# Jump-induced mixed-mode oscillations through piecewise-affine maps 

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#### Abstract

Mixed-mode oscillations (MMOs) are complex oscillatory patterns in which largeamplitude oscillations (LAOs) of relaxation type alternate with small-amplitude oscillations (SAOs). MMOs are found in singularly perturbed systems of ordinary differential equations of slow-fast type, and are typically related to the presence of socalled folded singularities and the corresponding canard trajectories in such systems. Here, we introduce a canonical family of three-dimensional slow-fast systems that exhibit MMOs which are induced by relaxation-type dynamics, and which are hence based on a "jump mechanism", rather than on a more standard canard mechanism. In particular, we establish a correspondence between that family and a class of associated one-dimensional piecewise affine maps (PAMs) which exhibit MMOs with the same signature. Finally, we give a preliminary classification of admissible mixedmode signatures, verifying results of [Rajpathak, Pillai, and Bandyopahdyay (2012)] in the process, and we illustrate our findings with numerical examples.


Keywords: mixed-mode oscillation, relaxation oscillation, piecewise affine map, slow-fast system, geometric singular perturbation theory.

## 1. Introduction

In the theory of dynamical systems, one is generally interested in the qualitative behaviour of solutions of differential equations. Thus, for instance, one investigates bifurcations of equilibria and periodic orbits in dependence of parameters in these systems. In this paper, we focus on singularly perturbed three-dimensional systems of "slow-fast" type, with two slow variables and one fast variable. Such systems are characterised by the variables evolving on different time-scales, which can, in some circumstances, give rise to canard phenomena. Canards [2] arise when trajectories of a singularly perturbed system follow an attracting manifold, pass through a folded singularity and then - somewhat counterintuitively - stay close to a repelling slow manifold for some time. In planar slow-fast systems, the canard phenomenon is often linked to the presence of a (singular) Hopf bifurcation at a turning (fold) point; one typical example is given by the singularly perturbed van der Pol equation

[^0][5, 9]. Canards have been studied extensively over the past decades; their study has mainly been based on non-standard analysis [3, 7, 8], matched asymptotic expansions [11, 26, 27], and a geometric approach that combines Fenichel's geometric singular perturbation theory (GSPT) [12, 13] and the so-called blow-up technique, which was introduced in the pioneering works of Dumortier and Roussarie [9], as well as of Krupa and Szmolyan [22].

In three-dimensional slow-fast systems with two slow variables, the canard phenomenon can give rise to mixed-mode oscillatory dynamics. Mixed-mode oscillations (MMOs) typically consist of large-amplitude oscillations (LAOs) of relaxation type, followed by small-amplitude oscillations (SAOs). While no generally accepted rigorous definition of MMOs seems to exist, a clear, intuitive separation between LAOs and SAOs seems to be evident in most cases; what draws immediate attention is the pattern that emerges in the alternation between oscillations of distinct amplitudes. Specifically, a (periodic) MMO is said to have signature $L^{s}$ if the corresponding orbit undergoes $s$ SAOs, followed by LAOs, at which point that sequence repeats. See [5] for a recent review of this complex oscillatory dynamics, as well as $[14,18,20,24,25,33]$ for a small selection of biological, chemical, and physical models in which a variety of MMO patterns have been observed.

Among the numerous mechanisms that have been proposed to explain mixedmode oscillatory dynamics in singularly perturbed systems of slow-fast type, the canard-based mechanism [4, 21, 25, 34] has been among the most popular. Roughly speaking, it combines local passage through the vicinity of a folded singularity which explains the SAO component of the corresponding MMO - with a global return mechanism which results in relaxation (LAO), returning the flow to the basin of attraction of the folded singularity [5]. In the present paper, we introduce an alternative mechanism for the generation of mixed-mode dynamics in three-dimensional slow-fast systems, which we will refer to as the "jump mechanism". In the process, we will show that the occurrence of MMOs is not necessarily caused by the presence of a folded singularity, as in the canard-based mechanism; in fact, the main characteristic of the MMOs studied in this paper is that both LAOs and SAOs are now of relaxation type and that the amplitude of the latter is thus of order $O(1)$ in the singular perturbation parameter.

Our study is inspired by previous work of Szmolyan and Wechselberger [31], Krupa, Popović, and Kopell [21], and Rajpathak, Pillai, and Bandyopahdyay [29]. By considering a prototypical family of slow-fast systems which incorporates two jump mechanisms of the type studied in [31], we reproduce MMOs that alternate between LAOs and SAOs of relaxation type; see Figure 1 for an illustration of the resulting geometry, as well as Section 2 for a precise definition of said family. As we rely on established results from [31], we do not explicitly need to perform a family blow-up in order to desingularise the flow near fold curves along which normal hyperbolicity is lost. In the process, we reduce the study of mixed-mode dynamics in our prototypical family to that of one-dimensional piecewise affine maps (PAMs) [1, 10, 28, 35]. In particular, we show that the singular limit of the corresponding first return (or Poincaré) map yields a PAM; see Proposition 1. Piecewise maps [6] have been popularised in the study of dynamical systems in recent decades, with a particular focus on models for switching phenomena such as electrical circuits [15, $17,30]$ and neurons $[16,18,32]$; such maps are naturally related to the corresponding Poincaré maps in oscillatory systems [23].

Thus, we establish a natural two-way correspondence between the family of three-dimensional slow-fast systems studied here and a suitably defined class of one-dimensional PAMs which is induced by the reduced flow on the critical manifold of that family. Specifically, we show that a slow-fast system which satisfies the assumptions in Section 2 exhibits a periodic MMO of a given signature $L^{s}$ if the PAM which is associated with that system exhibits a periodic MMO with the same signature (Theorem 3.2). Conversely, we show that any PAM within a very broad class of maps is associated with a slow-fast system within the family defined in Section 2 (Theorem 3.3). That family is characterised by the presence of two perturbation parameters; as will become clear in the following, the interplay between those - as expressed through their relative magnitudes - crucially impacts on the resulting mixed-mode dynamics. However, we emphasise that the scenario considered here differs substantively from the one studied in [21], where the second parameter introduces an additional (third) timescale.

This paper is organised as follows. In Section 2, we define the three-dimensional family of slow-fast systems which underlies our results; in particular, we introduce the jump-type mechanism that gives rise to mixed-mode oscillatory dynamics in our context. We state the main results of the paper in Section 3; before proving those in Section 5, we explain how to compute the associated PAM (Section 4). In Section 6, we consider a particular representative from our family of systems to verify our findings numerically, and we present relevant simulations; in the process, we identify parameter regimes which allow for robust mixed-mode dynamics. We find that our results are in accordance with [29]; moreover, we give numerical evidence of mixed, "crossover" signatures. In Section 7, we conclude with a discussion of our findings, as well as with an outlook to potential future research endeavours.

## 2. Slow-fast model and assumptions

We study MMOs in the context of the following well-known three-dimensional family of slow-fast systems in the standard form of geometric singular perturbation theory,

$$
\begin{align*}
& x^{\prime}=y-F(x, z, \epsilon, \delta)=: f(x, y, z, \epsilon, \delta),  \tag{2.1a}\\
& y^{\prime}=\epsilon g_{1}(x, y, z, \epsilon, \delta),  \tag{2.1b}\\
& z^{\prime}=\epsilon g_{2}(x, y, z, \epsilon, \delta) ; \tag{2.1c}
\end{align*}
$$

here, $f, g_{1}$, and $g_{2}$ are $C^{\infty}$-smooth functions in their arguments that will be specified in the following and $\epsilon \geq 0$ is a (small) singular perturbation parameter. We emphasise that (2.1) contains an additional parameter $\delta$, the relevance of which will become evident below. Correspondingly, $x \in \mathbb{R}$ is a fast variable, while $(y, z) \in \mathbb{R}^{2}$ are slow variables, all of which depend on the fast time $t$. To avoid unnecessary abstraction, we have assumed that $f$ is of the specific form $f=y-F$ in (2.1a); that assumption does not represent a major restriction. Our study will chiefly be based on Fenichel's geometric singular perturbation theory (GSPT) [12, 13]; an excellent introduction can be found in [19].

Rewriting the above fast system in terms of the slow time variable $\tau=\epsilon t$, we
obtain the equivalent slow system

$$
\begin{align*}
\epsilon \dot{x} & =y-F(x, z, \epsilon, \delta),  \tag{2.2a}\\
\dot{y} & =g_{1}(x, y, z, \epsilon, \delta),  \tag{2.2b}\\
\dot{z} & =g_{2}(x, y, z, \epsilon, \delta), \tag{2.2c}
\end{align*}
$$

where the overdot denotes differentiation with respect to $\tau$.
In the singular limit of $\epsilon=0$, the above systems yield the layer problem

$$
\begin{align*}
& x^{\prime}=y-F(x, z, 0, \delta),  \tag{2.3a}\\
& y^{\prime}=0  \tag{2.3b}\\
& z^{\prime}=0 \tag{2.3c}
\end{align*}
$$

and the reduced problem

$$
\begin{align*}
& 0=y-F(x, z, 0, \delta),  \tag{2.4a}\\
& \dot{y}=g_{1}(x, y, z, 0, \delta),  \tag{2.4b}\\
& \dot{z}=g_{2}(x, y, z, 0, \delta), \tag{2.4c}
\end{align*}
$$

respectively. In particular, (2.4) allows us to define the ( $\delta$-family of) critical manifolds $\mathcal{S}:=\left\{(x, y, z) \in \mathbb{R}^{3}: f(x, y, z, 0, \delta)=0\right\}$, which is of central importance in GSPT: the sign of $\frac{\partial f}{\partial x}$ determines the stability of the steady states of the layer problem in (2.3), which are located on $\mathcal{S}$. Specifically, orbits that are initiated away from $\mathcal{S}$ will converge to attracting segments of the critical manifold under the layer flow of (2.3); on $\mathcal{S}$, they will then be subject to the reduced flow of (2.4). Away from zeros of $\frac{\partial f}{\partial x}$, the critical manifold $\mathcal{S}$ is normally hyperbolic; by Fenichel's First Theorem, normally hyperbolic segments of $\mathcal{S}$ will perturb, for $\epsilon$ positive and sufficiently small, to a slow manifold $\mathcal{S}_{\epsilon}$ [13]. Correspondingly, the reduced flow on $\mathcal{S}$ will perturb in a regular fashion to the slow flow on $\mathcal{S}_{\epsilon}$. Likewise, the fast flow of (2.1) will be a regular perturbation of the layer flow off $\mathcal{S}$.

A canonical scenario in which normal hyperbolicity is lost is found at so-called fold curves in (2.1), where orbits exhibit jumping behaviour. The reduced flow on $\mathcal{S}$ is directed towards (attraction) or away from (repulsion) these fold curves, which results in orbits having to jump to a different segment of the critical manifold there. When such behaviour occurs in a periodic fashion, relaxation oscillation is observed. The emergence of fold-induced relaxation oscillation in three-dimensional slow-fast systems was studied in detail in [31], where the desingularisation technique known as "blow-up" [9] was applied to remedy the loss of normal hyperbolicity.

We follow the same approach here and proceed to make the following (analogous) assumptions on the singular geometry of Equation (2.1).

Assumption 1. Possibly restricting to a part of phase space for $z \in\left(z_{\min }, z_{\max }\right)$, with $z_{\min }<z_{\max }$, we assume that the critical manifold $\mathcal{S}$ is $\xi$-shaped, or "Bactrian"shaped; see Figure 1. In other words, $\mathcal{S}$ can be written as

$$
\mathcal{S}=\mathcal{S}_{a_{1}} \cup L_{1} \cup \mathcal{S}_{r_{1}} \cup L_{2} \cup \mathcal{S}_{a_{2}} \cup L_{3} \cup \mathcal{S}_{r_{2}} \cup L_{4} \cup \mathcal{S}_{a_{3}},
$$

where $\mathcal{S}_{a_{1}} \cup \mathcal{S}_{a_{2}} \cup \mathcal{S}_{a_{3}}=\mathcal{S} \cap\left\{\frac{\partial f}{\partial x}(x, y, z, 0, \delta)<0\right\}$ and $\mathcal{S}_{r_{1}} \cup \mathcal{S}_{r_{2}}=\mathcal{S} \cap\left\{\frac{\partial f}{\partial x}(x, y, z, 0, \delta)>\right.$ $0\}$ denote the normally attracting and normally repelling segments of $\mathcal{S}$, respectively,
which are divided by four ( $\delta$-families of) fold curves along which normal hyperbolicity is lost, denoted $L_{1}, L_{2}, L_{3}$ and $L_{4}$ from left to right. These fold curves can be written as graphs

$$
L_{i}:=\left\{(x, y, z) \in \mathcal{S}:(x, y, z)=\left(\nu_{i}(z, \delta), \phi\left(\nu_{i}(z, \delta), z\right), z\right)\right\} \quad \text { for } i=1,2,3,4
$$

here, $\phi$ and $\nu_{i}(i=1,2,3,4)$ are appropriately defined functions along which the non-degeneracy conditions

$$
\frac{\partial f}{\partial x}\left(\nu_{i}(z, \delta), \phi\left(\nu_{i}(z, \delta), z\right), z, 0, \delta\right)=0 \quad \text { and } \quad \frac{\partial^{2} f}{\partial x^{2}}\left(\nu_{i}(z, \delta), \phi\left(\nu_{i}(z, \delta), z\right), z, 0, \delta\right) \neq 0
$$

are satisfied.
As $\nu_{1}<\nu_{2}<\nu_{3}<\nu_{4}$, by assumption, the normally attracting segments of $\mathcal{S}$ can equally be represented as

$$
\begin{aligned}
& \mathcal{S}_{a_{1}}:=\left\{(x, y, z) \in \mathcal{S} \cap\left\{x<\nu_{1}\right\}\right\}, \quad \mathcal{S}_{a_{2}}:=\left\{(x, y, z) \in \mathcal{S} \cap\left\{\nu_{2}<x<\nu_{3}\right\}\right\}, \quad \text { and } \\
& \mathcal{S}_{a_{3}}:=\left\{(x, y, z) \in \mathcal{S} \cap\left\{x>\nu_{4}\right\}\right\},
\end{aligned}
$$

while the normally repelling ones are given by

$$
\mathcal{S}_{r_{1}}:=\mathcal{S} \cap\left\{(x, y, z): \nu_{1}<x<\nu_{2}\right\} \quad \text { and } \quad \mathcal{S}_{r_{2}}:=\mathcal{S} \cap\left\{(x, y, z): \nu_{3}<x<\nu_{4}\right\} .
$$

Assumption 2 (Normal switching condition). We assume that

$$
\begin{equation*}
f_{y} g_{1}+\left.f_{z} g_{2}\right|_{p \in L_{i}} \neq 0 \quad \text { for } i=1,2,3,4 \tag{2.5}
\end{equation*}
$$

i.e., that any fold point $p$ on $L_{i}$ is a jump point. In other words, (2.5) asserts that the reduced flow of (2.4) is unbounded on the fold lines $L_{i}$ and that orbits must hence jump there. It is therefore required that the reduced flow on both sides of the fold lines $L_{i}(i=1,3,4)$ is transverse to, and directed towards, the fold lines $L_{i}$ at all times.

We define by $\omega\left(L_{2}\right)$ the projection of the fold line $L_{2}$ onto the attracting sheet $\mathcal{S}_{a_{1}}$ of $\mathcal{S}$; moreover, we define $\omega\left(L_{1}\right)$ and $\omega\left(L_{3}\right)$ as the projections of the fold lines $L_{1}$ and $L_{3}$ onto $\mathcal{S}_{a_{3}}$. Likewise, we define the projection $\omega\left(L_{4}\right)$ of the fold line $L_{4}$ onto both $\mathcal{S}_{a_{1}}$ and $\mathcal{S}_{a_{2}}$. (In spite of the suggestive notation, these projections should not be confused with $\omega$-limit sets of the corresponding fold lines.) Then, we assume the following.

Assumption 3 (Transversality of reduced flow). For $\delta=0$, the reduced flow of (2.4) is transverse to $\omega\left(L_{1}\right)$ and $\omega\left(L_{3}\right)$ on $\mathcal{S}_{a_{3}}$, transverse to $\omega\left(L_{2}\right)$ on $\mathcal{S}_{a_{1}}$, and transverse to $\omega\left(L_{4}\right)$ on both $\mathcal{S}_{a_{1}}$ and $\mathcal{S}_{a_{2}}$.

Let $\Delta$ denote a section between $L_{2}$ and $L_{4}$ that is transverse to the layer flow of (2.3); specifically, we take $\Delta \subset\left\{x=x_{0}\right\}$, where $\nu_{2}(z, \delta)<x_{0}<\nu_{4}(z, \delta)$, with $\nu_{2}(z, \delta)$ and $\nu_{4}(z, \delta)$ as in Assumption 1. (The section $\Delta$ is defined such that when $\epsilon$ is sufficiently small, but non-zero, all trajectories that jump at the fold line $L_{4}$ must intersect $\Delta$ before reaching the opposite attracting sheet of $\mathcal{S}$, be it $\mathcal{S}_{a_{1}}$ or $\mathcal{S}_{a_{2}}$.) Also, let $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$ denote the projections of the fold lines $L_{2}$ and $L_{4}$, respectively, onto $\Delta$. These two projections intersect transversely in a point $P_{c}:=P\left(L_{2}\right) \cap P\left(L_{4}\right)$, as indicated in Figure 1. Then, we make the following assumption:

Assumption 4 (Breaking mechanism). For $\delta>0$ sufficiently small and $\epsilon=0$, the $z$-parametrised curves $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$ intersect transversely at some $z$-value $z_{0}(\delta)$, with $z_{0}(0)=z_{0}$ and $z_{0}^{\prime}(0)=0$; in particular, the point of intersection $P_{c}$ between the curves thus depends on $\delta$. Furthermore, we assume that for $z<z_{0}(\delta)$, the evolution of $P\left(L_{4}\right)$ in forward time remains below the normally hyperbolic sheets $\mathcal{S}_{r_{1}}$ and $\mathcal{S}_{a_{2}}$ - in the sense that its $y$-component satisfies $y<\phi\left(\nu_{2}(z, \delta), z\right)$, with $\phi\left(\nu_{2}(z, \delta), z\right)$ as in Assumption 1 - and that it lands directly on the opposite attracting sheet $\mathcal{S}_{a_{1}}$. On the other hand, for $z>z_{0}(\delta)$, the evolution of $P\left(L_{4}\right)$ lands on $\mathcal{S}_{a_{2}}$ away from the fold line $L_{2}$, with the corresponding $y$-component satisfying $y>\phi\left(\nu_{2}(z, \delta), z\right)$; see again Figure 1.

We assume that $z_{0}^{\prime}(0)=0$ to ensure that the one-dimensional piecewise affine map (PAM) associated with (2.1), as introduced in Section 3, has a jump at the origin. As will become clear there, that assumption is made without loss of generality: the general case yields a piecewise affine map with a jump at non-zero $z_{0}^{\prime}(0)$, which can be studied in an analogous fashion after a translation.

Under the above assumptions, we can already give a partial slow-fast analysis of the system in (2.1); it will be convenient to do so now in order to explain our final assumption. Consider first the fold curve $L_{4}$ where we assume all orbits to jump. Given Assumption 4, the $z$-coordinate of a given orbit will determine whether it is attracted to $\mathcal{S}_{a_{1}}$ or to $\mathcal{S}_{a_{2}}$ under the layer flow. For $z=z_{0}$, the fate of orbits cannot be decided; consideration of the perturbation terms in (2.1) and a blow-up of $L_{2}$ would be necessary to describe the flow in that case. We expect that in general, canard phenomena are possible near $z=z_{0}$ where orbits will follow part of $\mathcal{S}_{r_{1}}$; here, we do not consider that scenario. However, as is clear from the above discussion, the $z$-value $z_{0}$ will nevertheless play a central role in our analysis. We will highlight two possible singular orbits passing through the transcritical point of intersection $P_{c}=P\left(L_{2}\right) \cap P\left(L_{4}\right)$ at $z=z_{0}$ : one orbit will continue along $\mathcal{S}_{a_{1}}$, while the other will follow $\mathcal{S}_{a_{2}}$. In both instances, we will assume that the sought-after singular orbit follows the reduced flow until a fold line is reached - $L_{1}$ in the former case and $L_{3}$ in the latter - at which point the orbit jumps to $\mathcal{S}_{a_{3}}$ and ultimately reaches $L_{4}$, following again the reduced flow. This "ambiguous" behaviour of the singular flow is a key point in our study and motivates the following Assumption 5.

Assumption 5 (Ambiguous singular orbit). There exists a singular closed orbit $\Gamma_{0}^{L}$ that is defined by concatenating the reduced flow on $\mathcal{S}_{a_{1}}$ and on $\mathcal{S}_{a_{3}}$ with the layer flow between $L_{1}$ and $\mathcal{S}_{a_{3}}$ and between $L_{4}$ and $\mathcal{S}_{a_{1}}$, respectively. Further, there exists a singular closed orbit $\Gamma_{0}^{S}$ that is defined by concatenating the reduced flow on $\mathcal{S}_{a_{2}}$ and on $\mathcal{S}_{a_{3}}$ with the layer flow between $L_{3}$ and $\mathcal{S}_{a_{3}}$ and between $L_{4}$ and $\mathcal{S}_{a_{2}}$, respectively. Both $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$ contain the point of intersection $P_{c}$ defined in Assumption 4; see Figure 1. Finally, we define $\Gamma_{0}=\Gamma_{0}^{L} \cup \Gamma_{0}^{S}$ as the "ambiguous" singular orbit.

Remark 1. Any orbit $\Gamma_{0}$ in the above family forms a natural boundary between oscillations of different amplitude.

While Assumption 5 may seem restrictive at first glance, the appearance of a single (large-amplitude) relaxation oscillation through $P_{c}$, corresponding to $\Gamma_{0}^{L}$, is a so-called codimension-1 problem: one parameter suffices to guarantee that the orbit will close after one cycle. Similarly, another parameter is required to ensure the existence of a relaxation orbit that is associated with $\Gamma_{0}^{S}$, resulting in a codimension-2
problem overall. In sum, we can conclude that Assumption 5 will generically be satisfied in two-parameter families of vector fields of the type in (2.1). However, as our analysis will necessitate the evaluation of certain integrals along the singular orbit $\Gamma_{0}=\Gamma_{0}^{L} \cup \Gamma_{0}^{S}$, we will make the following simplifying assumption: we will assume that both orbits $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$ lie in a plane $\left\{z=z_{0}\right\}$. It follows that $\left.\dot{z}\right|_{(z, \delta)=\left(z_{0}, 0\right)}=0$ and, hence, that we can write

$$
\begin{equation*}
g_{2}(x, y, z, \epsilon, \delta)=\delta G(x, y, z, \epsilon, \delta)+\left(z-z_{0}\right) H(x, y, z, \epsilon, \delta) \tag{2.6}
\end{equation*}
$$

in (2.1c). Our aim in this paper is to study the behaviour of orbits near $\Gamma_{0}$. To that end, we will formulate a first return map on the section $\Delta$ defined above which is transverse to the layer flow. We will present a partial study here: to be precise, we will restrict to characterising orbits that are sufficiently close, but not too close, to the point $P_{c}=P\left(L_{2}\right) \cap P\left(L_{4}\right)$; in other words, we will consider orbits in a sufficiently small neighbourhood of $P_{c}$ inside $\Delta$, uniformly away from $P\left(L_{2}\right)$. Our analysis will rely in part on [31], which will allow us to describe the persistence of both $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$.


Figure 1: Critical manifold $\mathcal{S}$ with "ambiguous" singular orbit $\Gamma_{0}=\Gamma_{0}^{L} \cup \Gamma_{0}^{S}$ and section $\Delta$.

## 3. Statement of results

Recall the definition of the section $\Delta$ which is located between the fold lines $L_{2}$ and $L_{4}$ and which is transverse to the layer flow of Equation (2.3). Also, recall that
the projection of the fold lines $L_{2}$ and $L_{4}$ onto $\Delta$ is denoted by $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$, respectively. Our first result concerns the well-definedness of the first return map from $\Delta$ to itself under the flow of Equation (2.1) and is relatively straightforward, since most of the relevant dynamics occurs along hyperbolically attracting parts of the critical manifold $\mathcal{S}$. We do, however, need to take additional care in a neighbourhood of $P\left(L_{2}\right)$ on $\Delta$, as the fate of orbits sufficiently close to the fold curve $L_{2}$ is difficult to predict: such orbits could either jump onto an attracting sheet or follow a repelling sheet of the critical manifold $\mathcal{S}$ after passing near $L_{2}$, which would give rise to canard behaviour. As stated above, we will avoid this unpredictability here; we will therefore formulate a result on the first return map that excludes a neighbourhood of $P\left(L_{2}\right)$.

First, we note that the form of the vector field in (2.1) allows us to conclude that $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$ are smooth $\delta$-families of graphs

$$
y=\phi_{L_{2}}(z, \delta) \quad \text { and } \quad y=\phi_{L_{4}}(z, \delta)
$$

which intersect in the point $P_{c}$ given by $\left(y_{0}, z_{0}, 0\right)+O(\delta)$. Next, for $r>0$, we define the open neighbourhood

$$
B_{r}\left(P\left(L_{2}\right)\right)=\left\{(y, z) \in \Delta:\left|y-\phi_{L_{2}}(z, \delta)\right|<r\right\}
$$

of $P\left(L_{2}\right)$. Then, we have the following result.
Theorem 3.1. There exists an open neighborhood $\mathcal{U}$ of the point $P_{c}$ in $\Delta$ such that, for each $\delta>0$ and $r>0$ sufficiently small, there exists $\epsilon>0$ small enough such that the first return map

$$
\Pi: \mathcal{U} \backslash B_{r}\left(P\left(L_{2}\right)\right) \subset \Delta \rightarrow \Delta:(y, z) \mapsto(\mathcal{Y}(y, z, \epsilon, \delta), \mathcal{Z}(y, z, \epsilon, \delta))
$$

is well-defined. Here, we write

$$
\mathcal{Y}(y, z, \epsilon, \delta)=\phi_{L_{4}}(\mathcal{Z}(y, z, 0, \delta), \delta)+\mathcal{E}(y, z, \epsilon, \delta)
$$

for some function $\mathcal{E}$ that is uniformly o(1) as $\epsilon \rightarrow 0$.
To be specific, Theorem 3.1 states that, given $\delta>0$ and $r>0$ small, there exists $\epsilon_{0}(\delta, r)>0$ such that the assertion of the theorem holds for all $\epsilon \in\left(0, \epsilon_{0}(\delta, r)\right]$.

Theorem 3.1 will be proved in Section 5.1. Let us now give some heuristics on how the return map $\Pi$ can be related to a suitably defined PAM, which we require in order to formulate our next result. We will give full proofs in Sections 4 and 5 below.

In the singular limit of $\epsilon=0$, the map $\Pi$ is given by

$$
\left.\Pi\right|_{\epsilon=0}:(y, z) \mapsto\left(\phi_{L_{4}}(\mathcal{Z}(y, z, 0, \delta), \delta), \mathcal{Z}(y, z, 0, \delta)\right)
$$

since the image of that map lies on $P\left(L_{4}\right)$, it makes sense to also restrict its domain to the graph $y=\phi_{L_{4}}(z, \delta)$, reducing it in essence to a one-dimensional map

$$
\Pi_{0}: z \mapsto \mathcal{Z}_{1}(z, \delta):=\mathcal{Z}\left(\phi_{L_{4}}(z, \delta), z, 0, \delta\right)
$$

The map $\Pi_{0}$ is only defined for values of $z$ that are at least an $O(r)$-distance away from $z_{0}(\delta)$; however, since we can apply Theorem 3.1 for any choice of $r$, the $\epsilon=0$ limit of $\Pi$ is actually defined for all $z \neq z_{0}(\delta)$. Let us now consider a neighbourhood
of $z=z_{0}$ by writing $z=z_{0}+\delta Z$. Recalling that $z_{0} \mapsto z_{0}$ for $\delta=0$, which marks the transverse intersection point $P_{c}$ of $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$ in Assumption 4, we arrive at

$$
\tilde{\Pi}_{0}: Z \mapsto \tilde{\mathcal{Z}}_{1}(Z, \delta):=\frac{\mathcal{Z}_{1}\left(z_{0}+\delta Z, \delta\right)-\mathcal{Z}_{1}\left(z_{0}, 0\right)}{\delta} \quad \text { for } z_{0}+\delta Z \neq z_{0}(\delta)
$$

as $z_{0}=\mathcal{Z}_{1}\left(z_{0}, 0\right)$. Using Assumption 4 once more, we can assume the map $\tilde{\Pi}_{0}$ to have a well-defined limit

$$
\tilde{\Pi}_{00}: Z \mapsto \lim _{\delta \rightarrow 0} \tilde{\mathcal{Z}}_{1}(Z, \delta) \quad \text { for } Z \neq 0
$$

Since $O\left(Z^{2}\right)$-terms are scaled away in the above limit, one expects the map $\tilde{\Pi}_{00}$ to be piecewise affine. While the above argument is heuristic, it can be made rigorous by relation to the vector field in (2.1) and on the basis of the two limiting systems that are obtained therefrom for $\epsilon=0$. Below, we express this correspondence in terms of a definition; a rigorous proof of our heuristics can be found in Proposition 1.

Definition 3.1. Let $\gamma$ be a curve on a normally hyperbolic segment of the critical manifold $\mathcal{S}$ for Equation (2.1) that is parametrised by $x$. Then, we define the affine map $M_{\gamma}$ associated with $\gamma$ as

$$
M_{\gamma}(Z)=\left(\mathrm{e}^{\int_{\gamma} p(x) d x}\right) Z+\left(\int_{\gamma} q(x) \mathrm{e}^{\int_{\gamma_{x}} p\left(x^{\prime}\right) d x^{\prime}} d x\right)
$$

with

$$
p(x):=\frac{H(x, y, z, 0,0)}{g_{1}(x, y, z, 0,0)} F_{x}(x, z) \quad \text { and } \quad q(x):=\frac{G(x, y, z, 0,0)}{g_{1}(x, y, z, 0,0)} F_{x}(x, z)
$$

where we substitute $y$ and $z$ with the $y$-coordinate and the $z$-coordinate of $\gamma$, respectively. (In particular, we note that $y=F(x, z):=F(x, z, 0,0)$.)

We remark that the coefficient of $Z$ in the definition of $M_{\gamma}$ is strictly positive, as well as that the last integral is defined along a curve $\gamma_{x}$ that is parametrised from $x$ until the end of the curve $\gamma$; see (4.6) below.

Next, we apply Definition 3.1 in the context of Equation (2.1), i.e., to the slow portions of the singular orbits $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$, which are given by $\Gamma_{0}^{L} \cap \mathcal{S}_{a_{1}}, \Gamma_{0}^{L} \cap \mathcal{S}_{a_{3}}$, $\Gamma_{0}^{S} \cap \mathcal{S}_{a_{2}}$, and $\Gamma_{0}^{S} \cap \mathcal{S}_{a_{3}}$.
Definition 3.2. We define the piecewise affine map (PAM)

$$
M(Z)= \begin{cases}M_{\Gamma_{0}^{L} \cap \mathcal{S}_{a_{3}}} \circ M_{\Gamma_{0}^{L} \cap \mathcal{S}_{a_{1}}}(Z)=: M_{1}(Z) & \text { for } Z<0  \tag{3.1}\\ M_{\Gamma_{0}^{S} \cap \mathcal{S}_{a_{3}}} \circ M_{\Gamma_{0}^{S} \cap \mathcal{S}_{a_{2}}}(Z)=: M_{2}(Z) & \text { for } Z>0,\end{cases}
$$

and we say that $M$ is associated with the vector field in (2.1).
We refer to Section 4 for specific expressions for $M$ in the context of (2.1).
Remark 2. We note that the Z-coefficient in the PAM M is strictly positive, as was the case in Definition 3.1. However, the image of the map could contain $Z=0$; in fact, the situation where the images $\left.M\right|_{Z<0}$ and $\left.M\right|_{Z>0}$ partly overlap represents the most interesting scenario here. We will see that the heuristics following the statement of Theorem 3.1 can be proved rigorously, and we will show that the limiting map $\tilde{\Pi}_{00}$ is precisely the associated PAM M. The fact that the singular limit of the first return map $\Pi$ is not one-to-one causes parts of the remaining analysis to differ from [31].

The following is our second main result:
Theorem 3.2. Given a slow-fast system of the form in (2.1) that satisfies Assumptions 1 through 5, assume that its associated PAM M, as defined in (3.1), exhibits a stable periodic MMO with signature $L_{1}{ }^{s_{1}} L_{2}{ }^{s_{2}} \ldots L_{k}{ }^{s_{k}}$, for some $k \in \mathbb{N}$. Further, assume that this MMO avoids the discontinuity at $Z=0$. Then, (2.1) exhibits a stable MMO with the same signature $L_{1}{ }^{s_{1}} L_{2}{ }^{s_{2}} \ldots L_{k}{ }^{s_{k}}$, for $\epsilon, \delta>0$ sufficiently small.

The requirement that $\epsilon$ and $\delta$ be (sufficiently) small in Theorem 3.2 signifies that for every $\delta>0$ small, there exists $\epsilon_{0}(\delta)>0$ small such that the result is true for all $\epsilon \in\left(0, \epsilon_{0}(\delta)\right]$; recall the statement of Theorem 3.1. We prove Theorem 3.2 in Section 5.3.

We emphasise that Theorem 3.2 concerns MMOs which are stable both for the PAM $M$ and the associated slow-fast system in (2.1). Given the definition of $M$ in (3.1), we hence require that $M_{1}^{\prime}(Z)^{L} M_{2}^{\prime}(Z)^{s}<1$, with $L=\sum_{i=1}^{k} L_{i}$ and $s=\sum_{i=1}^{k} s_{i}$, as per the above notation. As will become evident in Section 5.3, that requirement will equally guarantee stability of the corresponding MMO for (2.1) given the form of the induced Poincaré map $\Pi$. We further emphasise that a mixed-mode trajectory will clearly be stable if the two components thereof corresponding to the branches $M_{1}$ and $M_{2}$ of $M$ are stable, i.e., if $0<M_{1}^{\prime}(Z), M_{2}^{\prime}(Z)<1$. Since that condition is for instance assumed in [29], as discussed in Section 6.2 below, we also impose it henceforth.

Remark 3. We note that even if one of the two components corresponding to $M_{1}$ or $M_{2}$ is unstable, with either $M_{1}^{\prime}(Z)>1$ or $M_{2}^{\prime}(Z)>1$, a given mixed-mode trajectory may be stable overall provided that the other component is stable and that $M_{1}^{\prime}(Z)^{L} M_{2}^{\prime}(Z)^{s}<1$ holds. Only when $M_{1}^{\prime}(Z), M_{2}^{\prime}(Z)>1$ will the trajectory necessarily be unstable. The systematic consideration of these alternative scenarios - which may even include neutral stability - is left for future research.

Our theoretical results are complemented by an "inverse" theorem: any PAM is associated with a suitably chosen slow-fast system of the form in (2.1). In the proof of the following result, we will make specific choices for the functions $F, g_{1}$, and $g_{2}$ therein, which will allow us to obtain convenient expressions for the corresponding vector field.

Theorem 3.3. Assume we are given any piecewise affine map (PAM) of the form

$$
M(Z)= \begin{cases}a_{11} Z+a_{12} & \text { for } Z<0 \\ a_{21} Z+a_{22} & \text { for } Z>0\end{cases}
$$

with $a_{11}, a_{21}>0$. Then, there exists a slow-fast system of the form in Equation (2.1) which satisfies Assumptions 1 through 5 such that the PAM associated with the vector field is given by $M$.

Again, Theorem 3.3 will be proved in Section 5.4.

## 4. Computation of associated PAM

In this section, we obtain expressions for the PAM $M$ defined in (3.1) that is associated with the vector field in (2.1). Moreover, we establish formally that $M$
equals the limit $\tilde{\Pi}_{00}$ of the first return map $\Pi$; see Proposition 1. Recall the definition of the system in (2.1), which satisfies Assumptions 1 through 5. In particular, Assumption 1 implies that the function $F(x, z, 0, \delta)$ has four distinct local extrema with respect to the variable $x$, which we denote by

$$
x_{1}(z, \delta)<x_{2}(z, \delta)<x_{3}(z, \delta)<x_{4}(z, \delta) ;
$$

see Figure 2 for an illustration.


Figure 2: The critical manifold $\mathcal{S}=\left\{(x, y, z) \in \mathbb{R}^{3}: y=F(x, z, 0, \delta)\right\}$ for $z=z_{0}$ and $\delta=0$. The function $F\left(x, z_{0}\right):=F\left(x, z_{0}, 0,0\right)$ assumes the same value at $\hat{x}_{4}, x_{2}$, and $x_{4}$; see Assumption 4. Here, $\hat{x}_{3}$ and $\hat{x}_{1}$ are the $x$-coordinates of the points whose $y$-coordinates are $F\left(x_{3}, z_{0}\right)$ and $F\left(x_{1}, z_{0}\right)$, respectively, which both lie to the right of $x_{4}$.

Next, we recall that in order to simplify our analysis, we assume that $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$ lie in the plane $\left\{z=z_{0}\right\}$; see (2.6) and the text above that equation. Hence, we can make the general definition of the PAM $M$ somewhat more explicit in the present context. Specifically, the assumption in (2.6) allows us to introduce the following rescaling: since we are interested in $z \approx z_{0}$ and $\delta$ small in (2.1), we define the transformation

$$
\begin{equation*}
z=z_{0}+\delta Z \tag{4.1}
\end{equation*}
$$

Substituting (4.1) and (2.6) into (2.1), we have

$$
\begin{aligned}
x^{\prime} & =y-F\left(x, z_{0}+\delta Z, \epsilon, \delta\right) \\
y^{\prime} & =\epsilon g_{1}\left(x, y, z_{0}+\delta Z, \epsilon, \delta\right) \\
Z^{\prime} & =\epsilon\left[G\left(x, y, z_{0}+\delta Z, \epsilon, \delta\right)+Z H\left(x, y, z_{0}+\delta Z, \epsilon, \delta\right)\right]
\end{aligned}
$$

which implies

$$
\begin{aligned}
x^{\prime} & =y-F\left(x, z_{0}\right)+\mathcal{O}(\epsilon, \delta) \\
y^{\prime} & =\epsilon\left[g_{1}\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\epsilon, \delta)\right], \\
Z^{\prime} & =\epsilon\left[G\left(x, y, z_{0}, 0,0\right)+Z H\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\epsilon, \delta)\right]
\end{aligned}
$$

after Taylor expansion of $F, g_{1}, G$, and $H$. (Here, we again write $F\left(x, z_{0}\right)=$ $F\left(x, z_{0}, 0,0\right)$.) Reverting to the "slow time" $\tau$ in the above, we find

$$
\begin{align*}
\epsilon \dot{x} & =y-F\left(x, z_{0}\right)+\mathcal{O}(\epsilon, \delta)  \tag{4.2a}\\
\dot{y} & =g_{1}\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\epsilon, \delta)  \tag{4.2b}\\
\dot{Z} & =G\left(x, y, z_{0}, 0,0\right)+Z H\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\epsilon, \delta) \tag{4.2c}
\end{align*}
$$

In the singular limit of $\epsilon=0$, we obtain

$$
\begin{align*}
y & =F\left(x, z_{0}\right)+\mathcal{O}(\delta)  \tag{4.3a}\\
\dot{y} & =g_{1}\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\delta)  \tag{4.3b}\\
\dot{Z} & =G\left(x, y, z_{0}, 0,0\right)+Z H\left(x, y, z_{0}, 0,0\right)+\mathcal{O}(\delta) \tag{4.3c}
\end{align*}
$$

Differentiating (4.3a) with respect to $\tau$, we find

$$
\dot{y}=F_{x}\left(x, z_{0}\right) \dot{x}+\mathcal{O}(\delta)
$$

which, together with the $(y, Z)$-subsystem of (4.3), yields the projection of the reduced flow (in $\epsilon$ ) onto the ( $x, Z$ )-plane:

$$
\begin{align*}
& \dot{x}=\frac{g_{1}\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)}{F_{x}\left(x, z_{0}\right)}+\mathcal{O}(\delta),  \tag{4.4a}\\
& \dot{Z}=G\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)+Z H\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)+\mathcal{O}(\delta) . \tag{4.4b}
\end{align*}
$$

The important observation now is that Equation (4.4) is partially decoupled for $\delta=0$. As a consequence, Assumption 5 on the existence of a singular orbit $\Gamma_{0}$ actually implies that $g_{1}\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)$ is non-zero. In other words, we may parametrise the reduced flow by the variable $x$. Hence, introducing $x$ as the independent variable in (4.4) and noting that $F_{x}\left(x, z_{0}\right) \neq 0$ away from $L_{i}$, we obtain an ordinary differential equation

$$
\begin{equation*}
\frac{d Z}{d x}=p(x) Z+q(x)+O(\delta) \tag{4.5}
\end{equation*}
$$

with
$p(x):=\frac{H\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)}{g_{1}\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)} F_{x}\left(x, z_{0}\right)$ and $q(x):=\frac{G\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)}{g_{1}\left(x, F\left(x, z_{0}\right), z_{0}, 0,0\right)} F_{x}\left(x, z_{0}\right)$, which is linear with respect to $Z$ when $\delta=0$.

In that limit, (4.5) can hence be solved exactly, with initial condition $Z\left(x_{\text {init }}\right)=$ $Z_{\text {init }}$ :

$$
\begin{equation*}
Z\left(x, x_{\text {init }}, Z_{\text {init }}\right)=\left(\mathrm{e}^{\int_{x_{\text {init }}^{x}}^{x} p(s) d s}\right) Z_{\text {init }}+\left(\int_{x_{\text {init }}}^{x} q(u) \mathrm{e}^{\int_{u}^{x} p(s) d s} d u\right) \tag{4.6}
\end{equation*}
$$

Note that (4.6) is precisely the affine map defined in Definition 3.1 that is associated with the slow portion of $\Gamma_{0}^{L}$ or $\Gamma_{0}^{S}$ between $x_{\text {init }}$ and $x$. Now, we make use of (4.6) to define a map that encodes the mixed-mode dynamics of our canonical system, Equation (2.1). The discussion underneath Assumptions 3 and 4 implies that the sought-after map will have two branches which describe oscillations with different amplitudes as we pass through $Z_{\text {init }}=0$. Specifically, for $Z_{\text {init }}<0$, we observe
large-amplitude oscillations (LAOs), while for $Z_{\text {init }}>0$, we have small-amplitude oscillations (SAOs); we hence proceed to define the following one-dimensional PAM associated with (2.1),

$$
M\left(Z_{\text {init }}\right)= \begin{cases}Z\left(x_{4}, \hat{x}_{1}, Z\left(x_{1}, \hat{x}_{4}, Z_{\text {init }}\right)\right)=M_{1}\left(Z_{\text {init }}\right) & \text { if } Z_{\text {init }}<0,  \tag{4.7}\\ Z\left(x_{4}, \hat{x}_{3}, Z\left(x_{3}, x_{2}, Z_{\text {init }}\right)\right)=M_{2}\left(Z_{\text {init }}\right) & \text { if } Z_{\text {init }}>0,\end{cases}
$$

where $\hat{x}_{4}, \hat{x}_{3}$ and $\hat{x}_{1}$ are defined as in Figure 2. Given (4.6), we find the expressions for the affine maps defined in (4.7) or, equivalently, in (3.1):

$$
\begin{align*}
& Z\left(x_{4}, \hat{x}_{1}, Z\left(x_{1}, \hat{x}_{4}, Z_{\text {init }}\right)\right)=\left(\mathrm{e}^{\left(\int_{\hat{x}_{4}}^{x_{1}}+\int_{\hat{x}_{1}}^{x_{4}}\right) p(s) d s}\right) Z_{\text {init }} \\
& \quad+\left(\int_{\hat{x}_{4}}^{x_{1}} q(u) \mathrm{e}^{\left(\int_{u}^{x_{1}}+\int_{\hat{x}_{1}}^{x_{4}}\right) p(s) d s} d u+\int_{\hat{x}_{1}}^{x_{4}} q(u) \mathrm{e}^{x_{u}^{x_{4}} p(s) d s} d u\right) \tag{4.8}
\end{align*}
$$

and

$$
\begin{align*}
& Z\left(x_{4}, \hat{x}_{3}, Z\left(x_{3}, x_{2}, Z_{\text {init }}\right)\right)=\left(\mathrm{e}^{\left(\int_{x_{2}}^{x_{3}}+\int_{\hat{x}_{3}}^{x_{4}}\right) p(s) d s}\right) Z_{\text {init }} \\
& +\left(\int_{x_{2}}^{x_{3}} q(u) \mathrm{e}^{\left(\int_{u}^{x_{3}}+\int_{\hat{x}_{3}}^{x_{4}}\right) p(s) d s} d u+\int_{\hat{x}_{3}}^{x_{4}} q(u) \mathrm{e}^{\int_{u}^{x_{4}} p(s) d s} d u\right) . \tag{4.9}
\end{align*}
$$

Proposition 1. Let the map $\tilde{\Pi}_{0}$ be defined as in Section 3. For each $Z \neq 0, \tilde{\Pi}_{0}$ is well-defined for $\delta>0$ sufficiently small; moreover, the limit $\tilde{\Pi}_{00}(Z)=\lim _{\delta \rightarrow 0} \tilde{\Pi}_{0}(Z)$ exists and is equal to the piecewise affine map given in (4.7), i.e., $\tilde{\Pi}_{00}\left(Z_{\text {init }}\right)=$ $M\left(Z_{\text {init }}\right)$, with $Z_{\text {init }} \neq 0$.

Proposition 1 will be proved in Section 5.2. In the proof, we will use an important observation made in Section 5.1: the $\mathcal{Z}$-component of the return map $\Pi$ defined in Theorem 3.1 is a small $\epsilon$-perturbation of the return map induced by the reduced flow of (2.1) near $\Gamma_{0}^{L}$ and $\Gamma_{0}^{S}$, respectively, provided we are below and above $B_{r}\left(P\left(L_{2}\right)\right)$, respectively. (As in Assumption 4 introduced in Section 2, we hence have either $y<\phi_{L_{2}}(z, \delta)-r$ or $y>\phi_{L_{2}}(z, \delta)+r$ in $\Delta$.)

## 5. Proof of main results

In this section, we present rigorous proofs for the main results of the paper, as introduced in Section 3.

### 5.1. Proof of Theorem 3.1

We first prove Theorem 3.1. To that end, we consider Equation (2.1) under Assumptions 1 through 5 to show that there exists an open neighborhood $\mathcal{U}$ of the intersection point $P_{c}$ of $P\left(L_{2}\right)$ and $P\left(L_{4}\right)$ such that, for all $\delta>0$ and $r>0$ small and fixed, the Poincaré map $\Pi: \mathcal{U} \backslash B_{r}\left(P\left(L_{2}\right)\right) \subset \Delta \rightarrow \Delta$ induced by (2.1) is well-defined for $\epsilon$ sufficiently small.

The proof is based on the techniques developed in [31], as indicated in Figure 3: "fast" orbits of (2.1) passing through $\tilde{\Delta}_{\text {out }}^{3}(=\Delta)$ below the tubular neighbourhood $B_{r}\left(P\left(L_{2}\right)\right)$ are attracted to $\mathcal{S}_{a_{1}}$, and therefore give rise to an LAO in the resulting mixed-mode time series; similarly, orbits passing through $\tilde{\Delta}_{\text {out }}^{3}$ above $B_{r}\left(P\left(L_{2}\right)\right)$ are attracted to $\mathcal{S}_{a_{2}}$, resulting in an SAO. Considered separately, each of these two


Figure 3: The return map $\Pi$ induced by (2.1) on $\tilde{\Delta}_{\text {out }}^{3}$ is a composition of transition maps. The projection $P\left(L_{2}\right)$ divides $\tilde{\Delta}_{\text {out }}^{3}$ into two portions; in one portion, the fast flow is attracted to $\mathcal{S}_{a_{1}}$, whereas in the other, it tends to $\mathcal{S}_{a_{2}}$.
scenarios can clearly be reduced to the return map studied in [31], for fixed $\delta>0$ and $r>0$. (Recall that we stay uniformly away from the fold line $L_{2}$.) We focus on the first, LAO-generating scenario here; the second scenario can be studied in an analogous fashion.

Fundamentally, we need to show that, for $\delta>0$ and $r>0$ sufficiently small, the flow of (2.1) stays close to the singular closed orbit $\Gamma_{0}^{L}$ such that the return map $\Pi$ exists for $\epsilon$ small. Following [31], the map $\Pi$ is essentially composed of three different types of transition map: $\pi_{T}, \pi_{S_{a_{1}}}$, and $\tilde{\pi}_{L_{1}}$, as illustrated in Figure 3. Here, the map $\pi_{T}$ is defined by following the fast flow towards the attracting portion $\mathcal{S}_{a_{1}}$ of $\mathcal{S}$, while $\pi_{S_{a_{1}}}$ describes the passage near $\mathcal{S}_{a_{1}}$ away from the fold line $L_{1}$; the study of $\pi_{T}$ and $\pi_{S_{a_{1}}}$ is based on Fenichel's standard GSPT. The map $\tilde{\pi}_{L_{1}}$, which describes the passage near the fold line $L_{1}$, is studied via geometric desingularisation, or "blowup". Let us now consider the "half-return" map $\Pi_{H_{\alpha}}=\tilde{\pi}_{L_{1}} \circ \pi_{S_{a_{1}}} \circ \pi_{T}$ from the portion of $\mathcal{U}$ below $B_{r}\left(P\left(L_{2}\right)\right)$, which is defined by $y<\phi_{L_{2}}(z, \delta)-r$, as before, to a section $\tilde{\Delta}_{\text {out }}^{1}$ transverse to $\Gamma_{0}^{L}$. Following Theorem 2 in [31], the half-return map $\Pi_{H_{\alpha}}$ is given by

$$
\Pi_{H_{\alpha}}(y, z)=\left(\mathcal{Y}_{\alpha}(y, z, \epsilon, \delta), \mathcal{Z}_{\alpha}(y, z, \epsilon, \delta)\right)
$$

with $\mathcal{Z}_{\alpha}(y, z, \epsilon, \delta)=\mathcal{Z}_{\alpha}(y, z, \delta)+O(\epsilon \ln \epsilon)$, where $\mathcal{Z}_{\alpha}(y, z, \delta)$ is defined by following the orbit of the reduced flow on the attracting portion $\mathcal{S}_{a_{1}}$ between the $\omega$-limit of the point $(y, z)$ and $L_{1}$. Moreover, we have $\mathcal{Y}_{\alpha}(y, z, \epsilon, \delta)=\phi_{\alpha}\left(\mathcal{Z}_{\alpha}(y, z, \delta), \delta\right)+o(1)$, where $y=\phi_{\alpha}(z, \delta)$ describes the projection of the fold $L_{1}$ onto $\tilde{\Delta}_{\text {out }}^{1}$ and where the $o(1)$-term tends uniformly to zero as $\epsilon \rightarrow 0$.

The half-return map $\Pi_{H_{\beta}}=\left(\mathcal{Y}_{\beta}, \mathcal{Z}_{\beta}\right)$ from $\tilde{\Delta}_{\text {out }}^{1}$ back to $\Delta$ can be studied in a similar fashion, as a composition of transition maps that are of the same type as in $\Pi_{H_{\alpha}}$. Combining the two, the return map $\Pi=\Pi_{H_{\beta}} \circ \Pi_{H_{\alpha}}$, which is defined for
$y<\phi_{L_{2}}(z, \delta)-r$, i.e., below $B_{r}\left(P\left(L_{2}\right)\right)$, can be written as

$$
\Pi(y, z)=(\mathcal{Y}(y, z, \epsilon, \delta), \mathcal{Z}(y, z, \epsilon, \delta))
$$

with $\mathcal{Z}(y, z, \epsilon, \delta)=\mathcal{Z}(y, z, \delta)+O(\epsilon \ln \epsilon)$, where

$$
\begin{equation*}
\mathcal{Z}(y, z, \delta)=\mathcal{Z}_{\beta}\left(\phi_{\alpha}\left(\mathcal{Z}_{\alpha}(y, z, \delta), \delta\right), \mathcal{Z}_{\alpha}(y, z, \delta), \delta\right) \tag{5.1}
\end{equation*}
$$

is the return map defined by the reduced flow on $\mathcal{S}_{a_{1}}$ and $\mathcal{S}_{a_{3}}$. We can also conclude that the function $\mathcal{Y}$ has the property given in Theorem 3.1, which completes the proof.

### 5.2. Proof of Proposition 1

Next, we prove Proposition 1. Recall that the map

$$
\tilde{\Pi}_{0}\left(Z_{\text {init }}\right)=\frac{\mathcal{Z}_{1}\left(z_{0}+\delta Z_{\text {init }}, \delta\right)-z_{0}}{\delta}
$$

with $\mathcal{Z}_{1}(z, \delta)=\mathcal{Z}\left(\phi_{L_{4}}(z, \delta), z, 0, \delta\right)$, is defined for $z \neq z_{0}(\delta)$; cf. Section 3. If $Z_{\text {init }} \neq 0$ is fixed, then $z=z_{0}+\delta Z_{\text {init }} \neq z_{0}(\delta)$ for $\delta>0$ sufficiently small due to $z_{0}^{\prime}(0)=0$; see Assumption 4. Thus, $\tilde{\Pi}_{0}$ is well-defined for $Z_{\text {init }} \neq 0$ provided that $\delta>0$ is small.

First, let us consider $Z_{\text {init }}<0$ and fixed. Then, we have that $z=z_{0}+\delta Z_{\text {init }}<$ $z_{0}(\delta)$ for $\delta>0$ small, i.e., the point $\left(\phi_{L_{4}}(z, \delta), z\right)$ is attracted to $\mathcal{S}_{a_{1}}$; see again Assumption 4. We therefore observe LAOs and $\mathcal{Z}_{1}(z, \delta)=\mathcal{Z}\left(\phi_{L_{4}}(z, \delta), z, \delta\right)$, where the function $\mathcal{Z}(y, z, \delta)$ is defined in (5.1). Now, we note that the system in (4.4) is obtained by applying the coordinate transformation in (4.1) to the reduced flow in (2.4) in $(x, z)$-space, where $g_{2}$ is given in (2.6). It follows that the orbit of (4.4) which is initiated at $\left(\hat{x}_{4}\left(z_{0}+\delta Z_{\text {init }}, \delta\right), Z_{\text {init }}\right)$, with $F\left(\hat{x}_{4}(z, \delta), z, 0, \delta\right)=\phi_{L_{4}}(z, \delta)$, by Figure 2, intersects the projection of the fold line $L_{1}$ onto the $(x, Z)$-space in $\left(x_{1}\left(z_{0}+\delta Z_{\alpha}, \delta\right), Z_{\alpha}\right)$, where

$$
Z_{\alpha}=\frac{\mathcal{Z}_{\alpha}\left(\phi_{L_{4}}\left(z_{0}+\delta Z_{\text {init }}, \delta\right), z_{0}+\delta Z_{\text {init }}, \delta\right)-z_{0}}{\delta}
$$

(Here, $\mathcal{Z}_{\alpha}$ is defined as in Section 5.1.) Thus, $Z_{\alpha}$ converges to $Z\left(x_{1}, \hat{x}_{4}, Z_{\text {init }}\right)$ as $\delta \rightarrow$ 0 , with $Z\left(x, x_{\text {init }}, Z_{\text {init }}\right)$ given in (4.6), where we denote by $\hat{x}_{4}$ and $x_{1}$, respectively, the limit of $\hat{x}_{4}(z, \delta)$ and $x_{1}(z, \delta)$, respectively, as $(z, \delta) \rightarrow\left(z_{0}, 0\right)$. Here, we have used the fact that $\Gamma_{0}^{L}$ is located in the plane $\left\{z=z_{0}\right\}$ with $\delta=0$ and, thus, that $\delta Z_{\alpha} \rightarrow 0$ as $\delta \rightarrow 0$ in $x_{1}$. Moreover, we have exploited our observation in Section 4 that (4.4) is a $\delta$-perturbation of a linear (in $Z$ ) differential equation.

Similarly, the orbit of (4.4) that is initiated at $\left(\hat{x}_{1}\left(z_{0}+\delta Z_{\alpha}, \delta\right), Z_{\alpha}\right)$, with $F\left(\hat{x}_{1}(z, \delta)\right.$, $z, 0, \delta)=\phi_{\alpha}(z, \delta)=F\left(x_{1}(z, \delta), z, 0, \delta\right)$, again by Figure 2, intersects the $(x, Z)$ projection of the fold line $L_{4}$ in $\left(x_{4}\left(z_{0}+\delta Z_{\beta}, \delta\right), Z_{\beta}\right)$, with

$$
Z_{\beta}=\frac{\mathcal{Z}_{\beta}\left(\phi_{\alpha}\left(z_{0}+\delta Z_{\alpha}, \delta\right), z_{0}+\delta Z_{\alpha}, \delta\right)-z_{0}}{\delta}
$$

We therefore conclude that $Z_{\beta}$ converges to (4.8) as $\delta \rightarrow 0$. (As above, we use that $\delta Z_{\beta} \rightarrow 0$ as $\delta \rightarrow 0$ in $x_{4}$.) Now, it suffices to note that $\tilde{\Pi}_{0}\left(Z_{\text {init }}\right)=Z_{\beta}$, from (5.1).

The case where $Z_{\text {init }}>0$ can be studied in a similar fashion to show that $\tilde{\Pi}_{0}\left(Z_{\text {init }}\right)$ tends to (4.9) as $\delta \rightarrow 0$, as claimed, which completes the proof.

### 5.3. Proof of Theorem 3.2

For the sake of simplicity and readability, we first prove Theorem 3.2 for a MMO with signature $1^{0}$; then, we will indicate how the proof can be extended to the general case, i.e., to MMOs with signature $L_{1}^{s_{1}} L_{2}^{s_{2}} \cdots L_{k}^{s_{k}}$, for $k \geq 1$ integer.

Thus, we suppose that the PAM in (3.1) which is associated with the vector field in (2.1) has a stable periodic orbit that undergoes one LAO, i.e., that $M\left(Z^{*}\right)=Z^{*}$ for $Z^{*}<0$ as well as that $a_{11}^{1} a_{21}^{0}<1$, where $a_{11}$ and $a_{21}$ denote the coefficients of $Z$ in the definition of $M$; cf. also the statement of Theorem 3.3. Our aim is to prove that (2.1) admits a stable periodic orbit with one LAO for $\epsilon>0$ and $\delta>0$ small. Clearly, periodic orbits for (2.1) correspond to fixed points of the first return map $\Pi$ defined in Theorem 3.1. It can easily be seen that $(y, z)$ is a solution of $\Pi(y, z)-(y, z)=(0,0)$ if and only if $(y, Z)$, with $z=z_{0}+\delta Z$, is a solution of

$$
\begin{array}{r}
\phi_{L_{4}}\left(z_{0}+\delta \tilde{Z}(y, Z, 0, \delta), \delta\right)+\mathcal{E}\left(y, z_{0}+\delta Z, \epsilon, \delta\right)-y=0, \\
\tilde{Z}(y, Z, \epsilon, \delta)-Z=0, \tag{5.2}
\end{array}
$$

where

$$
\tilde{Z}(y, Z, \epsilon, \delta):=\frac{\mathcal{Z}\left(y, z_{0}+\delta Z, \epsilon, \delta\right)-z_{0}}{\delta}
$$

and $\phi_{L_{4}}, \mathcal{Z}$, and $\mathcal{E}$ are defined as in Theorem 3.1. The Implicit Function Theorem then implies that the system in (5.2) has a unique solution $\left(y_{\epsilon, \delta}^{*}, Z_{\epsilon, \delta}^{*}\right)$ for $\epsilon>0$ and $\delta>0$ sufficiently small, with $\left(y_{\epsilon, \delta}^{*}, Z_{\epsilon, \delta}^{*}\right)$ close to $\left(\phi_{L_{4}}\left(z_{0}, 0\right), Z^{*}\right)$. (An alternative approach is outlined in Remark 4.) Note that $\tilde{Z}\left(\phi_{L_{4}}\left(z_{0}+\delta Z, \delta\right), Z, 0, \delta\right)=\tilde{\Pi}_{0}(Z)$, where $\tilde{\Pi}_{0}(Z)$ tends to $M(Z)$ as $\delta \rightarrow 0$, by Proposition 1 . More generally, we have $\tilde{Z}(y, Z, 0, \delta) \rightarrow M(y, Z)$ as $\delta \rightarrow 0$, where $M(y, Z)$ is a PAM as in (4.7) or, equivalently, in (3.1), with $\hat{x}_{4}$ and $x_{2}$ depending on $y$. (This follows easily from the proof of Proposition 1.) Now, for $\epsilon \rightarrow 0$ followed by $\delta \rightarrow 0$, the system in (5.2) reduces to

$$
\begin{align*}
\phi_{L_{4}}\left(z_{0}, 0\right)-y & =0,  \tag{5.3}\\
M(y, Z)-Z & =0 .
\end{align*}
$$

Since $Z=Z^{*}$ is a fixed point of $M(Z)$ - or, equivalently, of $M\left(\phi_{L_{4}}\left(z_{0}, 0\right), Z\right)$ - it follows that $(y, Z)=\left(\phi_{L_{4}}\left(z_{0}, 0\right), Z^{*}\right)$ is a solution of (5.3). The Jacobian determinant of the left-hand side in (5.3) evaluated at this solution is $1-a_{11} \neq 0$, where we note that $M^{\prime}\left(Z^{*}\right)=a_{11}<1$ due to $Z^{*}<0$. The Implicit Function Theorem now implies the existence of a solution $\left(y_{\epsilon, \delta}^{*}, Z_{\epsilon, \delta}^{*}\right)$ of (5.2) for $\epsilon>0$ and $\delta>0$ small. Thus, $(y, z)=\left(y_{\epsilon, \delta}^{*}, z_{0}+\delta Z_{\epsilon, \delta}^{*}\right)$ is a fixed point of $\Pi$. As that point depends smoothly on $\epsilon$ and $\delta$, again by the Implicit Function Theorem, it is clearly stable. Hence, it follows that the corresponding periodic orbit for (2.1) is also stable, which completes the proof.

In the general case, where the given MMO has signature $L_{1}^{s_{1}} L_{2}^{s_{2}} \cdots L_{k}^{s_{k}}$, we have to study fixed points of the $\kappa$-th iterate of the first return map $\Pi$, where $\kappa:=\sum_{i=1}^{k}\left(L_{i}+s_{i}\right)$. In the limit of $\epsilon=0=\delta$, the $\kappa$-th iterate of $\Pi$ can be written as $(y, Z) \rightarrow\left(\phi_{L_{4}}\left(z_{0}, 0\right), M^{\kappa-1}(M(y, Z))\right)$ in $(y, Z)$-coordinates. The Jacobian determinant of the corresponding system $\left\{\phi_{L_{4}}\left(z_{0}, 0\right)-y=0, M^{\kappa-1}(M(y, Z))-Z=0\right\}$ is then equal to $1-a_{11}^{L} a_{21}^{s}$, with $L=\sum_{i=1}^{k} L_{i}$ and $s=\sum_{i=1}^{k} s_{i}$. Since we supposed that $M^{\kappa}\left(Z^{*}\right)=Z^{*}$ for some $Z^{*}<0$ with $a_{11}^{L} a_{21}^{s}<1$ (stability), the result easily follows.

Remark 4. Alternatively, Theorem 3.2 can be proved via the approach taken in [31]. For $\delta>0$ small, the first return map $\Pi$ from Theorem 3.1 contracts its domain to the curve $y=\phi_{L_{4}}(z, \delta)$, in the limit as $\epsilon \rightarrow 0$. Following Theorem 3 in [31], П admits a one-dimensional attracting invariant manifold $y=m_{\epsilon, \delta}(z)$; the dynamics of $\Pi$ on that manifold is given by $\Pi_{0}$ in the limit of $\epsilon \rightarrow 0$, with $\Pi_{0}$ as defined underneath Theorem 3.1. In $(y, Z)$-coordinates, $\Pi_{0}$ is given by the PAM M $(Z)$ for $\delta \rightarrow 0$; see Proposition 1. Now, it suffices to note that hyperbolic fixed points of $M$ persist under perturbation of $M$ in $\delta$ - which gives $\Pi_{0}$ - and, subsequently, under perturbation of $\Pi_{0}$ in $\epsilon$. Thus, we find a stable fixed point of the one-dimensional map $\Pi_{m_{\epsilon, \delta}}$; the $\kappa$-th iterate of $\Pi$ can be studied in a similar fashion.

### 5.4. Proof of Theorem 3.3

To prove Theorem 3.3, we introduce a specific subfamily of slow-fast systems of the form in (2.1) that satisfies Assumptions 1 through 5. Then, we will show that a given PAM $M$ can be associated with a representative system from that family. Specifically, we take

$$
F(x, z, \epsilon, \delta)=F(x, z), g_{1}(x, y, z, \epsilon, \delta)=J(x), \text { and } g_{2}(x, y, z, \epsilon, \delta)=\delta G(x)+z H(x)
$$

in (2.1). In particular, we assume $F(x, z)$ to be a polynomial of degree 9 in $x$, restricted to $(x, z) \in(-3,2) \times(-1,1)$; moreover, we choose the functions $G(x)$, $H(x), J(x)$, and $Q(x)$ such that the integrals to be evaluated in (4.8) and (4.9) are as simple as possible, with convenient substitutions inside the integrands. Also, for simplicity, we take $z_{0}=0$.

In sum, we hence have

$$
\begin{align*}
F(x, z) & =a_{9} x^{9}+\sum_{k=2}^{8} a_{k}(z) x^{k}, \text { with } a_{9}=\frac{184180}{67741437}, a_{8}(z)=\frac{1}{8}\left(\frac{138135}{90321916} z+\frac{3555512}{22580479}\right), \\
a_{7}(z) & =\frac{1}{7}\left(\frac{751493}{90321916} z+\frac{212863}{22580479}\right), a_{6}(z)=-\frac{1}{6}\left(\frac{2793109}{361287664} z+\frac{23361467}{2250479}\right), \\
a_{5}(z) & =-\frac{1}{5}\left(\frac{10284179}{180638332} z+\frac{124990}{225800799}\right), a_{4}(z)=\frac{1}{4}\left(\frac{2417921}{4516058} z+\frac{64933913}{22580479}\right), \\
a_{3}(z) & =\frac{1}{3}\left(\frac{45620545}{361287664} z+\frac{459587}{22580479}\right), \text { and } a_{2}(z)=-\frac{1}{2}\left(\frac{1}{8} z+2\right), \tag{5.4a}
\end{align*}
$$

$$
\begin{equation*}
J(x)=\frac{1}{2}-x, \rho(x)=p+x+q x^{2}, Q(x)=\int_{0}^{x} \rho(s) \frac{\partial F}{\partial s}\left(s, z_{0}\right) d s \tag{5.4b}
\end{equation*}
$$

$$
\begin{equation*}
G(x)=\left[\kappa+\lambda\left(\frac{\alpha Q(x)^{2}}{2}+\beta Q(x)\right)\right](\alpha Q(x)+\beta) \rho(x) J(x), \text { and } \tag{5.4c}
\end{equation*}
$$

$$
\begin{equation*}
H(x)=\rho(x)\left[\int_{0}^{x}\left(\rho(s) \frac{\partial F}{\partial s}\left(s, z_{0}\right) d s\right) \alpha+\beta\right] J(x) \tag{5.4d}
\end{equation*}
$$

While the choices in (5.4) seem far from simple at first glance, they are made for the sole purpose of simplifying the requisite calculations that follow; different choices are certainly possible as long as Assumptions 1 through 5 are satisfied. (A related system will also underlie the numerical simulations presented in the next Section 6; although that system will mostly be identical to the one in (5.4), the definition of the function $\rho(x)$ will differ for computational convenience.)

By Definition 3.1, the given PAM $M(Z)$ is determined by the coefficients $a_{i j}$, for $i, j=1,2$. We will prove that there exists a slow-fast system of the specific form in
(5.4) which is associated with $M$; to that end, we need to show that the system of equations

$$
\begin{equation*}
a_{i j}=f_{i j}(\alpha, \beta, \kappa, \lambda, p, q), \quad \text { with } i, j=1,2, \tag{5.5}
\end{equation*}
$$

has at least one solution $\left(\alpha_{*}, \beta_{*}, \kappa_{*}, \lambda_{*}, p_{*}, q_{*}\right)$ which fully determines the vector field in (5.4). (Here, the notation $f_{i j}$ is shorthand for the right-hand sides in the definition of $a_{i j}$ in Definition 3.1.)

In a first step, we note that $a_{11}$ and $a_{21}$ depend on $(\alpha, \beta, p, q)$ only, i.e., that

$$
a_{11}=g_{11}(\alpha, \beta, p, q) \quad \text { and } \quad a_{21}=g_{21}(\alpha, \beta, p, q)
$$

for some new functions $g_{11}$ and $g_{21}$, as well as that $a_{11}$ and $a_{21}$ are positive by definition:

$$
\begin{aligned}
& a_{11}=g_{11}(\alpha, \beta, p, q)=\exp \left(\mathcal{A}_{11}(p, q) \alpha+\mathcal{A}_{12}(p, q) \beta\right) \quad \text { and } \\
& a_{21}=g_{21}(\alpha, \beta, p, q)=\exp \left(\mathcal{A}_{21}(p, q) \alpha+\mathcal{A}_{22}(p, q) \beta\right)
\end{aligned}
$$

for some functions $\mathcal{A}_{i j}(p, q)$ which are, in fact, polynomial in $p$ and $q$. Taking logarithms, we find a linear system in the unknowns $(\alpha, \beta)$ whose principal matrix has determinant $D(p, q)$. With the aid of the computer algebra package Maple, we compute $D$ to be a polynomial of degree 3 with positive coefficients. Restricting to the parameter domain $\{p>0, q>0\}$, we can hence safely assume that $D$ is non-zero and, hence, that the above system has a solution

$$
\alpha=\alpha_{*}\left(p, q, a_{11}, a_{21}\right) \quad \text { and } \quad \beta=\beta_{*}\left(p, q, a_{11}, a_{21}\right) .
$$

The next part of the proof is more intricate, and again relies on symbolic computation in Maple. Substituting the above expressions for $(\alpha, \beta)=\left(\alpha_{*}, \beta_{*}\right)$ into (5.5), we obtain

$$
\begin{align*}
& a_{12}=f_{12}\left(\alpha_{*}, \beta_{*}, \kappa, \lambda, p, q\right),  \tag{5.6a}\\
& a_{22}=f_{22}\left(\alpha_{*}, \beta_{*}, \kappa, \lambda, p, q\right), \tag{5.6b}
\end{align*}
$$

which is a linear system in $(\kappa, \lambda)$ whose principal matrix has determinant $\tilde{D}\left(p, q, a_{11}, a_{21}\right)$. The expression for $\tilde{D}$ can be written as

$$
\tilde{D}\left(p, q, a_{11}, a_{21}\right)=\frac{N\left(p, q, a_{11}, a_{21}, \ln a_{11}, \ln a_{21}, \mathrm{e}^{E_{1}}, \mathrm{e}^{E_{2}}\right)}{D(p, q)}
$$

where $D$ is defined as above, the exponents of the exponential terms $\mathrm{e}^{E_{i}}$ are of the form $E_{i}=E_{i}(\alpha, \beta, p, q)$, and $N$ is polynomial in all its 8 arguments. (In fact, $N$ has degree 1 with respect to $\mathrm{e}^{E_{1}}$ and $\mathrm{e}^{E_{2}}$.) It now suffices to show that, for each choice of $\left(a_{11}, a_{21}\right)$, there is at least one choice of $(p, q)$, with $p>0$ and $q>0$, for which $N$ is non-zero. Given the complex algebraic form of $N$, that is a cumbersome task. However, it suffices to argue that almost any choice of $(p, q)$ will be admissible.

We will outline that argument here. First, we write $N=N_{0}+N_{1} \mathrm{e}^{E_{1}}+N_{2} \mathrm{e}^{E_{2}}$, where each $N_{i}$ is a polynomial expression in ( $p, q, a_{11}, a_{21}$ ). Using Maple, we verify that

$$
\lim _{p \rightarrow \pm \infty} \frac{E_{i}}{p}=\frac{R_{i}\left(q, \ln a_{11}, \ln a_{12}\right)}{S(q)}
$$

for some strictly positive degree- 2 polynomial $S$ and some degree-1 polynomials (in $q$ ) $R_{1}$ and $R_{2}$. As there is only one choice for $q$ where the asymptotics of $E_{1}$ coincides with that of $E_{2}$, we restrict to the generic case where the two limits are strictly different. Then, there are six possibilities,

$$
\begin{align*}
& \lim _{p \rightarrow \infty} \frac{E_{1}}{p}<\lim _{p \rightarrow \infty} \frac{E_{2}}{p}<0,  \tag{5.7a}\\
& \lim _{p \rightarrow \infty} \frac{E_{1}}{p}<0 \leq \lim _{p \rightarrow \infty} \frac{E_{2}}{p}, \quad \text { and }  \tag{5.7b}\\
& \quad 0 \leq \lim _{p \rightarrow \infty} \frac{E_{1}}{p}<\lim _{p \rightarrow \infty} \frac{E_{2}}{p}, \tag{5.7c}
\end{align*}
$$

as well as the three possibilities obtained by swapping $E_{1}$ and $E_{2}$. Let us consider the third case as an example: in that scenario, as $p \rightarrow \infty$, the contributions of $N_{0}$ and $N_{1}$ in $N$ become negligible, and it suffices to see whether or not one can find $q$ for which $N_{2}$ is non-zero. Equally, in the first two scenarios, we conclude that the contribution of $N_{1}$ becomes significant in the limit as $p \rightarrow-\infty$. It now suffices to observe that both $N_{1} / p$ and $N_{2} / p$ are asymptotic to a quadratic polynomial in $q$ for large $|p|$ and, hence, that there are many choices of $(p, q)$ for which these expressions are non-zero. Hence, at least for $|p|$ sufficiently large, one can solve Equation (5.6) for ( $\kappa, \lambda$ ), which, in sum, gives a solution ( $\alpha_{*}, \beta_{*}, \kappa_{*}, \lambda_{*}$ ) to (5.5). Hence, generically, given a PAM $M$, one can choose $\left(p_{*}, q_{*}\right)$ such that there exists a slow-fast vector field within the family defined by (5.4) to which $M$ is associated. This completes the proof.

Remark 5. In practice, one would not take $|p|$ too large, as that would introduce another layer of time scale separation in the system.

## 6. Numerical verification

Finally, in this section, we give a numerical verification of two of our main results, Theorems 3.2 and 3.3. To that end, we consider the family of one-dimensional PAMs of the form

$$
M(Z)= \begin{cases}a_{11} Z+a_{12} & \text { for } Z<0  \tag{6.1}\\ a_{21} Z+a_{22} & \text { for } Z>0\end{cases}
$$

where $a_{i j}=f_{i j}(\alpha, \beta, \kappa, \lambda, p, q)$ with $i, j=1,2$, as introduced in (5.5). For the calculations of the integrals appearing in (4.8) and (4.9), we require the following $x$-values, which are obtained from (5.4) with $z_{0}=0$ :

$$
\hat{x}_{4}=-\frac{5}{2}, x_{1}=-2, x_{2}=-1, x_{3}=0, x_{4}=1, \hat{x}_{3}=\frac{3}{2}, \text { and } \hat{x}_{1}=\frac{8}{5}
$$

see Figure 2. Next, and as outlined in the proof of Theorem 3.3 in Section 5.4, we have to choose a suitable function $\rho$ in (5.4). Rather than taking $\rho$ within the family specified there, we pick the numerically more convenient function

$$
\begin{equation*}
\rho(x)=\left(\frac{552540}{22580479} x^{4}+\frac{2453432}{22580479} x^{3}-\frac{4141461}{22580479} x^{2}-\frac{11520033}{22580479} x+1\right)^{-1} \tag{6.2}
\end{equation*}
$$

The choice in (6.2) allows us to determine the four pivotal quantities $\alpha, \beta, \kappa$, and $\lambda$ in (5.4), in agreement with our expectation that a wide range of functions $\rho(x)$
will yield an admissible solution $\left(\alpha_{*}, \beta_{*}, \kappa_{*}, \lambda_{*}\right)$. That solution then specifies a threedimensional slow-fast system from the family defined in (5.4) that is associated with the given PAM in (6.1).

Below, we showcase a number of examples which confirm that the resulting mixed-mode time series in that system have identical signature to the corresponding periodic orbits for the PAM $M$, thus verifying Theorem 3.2. Here, we note that the functions $\rho$ and $J$ in (5.4) are independent of $\alpha, \beta, \kappa$ and $\lambda$, and that they hence do not change with the signature. The functions $Q, G$, and $H$, on the other hand, are signature-dependent. At this point, we remark that the parameters $\epsilon$ and $\delta$, although seemingly independent, have to be calibrated carefully to allow for robust mixed-mode dynamics while maintaining computational efficiency: throughout this section, we fix $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$; see Section 7 for a more in-depth discussion of the interplay between $\epsilon$ and $\delta$.

Remark 6. Given that $F(x, z)$ in (5.4) is a ninth-degree polynomial in $x$, we rescaled $x$ and $y$ as

$$
x \mapsto \frac{2}{7} x \quad \text { and } \quad y \mapsto \frac{3}{2} y
$$

in our visualisation in order to restrict the area of interest in $x$ to the interval $[-1,1]$. Finally, we note that throughout this section, $z$ remains rescaled to $Z$, as defined by the change of coordinates in (4.1) which was used in both Sections 3 and 4. In particular, $z$ is hence highlighted as the slowest variable despite the seemingly larger scale (in Z) that appears in the numerical examples below.

### 6.1. Examples: MMOs of various signatures

In a first step, we fix the coefficients $a_{11}, a_{21}$, and $a_{22}$ in the definition of the PAM in (6.1), varying only $a_{12}$ as the "bifurcation parameter". In Table 1 below, we list two sequences of mixed-mode signatures that are obtained upon variation of $a_{12}$, with $a_{11}, a_{21}$, and $a_{22}$ fixed as stated there. For completeness, and to illustrate the two-way correspondence established in Theorems 3.2 and 3.3, we also give the corresponding pivotal quantities $\alpha, \beta, \kappa$ and $\lambda$ in the definition of the associated vector field in (5.4). (We note that, given $a_{11}, a_{21}$, and $a_{22}, \alpha$ and $\beta$ do not change as $a_{12}$ is varied, in contrast to $\kappa$ and $\lambda$, as is to be expected from the proof of Theorem 3.3.)

In particular, we thus observe an unfolding of a "regular" sequence of signatures which are either of the form $1^{s}$ or $L^{1}$ in the bifurcation parameter $a_{12}$. A selection of (periodic) mixed-mode trajectories, both for the PAM in (6.1) and the associated vector field, is illustrated graphically in Figures 4 through 9 below. We emphasise that we observe the same signature in all three (state) variables $x, y$, and $z$ in (2.1), which is due to the geometry of the underlying critical manifold $\mathcal{S}$; see again Figure 5, where we highlight the signature $1^{3}$ as one particular example.

### 6.2. The At Most $\xi^{3}$ At Least Lemma

In this section, we give conditions on the coefficients in the definition of the PAM $M$ in (6.1) that guarantee the occurrence of certain numbers of LAOs $(Z<0)$ or SAOs $(Z>0)$ in a periodic MMO generated by the PAM $M$ in (6.1). To that end, we apply results of [29]; in a first step, we transform $M$ into the form considered

| signature | $a_{11}$ | $a_{12}$ | $a_{21}$ | $a_{22}$ | $\alpha$ | $\beta$ | $\kappa$ | $\lambda$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $1^{1}$ | 0.3 | 1 | 0.9 | -2 | 0.8743 | 0.0240 | 27.2674 | -64.5764 |
| $1^{2}$ | 0.3 | 3 | 0.9 | -2 | 0.8743 | 0.0240 | 28.2364 | -73.1866 |
| $1^{3}$ | 0.3 | 7 | 0.9 | -2 | 0.8743 | 0.0240 | 30.1744 | -90.4070 |
| $1^{4}$ | 0.3 | 10 | 0.9 | -2 | 0.8743 | 0.0240 | 31.6279 | -103.3223 |
| $1^{5}$ | 0.3 | 12 | 0.9 | -2 | 0.8743 | 0.0240 | 32.5969 | -111.9325 |
| $1^{6}$ | 0.3 | 15 | 0.9 | -2 | 0.8743 | 0.0240 | 34.0504 | -124.8478 |
| $1^{7}$ | 0.3 | 20 | 0.9 | -2 | 0.8743 | 0.0240 | 36.4729 | -146.3733 |
| $1^{8}$ | 0.3 | 25 | 0.9 | -2 | 0.8743 | 0.0240 | 38.8954 | -167.8987 |
| $1^{1}$ | 0.9 | 3 | 0.4 | -3 | -0.5065 | 1.0238 | 3.2091 | $-7,7202$ |
| $2^{1}$ | 0.9 | 1.5 | 0.4 | -3 | -0.5065 | 1.0238 | 3.9766 | $-4,4118$ |
| $3^{1}$ | 0.9 | 1 | 0.4 | -3 | -0.5065 | 1.0238 | 4.2325 | -3.3088 |
| $4^{1}$ | 0.9 | 0.7 | 0.4 | -3 | -0.5065 | 1.0238 | 4.3860 | -2.6471 |
| $5^{1}$ | 0.9 | 0.5 | 0.4 | -3 | -0.5065 | 1.0238 | 4.4883 | -2.2059 |
| $6^{1}$ | 0.9 | 0.4 | 0.4 | -3 | -0.5065 | 1.0238 | 4.5395 | -1.9853 |
| $7^{1}$ | 0.9 | 0.3 | 0.4 | -3 | -0.5065 | 1.0238 | 4.6162 | -1.7647 |
| $8^{1}$ | 0.9 | 0.25 | 0.4 | -3 | -0.5065 | 1.0238 | 4.6418 | -1.6544 |

Table 1: Signatures of the form $L^{1}$ and $1^{s}$ generated by the vector field defined by (5.4) and the associated PAM in (6.1).


Figure 4: MMO of signature $1^{1}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.
(a)

(b)


(c)
(d)
(e)

Figure 5: MMO of signature $1^{3}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, with time series of $x, y$, and $Z$ in (c) through (e), for parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.


Figure 6: MMO of signature $3^{1}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.


Figure 7: MMO of signature $4^{1}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.


Figure 8: MMO of signature $1^{5}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.


Figure 9: MMO of signature $1^{8}$ : (a) piecewise affine map and (b) associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$.
there, via $a=a_{11}, b=a_{21}, \mu=a_{12}$, and $l=a_{22}-a_{12}$ :

$$
M(Z)= \begin{cases}a Z+\mu & \text { for } Z<0  \tag{6.3}\\ b Z+\mu+l & \text { for } Z>0\end{cases}
$$

Since $a, b>1$ would imply instability of the corresponding MMO, we assume that $a$ and $b$ take values in the interval $(0,1)$; recall Remark 3 as well as the proof of Theorem 3.2 in Section 5.3. Then, the parameter $l$ represents the height of the jump at $Z=0$, while the parameter $\mu$ will be varied. As explained in [29], we restrict to $0<\mu<-l$, in which case (6.3) has no fixed points and periodic orbits are possible.

The following result then gives conditions on the control parameter $\mu$ for at most, or at least, $L$ consecutive LAOs, respectively $s$ SAOs, to appear in a periodic MMO for $M$.

Proposition 2 (At Most \& At Least Lemma [29]). Let $M$ be as defined in (6.3). Then, the following statements hold true.
(1) When $\mu \leq \frac{-l a^{L-1}}{a^{L-1} b+\sum_{k=0}^{L-1} a^{k}}=: \mu_{1}$, then at least $L$ consecutive LAOs appear in a periodic MMO for $M$. When $\mu>\frac{-l a^{L}}{\sum_{k=0}^{L} a^{k}}=: \mu_{2}$, then at most $L$ consecutive LAOs appear in a periodic MMO for $\bar{M}$.
(2) When $\mu<\frac{-l \sum_{k=0}^{s-1} b^{k}}{\sum_{k=0}^{k} b^{k}}$, then at most $s$ consecutive SAOs appear in a periodic MMO for $M$. When $\mu \geq \frac{-l\left[\sum_{k=0}^{s-1} b^{k}+b^{s-1}(a-1)\right]}{b^{s-1} a+\sum_{k=0}^{s-1} b^{k}}$, then at least $s$ consecutive SAOs appear in a periodic MMO for $M$.
Remark 7. In the context of jump-induced MMOs considered here, signatures can be correlated to specific $\mu$-intervals by combining items (1) and (2) in Proposition 2 [29]. In the classical scenario of canard-induced MMOs, the corresponding parameter intervals are determined by pinpointing so-called "sectors of rotation" in the vicinity of a "folded singularity", and by estimating the width of these in terms of the singular perturbation parameter; see, e.g., [21] for details and references.

Given Proposition 2, it can be shown [29] that for $\mu \in\left(\mu_{2}, \mu_{1}\right.$ ], the only possible periodic MMO for $M$ is the one with signature $L^{1}$. Similarly, we can determine intervals for $\mu$ on which periodic MMOs with signature $1^{s}$ exist. We summarise a sample of mixed-mode signatures, and the corresponding parameter regimes, in Table 2 below. Here, the relevant $\mu$-intervals are obtained from Proposition 2; throughout, we find agreement between the theory ("Predicted $\mu$ ") and our numerics ("Actual $\mu$ ").

Hence, it is natural to ask whether MMOs with signature $L^{s}$ for $L>1$ and $s>1$ can be found in the present context. Following again [29], it can be shown that stable periodic MMOs with such signatures cannot occur; we outline the argument here for completeness. In [29], an orbit $\mathcal{O}$ is called admissible if the $\mu$-interval for which $\mathcal{O}$ exists is non-empty. Then, their Lemma 2 states that "for any admissible orbit $\mathcal{O}$, its pattern cannot contain consecutive Ls and consecutive Rs simultaneously", where $L$ and $R$ denote numbers of LAOs and SAOs in $\mathcal{O}$, respectively. The proof of Lemma 2 is by contradiction: if one assumes that an orbit with signature $L^{R}$ is actually possible, one concludes that, necessarily, $a, b>1$ in Equation (6.3); however, that contradicts the underlying assumption of $a, b \in(0,1)$ which is imposed both in [29] and in our analysis; cf. again Remark 3. Hence, we can equally rule out the existence of "exotic" stable periodic MMOs with general signature $L^{s}$ here.

| Signature | $a$ | $b$ | $l$ | Predicted $\mu$ | Actual $\mu$ | $\alpha$ | $\beta$ | $\kappa$ | $\lambda$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $1^{2}$ | 0.3 | 0.9 | -5 | $\mu \in(2.9262,3.5055]$ | 3 | 0.8743 | 0.0241 | 28.23 | 73.18 |
| $1^{3}$ | 0.3 | 0.9 | -9 | $\mu \in(6.5313,7.0921]$ | 7 | 0.8743 | 0.0241 | 30.1744 | 90.4070 |
| $1^{4}$ | 0.3 | 0.9 | -11 | $\mu \in(8.8076,9.2376]$ | 9 | 0.8743 | 0.0241 | 31.1434 | -99.0172 |
| $1^{8}$ | 0.3 | 0.9 | -2.28 | $\mu \in(2.0932,2.1197]$ | 2.1 | 0.8743 | 0.0241 | 3.4279 | -14.4651 |
| $1^{9}$ | 0.3 | 0.9 | -2.68 | $\mu \in(2.4955,2.5205]$ | 2.5 | 0.8743 | 0.0241 | 3.6217 | -16.1861 |
| $1^{25}$ | 0.5 | 0.9 | -15.25 | $\mu \in(14.9889,15.0064]$ | 15 | 0.5025 | 0.0152 | 15.8532 | -177.4797 |
| $2^{1}$ | 0.9 | 0.8 | -7.2 | $\mu \in[2.1520,2.4732)$ | 2.2 | -0.0610 | 0.2430 | 24.4916 | -96.1819 |
| $3^{1}$ | 0.9 | 0.8 | -6.5 | $\mu \in[1.3778,1.5678)$ | 1.5 | -0.0610 | 0.2430 | 24.5673 | -81.8569 |
| $6^{1}$ | 0.9 | 0.8 | -5.6 | $\mu \in[0.5704,0.6410)$ | 0.6 | -0.0610 | 0.2430 | 24.6646 | -63.4391 |
| $8^{1}$ | 0.9 | 0.9 | -6.5 | $\mu \in[0.4675,0.5075)$ | 0.5 | 0.0147 | 0.1104 | 65.5190 | -462.9354 |
| $9^{1}$ | 0.9 | 0.9 | -9.6 | $\mu \in[0.5710,0.6344)$ | 0.6 | 0.0147 | 0.1104 | 98.1512 | -683.7200 |

Table 2: Signatures of the form $L^{1}$ and $1^{s}$ generated by the vector field defined by (5.4) and the corresponding $\mu$-intervals, as determined from Proposition 2.

### 6.3. Crossover signatures

Given the numerical results in the previous two subsections, it is natural to ask what happens between two "consecutive" signatures, i.e., how the shape of an MMO changes as orbits cross over from a cycle of signature $L^{s}$ to one of signature $L^{s+1}$ or, equivalently, from one of signature $L^{s}$ to one of signature $(L+1)^{s}$. Motivated again by results of [29] - see, in particular, Lemma 4, Figure 3, and Note 2 therein - we observe the existence of so-called "crossover signatures" inside "intermediate neighbourhoods" for some of the corresponding parameters in the definition of the transformed PAM $M$ in (6.3). (In [29], the existence of similar regions, named "molecular regions" there, is concluded.) These observations lead to the conclusion that mixed-mode signatures are not arranged in a monotonic way, as far as the number of LAOs or SAOs therein is concerned. For illustration, we showcase a simple example here, namely, an MMO of signature $1^{4}$, which can be obtained from the following PAM,

$$
M_{1^{4}}(Z)= \begin{cases}0.9 Z+6 & \text { for } Z<0,  \tag{6.4}\\ 0.85 Z-1 & \text { for } Z>0\end{cases}
$$

with corresponding parameter values $\alpha \approx-0.0220, \beta \approx 0.1747, \kappa \approx 7.7321$, and $\lambda \approx-233.3068$ in the associated slow-fast vector field that is determined by (5.4). It is straightforward to obtain an MMO with the "consecutive signature", namely $1^{5}$, in the following PAM:

$$
M_{1^{5}}(Z)= \begin{cases}0.9 Z+7.2 & \text { for } Z<0,  \tag{6.5}\\ 0.85 Z-1 & \text { for } Z>0,\end{cases}
$$

with parameter values $\alpha \approx-0.0220, \beta \approx 0.1747, \kappa \approx 7.9239$, and $\lambda \approx-275.3021$.

Noting that the numerical values of the parameters $\alpha$ and $\beta$ that determine $M_{1^{4}}$ and $M_{1^{5}}$ are almost identical while $\kappa$ and $\lambda$ vary, we take $\kappa \in(7.7321,7.9239)$ and $\lambda \in(-275.3021,-233.3068)$, which generates the "crossover signature" $1^{4} 1^{5}$ for $\mu=6.5$, as shown in Figure 10(a).

Here, it is important to emphasise that these mixed signatures do not contradict the At Most \& At Least Lemma, Proposition 2. Rather, for a fixed choice of the pivotal quantities $\alpha, \beta, \kappa$, and $\lambda$, we obtain a hierarchy of disjoint $\mu$-intervals that correspond to mixed-mode signatures of the form $L^{s}$ from Proposition 2. "Crossover" signatures are found for $\mu$ in the complements of those intervals; from a practical point of view, the preferred choice of the pivotal quantities is guided by where the adjacent, "simple" signatures occur, whereupon $\mu$ can be fixed from the At Most \& At Least Lemma. See Table 2 for a specification of the corresponding $\mu$-intervals.

Following the same procedure as above, we were able to detect intermediate neighbourhoods for the signature $2^{1}$ crossing over to $3^{1}$; see Figure 11 for an illustration.

Again, we first consider a PAM which realises the signature $2^{1}$ :

$$
M_{2^{1}}(Z)= \begin{cases}0.9 Z+2.2 & \text { for } Z<0  \tag{6.6}\\ 0.8 Z-5 & \text { for } Z>0\end{cases}
$$

where $\alpha=-0.0610, \beta=0.2430, \kappa=24.4916$, and $\lambda=-96.1819$, as well as a map which generates the "consecutive" signature $3^{1}$ :

$$
M_{3^{1}}(Z)= \begin{cases}0.9 Z+1.5 & \text { for } Z<0  \tag{6.7}\\ 0.8 Z-5 & \text { for } Z>0\end{cases}
$$

with $\alpha=-0.0610, \beta=0.2430, \kappa=24.5673$, and $\lambda=-81.8569$.
Naturally, here too the numerical values of the parameters $\alpha$ and $\beta$ that determine $M_{2^{1}}$ and $M_{3^{1}}$ are almost indistinguishable. If we then pick $\kappa \in(24.4916,24.5673)$ and $\lambda \in(-96.1819,-81.8569)$, we observe an MMO with crossover signature $2^{1} 3^{1}$ for $\mu=1.8$, as shown in Figure 11.

Note that for both examples, we took $a_{11}=a<1$ and $a_{21}=b<1$, in accordance with [29]; a more general choice of coefficients, with $a^{L} b^{s}<1$, still yields stable mixed-mode orbits for the PAM $M$ in (6.3). However, as stated above, that scenario is excluded in [29] and hence cannot be considered within the framework of the At Most \& At Least Lemma, Proposition 2.

Remark 8. Numerical evidence suggests that we only encounter a combination of either one of the adjacent signatures between two "simple", consecutive signatures: for example, picking $1^{s}$ and $1^{s+1}$, in the "intermediate neighbourhoods" we can only expect MMOs of signature $\left(1^{s}\right)^{i}\left(1^{s+1}\right)^{j}$, with finite multiplicity $0<i, j<\infty$. This last assertion is easier to observe for small s, since the parameter intervals corresponding to such "crossover" signatures tend to shrink with increasing s. An analogous assertion applies to MMOs with signature $L^{1}$.


Figure 10: Transition from MMOs with signature $1^{4}$ to $1^{5}$ via the "crossover" signature $1^{4} 1^{5}$ for (a) the PAM defined in (6.1) and (b) its associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$. The signature $1^{4} 1^{5}$ was found for $\kappa=7.8120$ and $\lambda=-250.8049$, with $\mu=6.5$.


Figure 11: Transition from MMOs with signature $2^{1}$ to $3^{1}$ via the "crossover" signature $2^{1} 3^{1}$ for (a) the PAM defined in (6.1) and (b) its associated three-dimensional slow-fast system, with parameter values $\epsilon=10^{-7}$ and $\delta=5 \times 10^{-3}$. The signature $2^{1} 3^{1}$ was found for $\alpha=-0.0610, \beta=0.2430$, $\kappa=24.5348$, and $\lambda=-87.9962$, with $\mu=1.8$.

## 7. Conclusions

In this paper, we have introduced a novel "jump mechanism" for the generation of mixed-mode oscillations (MMOs) in a family of three-dimensional slow-fast systems of singular perturbation type. In marked contrast to the canard-based mechanism that is typically invoked to explain mixed-mode dynamics in such systems, we do not assume the presence of a folded singularity on any of the fold lines at which normal hyperbolicity is lost; in fact, we require all such fold lines to consist of jump points only. Correspondingly, the small-amplitude oscillation (SAO) component in the resulting mixed-mode trajectories is then also of relaxation type, with an amplitude that is $O(1)$ in the singular perturbation parameter $\epsilon$. At this point, we remark that it is possible to obtain quantitative information on the MMOs constructed here. In particular, the amplitudes of both components can be determined by observing the height of the fold lines $L_{1}$ and $L_{3}$ with respect to the $y$-coordinate. The corresponding periods can be approximated from the transition times on normally attracting portions of the critical manifold in a manner similar to that of Section 4; given a mixed-mode trajectory of signature $L^{s}$, the overall period would be found by multiplying the periods of one LAO and one SAO with $L$ and $s$, respectively, before adding them. The details are left to the interested reader.

As the principal result of this paper, we have established a two-way correspondence between our family of slow-fast systems and a class of one-dimensional piecewise affine maps (PAMs) which are naturally associated with each other. In particular, we have shown that for every such PAM that exhibits an MMO with a certain given signature, there exists a slow-fast system that can be associated with it and vice versa, provided that certain conditions are met; recall Theorems 3.2 and 3.3. Thus, we have reduced the study of MMOs in a particular family of threedimensional slow-fast vector fields to the well-developed theory of one-dimensional maps. We were able to verify our theoretical results numerically, showing that they are consistent with those obtained in [29] - and, in particular, with the At Most \& At Least Lemma - in the process.

Naturally, a number of questions arise from the present analysis. The first of these concerns an in-depth investigation of a neighbourhood of the singular orbit $\Gamma_{0}$ defined in Assumption 5, as well as of the corresponding discontinuity in the associated PAM, where canard phenomena could occur. We conjecture that this discontinuity gives rise to a canard explosion which determines the interchange between LAOs and SAOs in the resulting mixed-mode time series.

Next, it seems natural to comment on the interplay between $\epsilon$ and $\delta$ in our prototypical family of slow-fast systems in Equation (2.1), which we restate below for reference:

$$
\begin{aligned}
& x^{\prime}=y-F(x, z, \epsilon, \delta)=: f(x, y, z, \epsilon, \delta), \\
& y^{\prime}=\epsilon g_{1}(x, y, z, \epsilon, \delta), \\
& z^{\prime}=\epsilon g_{2}(x, y, z, \epsilon, \delta) .
\end{aligned}
$$

Under the additional simplifying assumption in (2.6), the form of the above twoparameter singular perturbation problem implies that the choice of $\delta$ will mostly affect the slow flow near normally hyperbolic (attracting) portions of the corresponding critical manifold $\mathcal{S}$, away from the fold lines $L_{i}$. Conversely, it also appears that
any restrictions on the magnitude of $\epsilon$ are only due to the jump behaviour at those lines; both observations are corroborated by numerical experimentation in Maple.

Now, the initial rescaling of $z$ with respect to $\delta$, as illustrated in Section 4, implies that the relevant $z$-window for our analysis is $\mathcal{O}(\delta)$ wide. While $Z<0-$ in the rescaled $Z$-variable - results in LAOs in the corresponding mixed-mode time series, whereas $Z>0$ yields SAOs, that classification is only true in the singular limit, i.e., for $\epsilon=0$. To specify the interplay between $\epsilon$ and $\delta$ away from that limit, one would need to "blow up" (desingularise) the flow of (2.1) in the vicinity of the degenerate point of intersection between the singular orbit $\Gamma_{0}$ and the fold line $L_{2}$; as $\epsilon$ would receive triple the weight of $\delta$ in the corresponding blow-up transformation, it would then follow that we not only have to avoid $P_{c}$ itself, but also an $\mathcal{O}\left(\epsilon^{1 / 3}\right)$ "hole" around that point, in order not to have to consider canard phenomena in our analysis. When $\epsilon^{1 / 3}>\delta$, on the other hand, the Poincaré map $\Pi$ associated with (2.1) is likely to return the flow inside this canard hole. Correspondingly, in Section 6 , we fixed $\epsilon=10^{-7}$ in the relevant numerics, as we had experimentally concluded that the optimal choice of $\delta$ is somewhere in the region of $5 \times 10^{-3}$, in agreement with the above reasoning.

The next question that comes to mind, which concerns the patterns that the resulting mixed-mode signatures follow, is motivated by results of Freire and Gallas in [14]. There, it is shown that the number of SAOs in a given mixed-mode orbit is not arbitrary, but that it is organised in a pattern dictated by a so-called SternBrocot tree. It would seem natural to investigate whether similar number-theoretical arguments can be applied in the context of the family of slow-fast systems studied in the present work. Preliminary analysis seems to suggest that the "crossover" signatures observed between simple patterns of the form $1^{s}$ or $L^{1}$ are relatively regular; recall Remark 8.

Our final remark concerns the prototypical family of slow-fast systems in (2.1), and specifically those which have the property in (2.6). We are confident that, near the fold lines $L_{1}, L_{3}$, and $L_{4}$, a suitable Fenichel-like normal form can be derived. Should that expectation be verified, it remains to be seen if our approach can be generalised by connecting the well-developed theory of one-dimensional PAMs with the vast family of slow-fast systems that can be brought into said normal form.

## Acknowledgements

The authors thank the School of Mathematics at the University of Edinburgh for its hospitality during several research visits. In particular, we are grateful to Panagiotis Kaklamanos for his fruitful and meticulous comments, as well as to the entire Edinburgh Dynamical Systems Study Group for general feedback on a draft version of the paper. We would also like to thank two anonymous reviewers whose comments improved the quality of the original manuscript.

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