behavior, we define the distance function Δ to be a function of *both* ε and $\tilde{c}$, and then compute the leading-order expansion for this new Δ. To that end, let

$$\Delta(\xi,\kappa,\varepsilon,\tilde{c}) = \big[(U,Z)^u(\xi,\varepsilon,\tilde{c}) - (U,Z)^s(\xi,\varepsilon,\tilde{c})\big] \wedge \big(UZ, -2\kappa U(1-U) - \sqrt{\kappa}Z - Z^2\big)^T, \tag{57}$$

where Δ measures the distance between $\mathcal{W}^s(\mathcal{L}_\varepsilon^+)$ and $\mathcal{W}^u(\mathcal{L}_\varepsilon^-)$ along the normal $\mathbf{n}$ to the unperturbed heteroclinic manifold formed by the coincidence of one branch of each of $\mathcal{W}^s(\mathcal{L}_0^+)$ and $\mathcal{W}^u(\mathcal{L}_0^-)$, see Figure 8. Also, we define

$$\Delta_{10}(\xi,\kappa) := \frac{\partial}{\partial\varepsilon}\Delta(\xi,\kappa,\varepsilon,\tilde{c})\Big|_{(\varepsilon,\tilde{c})=(0,0)} \quad \text{and} \quad \Delta_{01}(\xi,\kappa) := \frac{\partial}{\partial\tilde{c}}\Delta(\xi,\kappa,\varepsilon,\tilde{c})\Big|_{(\varepsilon,\tilde{c})=(0,0)};$$

here $\kappa = e^{-Y_0}$, as before.

The above definition of Δ will allow for a rigorous justification of the regular expansion of $c_{\mathrm{crit}}^2(\varepsilon)$ in powers of ε derived in [WOK01]:

Proof of Theorem 2. The function Δ defined in (57) is $\mathcal{C}^k$-smooth, for any $k \in \mathbb{N}$, by definition; moreover, $\Delta(0,\kappa,0,0) = 0$. Hence, it follows that, to leading order,

$$\Delta(0,\kappa,\varepsilon,\tilde{c}) = \Delta_{10}(0,\kappa)\,\varepsilon + \Delta_{01}(0,\kappa)\,\tilde{c} + \mathcal{O}(2)$$

for ε and $\tilde{c}$ sufficiently small. Here, $\mathcal{O}(2)$ denotes terms of second order and upwards in ε and $\tilde{c}$. We now compute Δ_{10} and Δ_{01}. All expressions that follow are evaluated at $(\varepsilon,\tilde{c}) = (0,0)$.

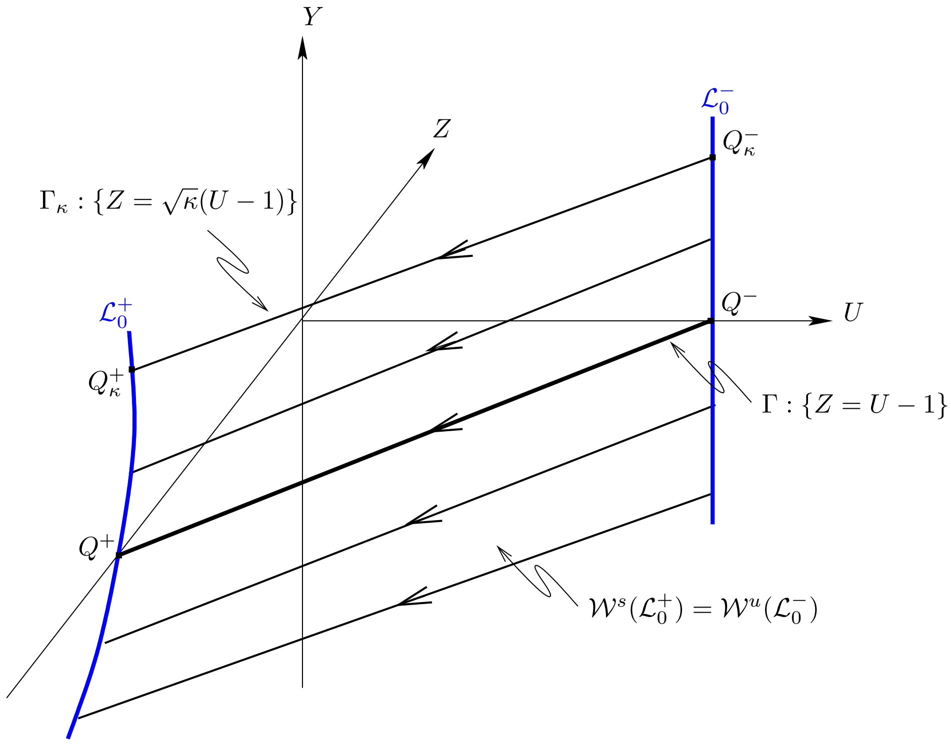


FIGURE 7. Geometry of (41) for $Y_0 \in \mathcal{Y}$.

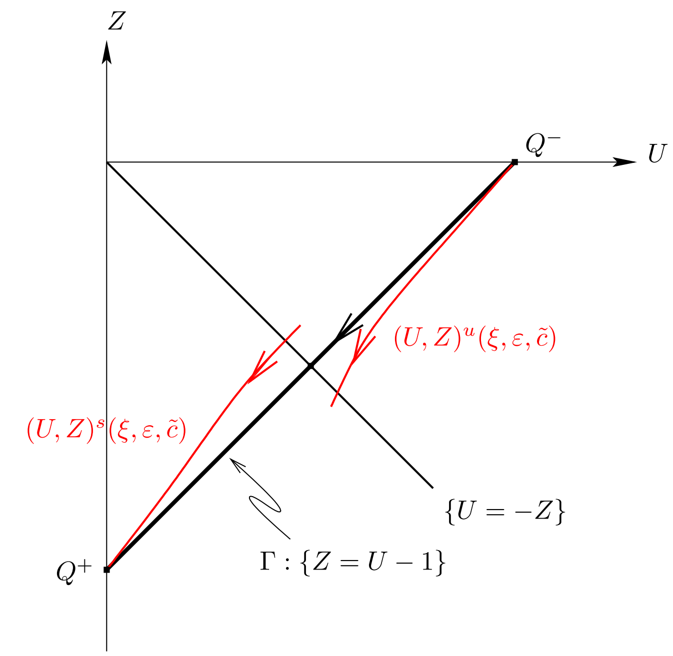


FIGURE 8. Projection of (41) onto the plane $\{Y = 0\}$.

For Δ_{10}, one can retrace the arguments in the proof of Proposition 4 to obtain

$$\begin{aligned}(58)\quad \Delta_{10}(0,\kappa) = -\int_{-\infty}^{\infty} &\big(0, 2\kappa U(1-U)\big)^T \ln U \\ &\wedge \big(UZ, -2\kappa U(1-U) - \sqrt{\kappa}Z - Z^2\big)^T \exp\Big[\int_0^\xi \big(\sqrt{\kappa}+Z(s)\big)\,ds\Big]\,d\xi\Big|_{(U,Z)=(U_0,Z_0)}\end{aligned}$$

(with $\mathbf{f}_0$ and $D\mathbf{f}_0$ given by (52) and (55), respectively). Here, (U_0, Z_0) is again defined as in (42), and we have made use of $Y = -\varepsilon \ln U$, whence

$$\frac{\partial Y}{\partial \varepsilon}(\xi)\Big|_{\varepsilon=0} = -\ln U_0(\xi).$$

To calculate (58), we first carry out the inner integration:

Lemma 8. *There holds*

$$\exp\Big[\int_0^\xi \big(\sqrt{\kappa}+Z_0(s)\big)\,ds\Big] = -\frac{2}{\sqrt{\kappa}}Z_0(\xi).$$

Proof. The result follows from (42) and the definition of Z, as

$$\int_0^\xi \big(\sqrt{\kappa}+Z_0(s)\big)\,ds = \sqrt{\kappa}\xi + \int_0^\xi \frac{U_0'(s)}{U_0(s)}\,ds = \sqrt{\kappa}\xi + \ln U_0(s)\Big|_0^\xi = \sqrt{\kappa}\xi + \ln\frac{2}{1+e^{\sqrt{\kappa}\xi}}.$$

■

Making use of the above lemma, as well as of $Z = \sqrt{\kappa}(U-1)$ and $\sqrt{\kappa}U(U-1) = \frac{dU}{d\xi}$, we can rewrite (58) as

$$\begin{aligned}(59)\quad \Delta_{10}(0,\kappa) = -4\sqrt{\kappa}\int_{-\infty}^{\infty} U^2(1-U)Z^2\ln U\,d\xi &= 4\kappa\int_{-\infty}^{\infty} U(U-1)^2\ln U\frac{dU}{d\xi}\,d\xi \\ &= 4\kappa\int_1^0 U(U-1)^2\ln U\,dU = \frac{13}{36}\kappa.\end{aligned}$$

The argument for Δ_{01} goes as follows: Define

$$\Delta_{01}^j(\xi) := \frac{\partial}{\partial\tilde{c}}(U,Z)^j \wedge \big(UZ, -2\kappa U(1-U) - \sqrt{\kappa}Z - Z^2\big)^T \quad \text{for } j\in\{u,s\},$$

and compute $\frac{d}{d\xi}\Delta_{01}^j$ as in the proof of Proposition 4 to obtain

$$\frac{d}{d\xi}\Delta_{01}^j = (0,-Z)^T \wedge \big(UZ, -2\kappa U(1-U) - \sqrt{\kappa}Z - Z^2\big)^T - \big(\sqrt{\kappa}+Z\big)\,\Delta_{01}^j;$$

here, the term $(0,-Z)^T$ is introduced by taking the derivatives of $U' = UZ$ and (51) with respect to $\tilde{c}$. After integration, one has

$$\Delta_{01}^u(0) = \Delta_{01}^u(\xi)\exp\Big[-\int_\xi^0 \big(\sqrt{\kappa}+Z(s)\big)\,ds\Big] + \int_\xi^0 UZ^2\exp\Big[-\int_s^0 \big(\sqrt{\kappa}+Z(r)\big)\,dr\Big]\,ds$$

for $\xi \in (-\infty, 0]$ and

$$\Delta_{01}^s(0) = \Delta_{01}^s(\xi)\exp\Big[\int_0^\xi \big(\sqrt{\kappa}+Z(s)\big)\,ds\Big] - \int_0^\xi UZ^2\exp\Big[\int_0^s \big(\sqrt{\kappa}+Z(r)\big)\,dr\Big]\,ds$$

for $\xi \in [0, \infty)$, respectively. Passing to the limit of $\xi \to \pm\infty$, we note that the first terms in both the above expressions vanish, as in the proof of Proposition 4. In fact, we now even have that $\frac{\partial}{\partial \tilde{c}}(U, Z)^j$ is bounded, i.e. $\mathcal{O}(1)$, as in the standard time-dependence case. Referring to Lemma 8 again, we get

$$\Delta_{01}(0, \kappa) = -\frac{2}{\sqrt{\kappa}} \int_{-\infty}^{\infty} UZ^3 \, d\xi = -2\sqrt{\kappa} \int_1^0 (U - 1)^2 \, dU = \frac{2}{3}\sqrt{\kappa}. \tag{60}$$

In sum, combining (59) and (60), we find

$$\Delta(0, \kappa, \varepsilon, \tilde{c}) = \frac{13}{36}\kappa\varepsilon + \frac{2}{3}\sqrt{\kappa}\tilde{c} + \mathcal{O}(2); \tag{61}$$

moreover, as $\Delta(0, \kappa, 0, 0) = 0$ and $\frac{\partial}{\partial \tilde{c}}\Delta(0, \kappa, 0, 0) = \Delta_{01}(0, \kappa) \neq 0$, by the Implicit Function Theorem there exists a $\mathcal{C}^k$-smooth function $\tilde{c}(\varepsilon)$ such that $\Delta(0, \kappa, \varepsilon, \tilde{c}(\varepsilon)) = 0$ for $\varepsilon \neq 0$ sufficiently small, which implies the existence of a solution to (40) with $c^2_{\text{crit}}(\varepsilon) = \sqrt{\kappa} + \tilde{c}(\varepsilon)$. Furthermore, from (61) we obtain that $\Delta = 0$ is equivalent to $\tilde{c}(\varepsilon) = -\frac{13}{24}\sqrt{\kappa}\varepsilon + \mathcal{O}(2)$. Hence, for $Y_0 = 0$ ($\sqrt{\kappa} = 1$), we have

$$c^2_{\text{crit}}(\varepsilon) = 1 + \tilde{c}(\varepsilon) = 1 - \frac{13}{24}\varepsilon + \mathcal{O}(\varepsilon^2),$$

which completes the proof of Theorem 2. ■

The proof of Theorem 2 implies the following: Not only will the orbit of (40) corresponding to $Y = 0$ persist for $\varepsilon \neq 0$ sufficiently small, but we have persistence of an entire two-dimensional surface of orbits obtained by varying $Y_0 \in \mathcal{Y}$ in (40). To state it differently: Just as $\mathcal{W}^s(\mathcal{L}_0^+)$ and $\mathcal{W}^u(\mathcal{L}_0^-)$ coincide for $U \in [0, 1]$ and $\varepsilon = 0$, so do the perturbed manifolds $\mathcal{W}^s(\mathcal{L}_\varepsilon^+)$ and $\mathcal{W}^u(\mathcal{L}_\varepsilon^-)$ for $0 < |\varepsilon| \ll 1$. The reason is that by picking $\tilde{c}$ for $Y = 0$ above, we simultaneously obtain the correct first-order correction to $c = e^{-\frac{Y}{2}}$ for any $Y \in \mathcal{Y}$. Due to the fact that we are dealing with a simple zero of Δ (as $\Delta_{01} \neq 0$), and since the equations in (40) are autonomous, the above claim is actually true to *any* order in ε.

Indeed, this should come as no surprise: The lifting of the two-dimensional system (37) in (U, V)-space to the equations in (40) in three-dimensional (U, Z, Y)-space is somewhat artificial. We are really only interested in $Y = 0$ in (40), but have to allow for an entire (open) set $\mathcal{Y}$ of possible parameter values. Alternatively, one can think of $\mathcal{Y}$ as a one-dimensional parameter space for (40) foliated by a family of differently scaled copies of what is essentially one set of equations, where we pick the copy corresponding to $Y = 0$.

Remark 13. *In* [Rob83]*, it is required that* $\frac{\partial Y}{\partial \varepsilon}(0) = 0$*. In our case, however, there holds* $\frac{\partial Y}{\partial \varepsilon}(0) = \ln 2$*, the reason being that we are dealing with a heteroclinic instead of with a symmetric homoclinic orbit here.*

Remark 14. *As we know from Lemma 7 that* $U \sim e^{-\sqrt{\kappa}\xi}$ *(*$U \sim 1 - e^{\sqrt{\kappa}\xi}$*) as* $\xi \to \infty$ *(*$\xi \to -\infty$*), it follows that* $Y \sim \varepsilon\sqrt{\kappa}\xi$ *(*$Y \sim \varepsilon e^{\sqrt{\kappa}\xi}$*) then. Thus, in the former case, we are exactly within the framework of* [Rob83]*, whereas in the latter case, we fare even better.*

Remark 15. *We now briefly return to the problem for* $m = 1 + \varepsilon$*, studied in Section 2, to show why the methods of this section do not seem to apply in that case. Even if the fixed point* $(U, Z) = (0, -\sqrt{2})$ *is not a hyperbolic saddle for* $n = 1$*, one might still hope that the results in* [Rob83] *are applicable. The fact that no explicit solutions to* (11) *seem to be known* *for $\varepsilon = 0$ should pose no difficulty, either, as one could still derive qualitative estimates on the corresponding Melnikov integrals.*

First, we remark that the vector field in (13) *again is not divergence free: Writing*

$$\mathbf{f}_0(U, Z, Y) = \left(UZ, -2e^{-Y}(1-U) - 2\sqrt{2}Z - Z^2\right)^T,$$

we see that $\mathrm{tr}D\mathbf{f}_0 = -2\sqrt{2} - Z$. *Retracing the proof of Proposition 4, however, it turns out that in* (50),

$$\Delta_1^s(t) \exp\Big[- \int_0^t \mathrm{tr}D\mathbf{f}_0(s)\, ds \Big] = \mathcal{O}(t) \to \infty \qquad \textit{as } t \to \infty$$

now, which is due to $\mathrm{tr}D\mathbf{f}_0 \to -\sqrt{2}$ *for* $Z \sim -\sqrt{2}$. *Therefore,* $\lambda^s - \mathrm{tr}D\mathbf{f}_0 \to 0$ *when* $t \to \infty$, *and the statement of Proposition 4 will no longer hold.*

This observation can be verified directly, as well: Given that $Z = \frac{U'}{U}$, *we have*

$$\exp\Big[- \int_0^\xi \mathrm{tr}D\mathbf{f}_0\, ds \Big] = e^{2\sqrt{2}\xi} \exp\Big[\int_{U(0)}^{U(\xi)} \frac{dU}{U} \Big] \propto e^{2\sqrt{2}\xi} U(\xi) \to \infty \qquad \textit{for } \xi \to \infty,$$

since $U(\xi) \sim e^{-\sqrt{2}\xi}$ *then, which implies that the integral in* (49) *will diverge. Hence, the methods employed here do not appear to be applicable to the problem analyzed in Section 2.*

Remark 16. *We chose to present the analysis of the case $m = 2+\varepsilon$ in Section 3 in the same framework used for the case $m = 1+\varepsilon$ in the previous section, namely, in the framework of system* (9)*, which involves both the projectivized coordinate Z and the embedding variable Y. This choice was made both to keep the presentation uniform and to bring out the geometry of the problem. The case $n = 2$ can probably also be analyzed using a more direct approach involving the Lyapunov-Schmidt method and the Melnikov condition associated to it, since the vector field in* (7) *is $\mathcal{C}^1$-smooth jointly in U and ε when $0 < |\varepsilon| \ll 1$. See* [CL90]*, where this approach was first developed for determining the existence of homoclinic and heteroclinic orbits, as well as of periodic orbits, in systems with fixed points that have a zero eigenvalue.*

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