A geometric analysis of front propagation in an integrable Nagumo equation with a linear cut-off

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Abstract

We investigate the effects of a linear cut-off on front propagation in the Nagumo equation at a so-called Maxwell point, where the corresponding front solution in the absence of a cut-off is stationary. We show that the correction to the propagation speed induced by the cut-off is positive in this case; moreover, we determine the leading-order asymptotics of that correction in terms of the cut-off parameter, and we calculate explicitly the corresponding coefficient. Our analysis is based on geometric techniques from dynamical systems theory and, in particular, on the method of geometric desingularization ('blow-up').

1. Introduction

In this article, we are concerned with front propagation in the classical Nagumo equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi(1-\phi)(\phi-\gamma), \tag{1}$$

where γ is a real parameter that is typically assumed to lie in $[-1, 1)$; see *e.g.* [1] and the references therein. Traveling front solutions of (1) maintain a fixed profile as they propagate through phase space, and are naturally studied in the framework of the traveling wave variable $\xi = x - ct$: with $u(\xi) = \phi(t, x)$, Equation (1) reduces to the traveling wave equation

$$u'' + cu' + u(1-u)(u-\gamma) = 0, \tag{2}$$

which, for any $\gamma \in [-\frac{1}{2}, \frac{1}{2}]$, supports the closed-form solution

$$u(\xi) = \frac{1}{1 + \mathrm{e}^{\frac{1}{\sqrt{2}}(\xi - \xi^-)}}; \tag{3}$$

here, $\xi^- > 0$ denotes an arbitrary phase. Since, clearly, $\lim_{\xi\to-\infty} u(\xi) = 1$ and $\lim_{\xi\to\infty} u(\xi) = 0$ in (3), the corresponding traveling front connects the rest states at 1 and 0 of (1). Moreover, the propagation speed is also known explicitly in this case, and is given by $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ [1].

Front propagation in the Nagumo equation can be classified in terms of the parameter γ: depending on the value of γ, the front defined by (3) is termed 'pulled,' 'pushed,' or 'bistable.' Correspondingly, the zero rest state of (1), which is unstable for negative γ, becomes metastable when γ is positive. While the propagation speed of pulled fronts is selected by linearization about that state, the corresponding selection mechanism in the pushed and bistable regimes is highly nonlinear; see *e.g.* [2, 3, 4] for details and references. In particular, the bistable regime in (1) is realized for $\gamma \in (0, \frac{1}{2})$, in which case c_0 is positive; at the so-called Maxwell point, which is defined by $\gamma = \frac{1}{2}$, the speed vanishes, and the front solution in (3) corresponds to a stationary (time-independent) solution of Equation (1), as $\xi = x$ then. (Past that point, *i.e.*, for $\gamma > \frac{1}{2}$, the front reverses direction, as the propagation speed c_0 becomes negative; in other words, the rest state at 1 becomes dominant [1].)

One aspect of front propagation in scalar reaction-diffusion systems that has received much recent attention concerns the effects of a 'cut-off' on the dynamics of traveling fronts and, in particular, on the propagation speed of these fronts. Cut-offs were introduced by Brunet and Derrida in the groundbreaking study [5], in the context of the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation; they are imposed in equations of the type in (1) to incorporate stochastic fluctuations which are due to the fact that the model underlying (1) is oftentimes a discrete, N-particle system: as there are no particles available to react if the concentration ϕ is less than $\varepsilon = N^{-1}$, the reaction terms in (1) are damped, or even canceled, in an ε-neighborhood of the zero rest state. In particular, Brunet and Derrida showed that, in the FKPP equation, a cut-off substantially reduces the front propagation speed that is realized in the absence of a cut-off, as well as that the first-order correction to that speed is of the order $O[(\ln \varepsilon)^{-2}]$, for a wide range of cut-off functions. Subsequently, Benguria, Depassier and collaborators [2, 6, 7] investigated the effects of a cut-off in a number of prototypical reaction-diffusion systems, including (1), showing that the leading-order asymptotics of the corresponding correction typically scales with fractional powers of ε; see also [8, 9].

A rigorous proof of some of the findings of Brunet and Derrida has been given in [10]: the results reported in [5] were obtained in the framework of matched asymptotic expansions, while the approach due to Benguria and Depassier [6] relies on a variational principle, in which the propagation speed is obtained as the supremum of an appropriately defined functional. By contrast, the analysis in [10] was based on geometric methods from dynamical systems theory and, in particular, on the 'blow-up' technique (also known as geometric desingularization) [11, 12, 13]. Blow-up is essentially a sophisticated coordinate transformation that desingularizes the system dynamics in a neighborhood of degenerate singularities, thus extending the validity of standard techniques, such as invariant manifold theory [14, 15]. In addition to confirming the leading-order (logarithmic) correction to the propagation speed that is induced by the cut-off, as well as the universality of the corresponding coefficient, the approach developed in [10] also explained the structure of the resulting asymptotics in terms of the linearization of the blown-up vector field at the zero rest state. Finally, it provided a constructive geometric proof for the existence and uniqueness of propagating fronts in the presence of a cut-off.

Geometric desingularization has since been successfully applied in the study of the effects of a cut-off on propagating fronts in the Nagumo equation: the bistable regime, with $\gamma \in (0, \frac{1}{2})$ in (1), was analyzed in [3], while the 'boundary' case where $\gamma = 0$ was discussed in detail in [4]. (Here, we remark that Equation (1) is also known as the Zeldovich equation in that case [16].) In both regimes, we proved the existence and uniqueness of front solutions to the corresponding cut-off equations, and we derived the leading-order correction to the propagation speed that is due to the cut-off; see [3, Theorem 2.1] and [4, Theorem 1.1], respectively.

In the present article, we study front propagation in a cut-off modification of Equation (1) at a Maxwell point, *i.e.*, for $\gamma = \frac{1}{2}$; our principal aim is to illustrate how the non-smoothness that is introduced into (1) by a cut-off can be resolved, using geometric singular perturbation theory [17] and blow-up. For illustration, we restrict ourselves to a linear cut-off here: we consider

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi(1-\phi)(\phi - \tfrac{1}{2})\Theta(\phi, \varepsilon), \tag{4}$$

where

$$\Theta(\phi, \varepsilon) = \frac{\phi}{\varepsilon} \quad \text{for } \phi \leq \varepsilon \quad \text{and} \quad \Theta(\phi, \varepsilon) \equiv 1 \quad \text{for } \phi > \varepsilon; \tag{5}$$

the corresponding cut-off traveling wave equation reads

$$u'' + cu' + u(1-u)(u - \tfrac{1}{2})\Theta(u, \varepsilon) = 0. \tag{6}$$

Other choices of cut-off function can be studied in a similar fashion; see [10] and Section 5 below for details. While most comparable results to date have been obtained in the context of the Heaviside cut-off, which cancels the reaction terms in (1) in a neighborhood of the zero rest state, our choice of a linear cut-off will still allow for an explicit (closed-form) analysis of the resulting dynamics.

Finally, we remark that, since the front defined by (3) is stationary in the absence of a cut-off, as $c_0 = 0$, and since the correction induced by the cut-off is positive, as we will show below, the cut-off effectively initiates front propagation in this case. (Similar effects were reported in [18], where it was observed that stochastic noise can significantly increase the propagation speed of stationary, deterministic fronts.)

Our main result is summarized in the following proposition:

Proposition 1. *For $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, there exists a unique value $\Delta c(\varepsilon)$ of c such that Equation (4) supports a unique traveling front solution which connects the rest states at 1 and 0. Moreover, Δc is a positive function that satisfies*

$$\Delta c(\varepsilon) = \frac{1}{\sqrt{2}}\varepsilon^2 + o(\varepsilon^2), \tag{7}$$

to leading order in ε.

This article is organized as follows. In Section 2, we introduce a geometric framework for the study of Equation (4); in Section 3, we prove the existence of a unique front solution for a unique value of the propagation speed c; in Section 4, we derive the leading-order ε-asymptotics of that speed; finally, in Section 5, we summarize and discuss our results and possible questions for future research.

2. Geometric framework

Following [3, 4, 10], we consider front propagation in the cut-off Nagumo equation in the framework of the equivalent first-order system that is obtained by introducing $u' = v$ in (6). Appending the trivial ε-dynamics to the resulting equations, we find

$$u' = v, \tag{8a}$$

$$v' = -cv - u(1-u)(u - \tfrac{1}{2})\Theta(u, \varepsilon), \tag{8b}$$

$$\varepsilon' = 0. \tag{8c}$$

For any $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, the points $Q_\varepsilon^- = (1, 0, \varepsilon)$ and $Q_\varepsilon^+ = (0, 0, \varepsilon)$ are equilibria for the extended system in (8); these points represent precisely the rest states at 1 and 0, respectively, of the reaction-diffusion equation in (4). Front solutions connecting the two states, with propagation speed c, thus correspond to heteroclinic connections between the equilibrium points Q_ε^- and Q_ε^+ that are realized for that same value of c. (The third equilibrium of (8), which is located at $Q_\varepsilon^\circ = (\frac{1}{2}, 0, \varepsilon)$, is of no interest in the propagation regime considered here.)

A linearization argument shows that Q_ε^-, when restricted to the (u, v)-plane for c and ε fixed, is a hyperbolic saddle with eigenvalues $\lambda_\pm^- = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{1}{2}}$. (One zero eigenvalue is trivially due to (8c).) The point Q_ε^+, on the other hand, is a semi-hyperbolic equilibrium, with eigenvalues $-c$ and 0, for ε positive and fixed. The limit as $\varepsilon \to 0^+$ in (8), however, is singular, since (8b) becomes ill-defined then; recall the definition of Θ in (5).

The proof of Proposition 1 is based on a geometric (phase space) analysis of the first-order system in (8): first, we will regularize the singular limit as $\varepsilon \to 0^+$ in that system, which will allow us to construct a singular heteroclinic connection Γ between Q_0^- and Q_0^+, with $c = c_0(= 0)$ in (8). Then, we will show that Γ persists, for $\varepsilon \in (0, \varepsilon_0]$ sufficiently small, as a connection between Q_ε^- and Q_ε^+, for a unique value $\Delta c(\varepsilon)$ of c. Finally, we will derive a necessary condition which will determine the ε-asymptotics of Δc to lowest order.

The blow-up transformation that will be applied to desingularize the dynamics of (8) in a neighborhood of the degenerate (non-hyperbolic) origin is defined by

$$u = \bar{r}\bar{u}, \quad v = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon}, \tag{9}$$

with $\bar{r} \in [0, r_0]$ for $r_0 > 0$ sufficiently small. Details can be found in [10], where that same transformation was used in the analysis of the cut-off FKPP equation; it was subsequently applied in [3] as well as in [4].

The transformation in (9) effectively 'blows up' the point Q_0^+ to the two-sphere

$$\mathbb{S}^2 = \{(\bar{u}, \bar{v}, \bar{\varepsilon}) \,|\, \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1\}$$

in three-space. (In fact, we only need to consider the quarter-sphere $\mathbb{S}_+^2$ here, with $\bar{u}$ and $\bar{\varepsilon}$ non-negative.) The blown-up vector field that is induced by (8) on the resulting invariant manifold is conveniently studied in two coordinate charts, which we denote by K_j $(j = 1, 2)$. The dynamics in the 'inner' region, where $u \leq \varepsilon$ (*i.e.*, where the vector field in (8) is cut-off), is described in the 'rescaling' chart K_2: setting $\bar{\varepsilon} = 1$ in (9), we obtain

$$u = r_2 u_2, \quad v = r_2 v_2, \quad \text{and} \quad \varepsilon = r_2. \tag{10}$$

Similarly, the 'phase-directional' chart K_1 is introduced to regularize the non-smooth transition, in $\{u = \varepsilon\}$, between the cut-off regime and the 'outer' region, where the dynamics of (8) is not affected by the cut-off. With $\bar{u} = 1$, the transformation in (9) reduces to

$$u = r_1, \quad v = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1 \tag{11}$$

in the corresponding 'intermediate' region.

Remark 1. While chart K_1 does cover the part of the phase space of (8) where $u = O(1)$, it is advantageous to describe the dynamics in that region in the original (u, v, ε)-variables; see Section 2.1 below.

Finally, the coordinate transformation between the two charts K_2 and K_1 (on their domain of overlap) can be written as follows:

$$r_1 = r_2 u_2, \quad v_1 = \frac{v_2}{u_2}, \quad \text{and} \quad \varepsilon_1 = \frac{1}{u_2}. \tag{12}$$

Remark 2. Given any object $\square$, we will denote the corresponding blown-up object by $\overline{\square}$, while that same object in chart K_j will be labeled $\square_j$, where necessary.

For future reference, we define the two lines ℓ^- and ℓ^+ via

$$\ell^- = \bigcup_{\varepsilon \in [0, \varepsilon_0]} Q_\varepsilon^- = \{(1, 0, \varepsilon) \,|\, \varepsilon \in [0, \varepsilon_0]\}, \tag{13a}$$

$$\ell^+ = \bigcup_{\varepsilon \in [0, \varepsilon_0]} Q_\varepsilon^+ = \{(0, 0, \varepsilon) \,|\, \varepsilon \in [0, \varepsilon_0]\}; \tag{13b}$$

by definition, ℓ^- and ℓ^+ are equilibrium states for (8) that are obtained from Q_ε^- and Q_ε^+, respectively, when ε is allowed to vary over the entire interval $[0, \varepsilon_0]$.

2.1. Outer region

In the outer region, which is defined by $u = O(1)$ in (6), the cut-off function Θ reduces to the identity; see (5). Hence, the first-order system in (8) is equivalent to Equation (2) (the traveling wave equation without cut-off) in this region. Rewriting (8) with u as the independent variable, *i.e.*, dividing (8b) (formally) by (8a), we obtain

$$v \frac{dv}{du} = -cv - u(1 - u)(u - \tfrac{1}{2}). \tag{14}$$

Remark 3. Clearly, Equation (14) is integrable when $c = 0$; correspondingly, the equilibrium at Q_ε° in (8) is a center when restricted to the (u, v)-plane, with eigenvalues $\pm\frac{i}{2}$. For $c \neq 0$, the integrability is destroyed: linearization shows that Q_ε° becomes a stable focus for any $0 < |c| < 1$.

By standard invariant manifold theory [15], the unstable manifold $\mathcal{W}^u(Q_\varepsilon^-)$ of Q_ε^- is analytic in u and v, as well as in the parameters c and ε, as long as $u > \varepsilon$. Writing $c = c_0 + \Delta c = \Delta c$, where Δc is assumed to be $o(1)$, but independent of ε for the time being, and taking into account that $c_0 = 0$ when $\gamma = \frac{1}{2}$, we can expand $\mathcal{W}^u(Q_\varepsilon^-)$ as follows:

$$v(u, c) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j v}{\partial c^j}(u, 0)(\Delta c)^j. \tag{15}$$

Remark 4. The structure of (14) implies that v can only depend on ε through c. Hence, the unstable manifold $\mathcal{W}^u(\ell^-)$ of ℓ^- is a trivial foliation in ε, with fibers $\mathcal{W}^u(Q_\varepsilon^-)$ that lie in planar sections through the three-dimensional phase space of (8).

The leading-order term in (15) corresponds precisely to the heteroclinic connection between Q_0^- and Q_0^+ that is obtained from the front solution in (3), in the absence of a cut-off: recalling that $v = u'$, we find

$$v(u, 0) = \frac{1}{\sqrt{2}} u(u - 1); \tag{16}$$

in particular, we note that $v(u, 0)$ is negative for $u \in (0, 1)$.

The next-order (linear) term in Δc can be found by considering the variational equation that is associated with (14), taken along the singular solution $v(u,0)$ defined in (16):

$$\frac{\partial}{\partial u}\Big(\frac{\partial v}{\partial c}(u,0)\Big) = -1 + \frac{2u-1}{u(1-u)}\frac{\partial v}{\partial c}(u,0). \tag{17}$$

Equation (17) can be solved explicitly, by variation of constants; see [3, Lemma 2.1] for the closed-form solution of that equation when $\gamma \in (0, \frac{1}{2})$.

Lemma 1. *Let* $u \in (0,1]$*; then, the unique solution to* (17) *that satisfies* $\frac{\partial v}{\partial c}(1,0) = 0$ *is given by*

$$\frac{\partial v}{\partial c}(u,0) = \frac{1-u}{2u}F(2,-1;3;1-u) = -\frac{2u^2-u-1}{6u}. \tag{18}$$

(Here, F denotes the hypergeometric function [19, Section 15].)

Finally, we introduce the section Σ^- in (u,v,ε)-space as follows: let

$$\Sigma^- = \{(\rho,v,\varepsilon)\,|\,(v,\varepsilon)\in[-v_0,0]\times[0,\varepsilon_0]\}, \tag{19}$$

where ρ and ε_0 are positive and small, but constant; similarly, $v_0 > \frac{1}{\sqrt{2}}$ denotes a fixed, positive constant. Since $\rho \geq \varepsilon$ for ε_0 sufficiently small, (16) defines the portion of the singular heteroclinic orbit Γ that is located in this outer region.

2.2. Inner region

For $u \leq \varepsilon$, the dynamics of Equation (6) is cut-off, as $\Theta(u,\varepsilon) = \frac{u}{\varepsilon}$ then; cf. (5). Since $\Theta(u_2,r_2) = u_2$ in the rescaled (u_2,v_2,r_2)-coordinates, by (10), the governing equations in this inner region read

$$u_2' = v_2, \tag{20a}$$
$$v_2' = -cv_2 + \tfrac{1}{2}u_2^2(1-3r_2u_2+2r_2^2u_2^2), \tag{20b}$$
$$r_2' = 0, \tag{20c}$$

where, moreover, (10) implies $u_2 \in [0,1]$. For $r_2(=\varepsilon)$ sufficiently small, the only equilibria of (20) are located on the portion of the r_2-axis that is given by $\ell_2^+ = \{(0,0,r_2)\,|\,r_2 \in [0,r_0]\}$, with $r_0 > 0$ as above. We note that ℓ_2^+ corresponds to the line ℓ^+ defined in (13b), after 'blow-down,' *i.e.*, after transformation to the original (u,v,ε)-coordinates. Linearization of (20) about $Q_2^+ = (0,0,r_2) \in \ell_2$, for r_2 fixed, shows that 0 is an eigenvalue of multiplicity three for $c = 0$ in (20), whereas it is a double eigenvalue for $c \neq 0$; the third eigenvalue is given by $-c$ in that case.

In the singular limit as $r_2 \to 0^+$, c reduces to $c_0(=0)$; the corresponding singular dynamics is found by considering (20) in that limit or, equivalently, by solving

$$\frac{dv_2}{du_2} = \frac{1}{2}\frac{u_2^2}{v_2} \qquad \text{with } v_2(0) = 0. \tag{21}$$

For negative v_2, Equation (21) has the unique closed-form solution

$$v_2(u_2) = -\frac{1}{\sqrt{3}}u_2\sqrt{u_2}; \tag{22}$$

hence, the portion of the singular orbit Γ that is located in this inner region, which we denote by Γ_2^+, is defined by the orbit corresponding to (22). We remark that Γ_2^+ equals $\mathcal{W}_2^s(Q_{0_2}^+)$, the stable manifold of the origin $Q_{0_2}^+$ in chart K_2, since u_2 and v_2 both decay to zero along Γ_2^+; cf. (20).

Remark 5. The singular dynamics that is obtained for $c = 0 = r_2$ in (20) is in fact Hamiltonian, with constant of motion $\mathcal{H}(u_2,v_2) = \frac{u_2^3}{6} - \frac{v_2^2}{2}$; in particular, the branch of the zero level curve of $\mathcal{H}$ that lies below the u_2-axis again yields the orbit Γ_2^+.

For c positive and sufficiently small, the stable manifold $\mathcal{W}_2^s(\ell_2^+)$ of ℓ_2^+ is a regular perturbation of Γ_2^+. (We do not consider c negative in (20), as we only allow for front propagation into the stable state at 0 here; recall Section 1.) Moreover, $\mathcal{W}_2^s(\ell_2^+)$ is analytic in (u_2,v_2), as well as in the parameters (c,r_2).

As in Section 2.1, we introduce a section Σ_2^+ for the flow of (20) via

$$\Sigma_2^+ = \{(1,v_2,r_2)\,|\,(v_2,r_2)\in[-v_0,0]\times[0,r_0]\}, \tag{23}$$

where v_0 and $r_0(=\varepsilon_0)$ are defined as in (19). Clearly, Σ_2^+ separates the inner region from the intermediate region; the point of intersection of Γ_2^+ with Σ_2^+ will be denoted by $P_{0_2}^+ = (1, v_{0_2}^+, 0)$, where $v_{0_2}^+ = -\frac{1}{\sqrt{3}}$ is found by evaluating (22) in Σ_2^+. Correspondingly, for c and r_2 fixed, we will write $P_2^+ = (1, v_2^+, 0)$ for the point of intersection of $\mathcal{W}_2^s(Q_2^+)$ with Σ_2^+, *i.e.*, we will suppress the parameter dependence of that point for convenience of notation.

The geometry in chart K_2 is illustrated in Figure 1.

Remark 6. The transformation to chart K_2 is a rescaling which naturally regularizes the singular limit as $\varepsilon \to 0^+$ in (8), for $u \leq \varepsilon$: by (10), that limit now corresponds to $r_2 \to 0^+$, with $u_2 \in [0,1]$.

Remark 7. The uniqueness of Γ_2^+ (and, correspondingly, of $\mathcal{W}_2^s(\ell_2^+)$) is a reflection of the fact that Equation (1) supports a front solution for precisely one value of c in the propagation regime discussed here. By contrast, the generalized notion of criticality introduced in [10] would imply the existence of an entire family of (pulled) fronts in the context of FKPP-type equations with a linear cut-off.

2.3. Intermediate region

The intermediate region, which is defined by $\varepsilon < u < O(1)$ in (6), provides the connection between the outer and inner regions; the dynamics in this region is governed by

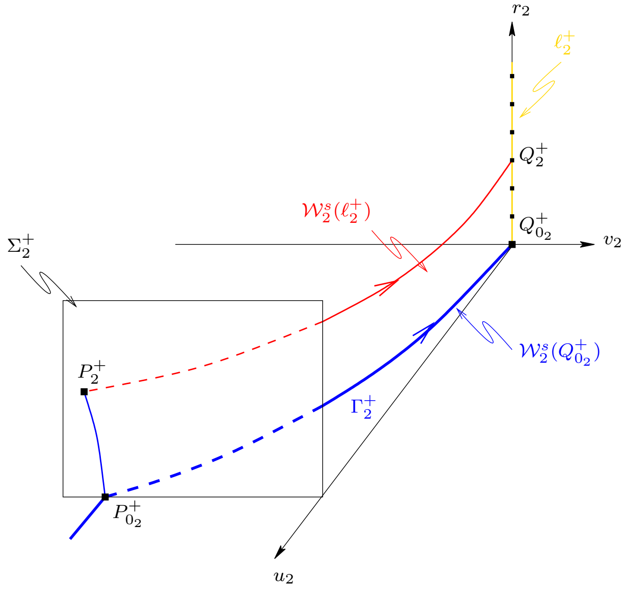

Figure 1: Geometry in chart K_2.

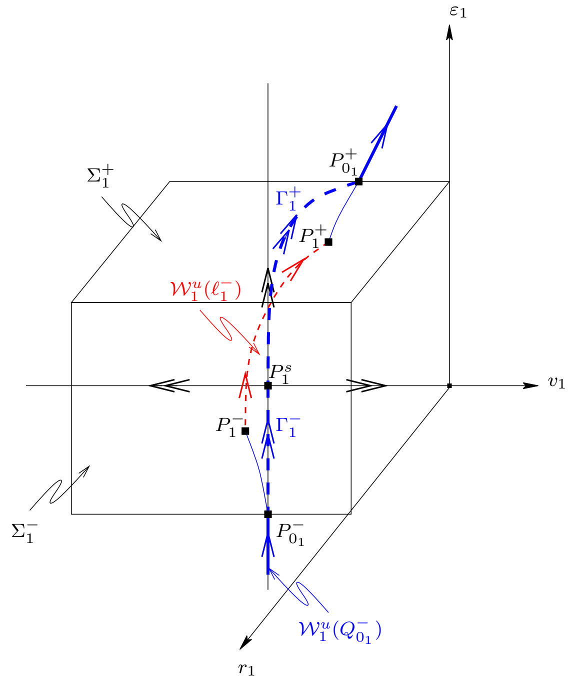

Figure 2: Geometry in chart K_1.

Equation (2), as in Section 2.1. For u small, that dynamics is conveniently studied in chart K_1: substituting the corresponding coordinates from (11) into (8), we find

$$r_1' = r_1 v_1, \tag{24a}$$
$$v_1' = -cv_1 - v_1^2 - (1 - r_1)(r_1 - \tfrac{1}{2}), \tag{24b}$$
$$\varepsilon_1' = -\varepsilon_1 v_1. \tag{24c}$$

Since $c = c_0(= 0)$ in (24) for $\varepsilon(= r_1\varepsilon_1) = 0$, the relevant two equilibria of these equations are located at $P_1^s = (0, -\frac{1}{\sqrt{2}}, 0)$ and $P_1^u = (0, \frac{1}{\sqrt{2}}, 0)$; both are hyperbolic saddle points, with eigenvalues $-\frac{1}{\sqrt{2}}$, $\sqrt{2}$, and $\frac{1}{\sqrt{2}}$; and $\frac{1}{\sqrt{2}}$, $-\sqrt{2}$, and $-\frac{1}{\sqrt{2}}$, respectively. In particular, we note the occurrence of a $(2, -1)$-resonance at both points; that resonance will generically give rise to logarithmic 'switchback' terms (in ε) in the transition through the intermediate region. (For a discussion of the switchback phenomenon from a geometric point of view, see *e.g.* [20] and the references therein.)

Remark 8. The transformation to chart K_1 is, in fact, a projectivization of the vector field in (8), since (11) implies $v_1 = \frac{v}{u} (= \frac{u'}{u})$. Consequently, the equilibria at P_1^s and P_1^u correspond to the stable and unstable eigendirections, respectively, of the linearization at the hyperbolic saddle point Q_0^+ of the first-order system that is equivalent to Equation (2). To state it differently, the decay behavior at the zero rest state in the absence of a cut-off is recovered in the transition between the inner and outer regions.

The dynamics of (24) in the singular limit as $\varepsilon \to 0^+$ is naturally described in the two (invariant) hyperplanes $\{\varepsilon_1 = 0\}$ and $\{r_1 = 0\}$. Hence, the portion Γ_1 of the singular orbit Γ that lies in the intermediate region is given as the union of two orbits Γ_1^- and Γ_1^+ that correspond to solutions of the respective limiting systems.

We first transform the sections Σ^- and Σ_2^+, as defined in (19) and (23), respectively, to chart K_1:

$$\Sigma_1^- = \{(\rho, v_1, \varepsilon_1) \,|\, (v_1, \varepsilon_1) \in [-v_0, 0] \times [0, 1]\}, \tag{25a}$$
$$\Sigma_1^+ = \{(r_1, v_1, 1) \,|\, (r_1, v_1) \in [0, \rho] \times [-v_0, 0]\}. \tag{25b}$$

Correspondingly, Σ_1^- represents a natural boundary between the outer and intermediate regions, while Σ_1^+ separates the intermediate from the inner region; see Figure 2.

Now, the orbit Γ_1^-, which is located in $\{\varepsilon_1 = 0\}$, can be found by observing that (24) is equivalent to (2) (the traveling wave equation without cut-off) as $\varepsilon_1 \to 0^+$. Recalling that a heteroclinic orbit is known explicitly for the corresponding first-order system, see (16), we have $v_1(r_1) = \frac{1}{\sqrt{2}}(r_1 - 1)$ for Γ_1^-, after transformation to K_1. Since, moreover, $\Gamma_1^- \to P_1^s$ for $r_1 \to 0^+$, that point is the equilibrium of (24) that is relevant to us here. Finally, we denote the point of intersection of Γ_1^- with Σ_1^- by $P_{0_1}^- = (\rho, v_{0_1}^-, 0)$, where $v_{0_1}^- = \frac{1}{\sqrt{2}}(\rho - 1)$, by the above.

To obtain Γ_1^+, we solve (24) in $\{r_1 = 0\}$: dividing (24b) (formally) by (24c) and taking the limit as $r_1 \to 0^+$, we find

$$\frac{dv_1}{d\varepsilon_1} = \frac{v_1^2 - \frac{1}{2}}{\varepsilon_1 v_1}. \tag{26}$$

The general solution to (26) (with v_1 negative) is given by

$$v_1(\varepsilon_1) = -\frac{1}{\sqrt{2}}\sqrt{1 + \alpha\varepsilon_1^2}, \tag{27}$$

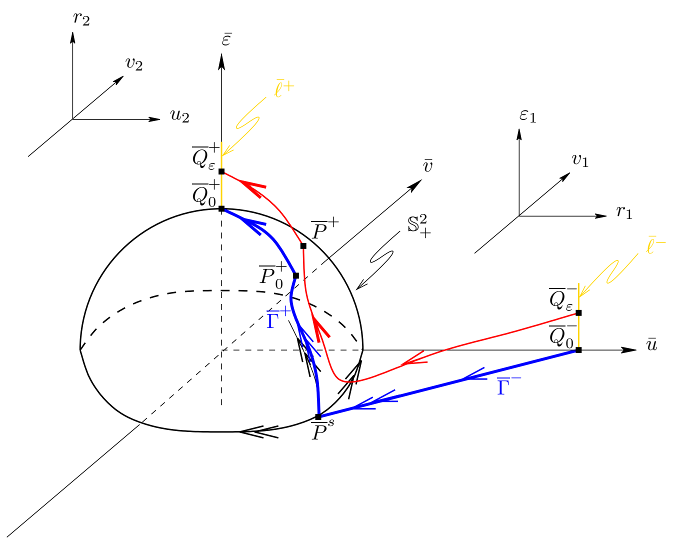

Figure 3: Global geometry of the blown-up vector field.

which is asymptotic to P_1^s as $\varepsilon_1 \to 0^+$, irrespective of the value of the constant α. That constant is uniquely determined by the requirement that the point of intersection of Γ_1^+ with Σ_1^+, which we denote by $P_{0_1}^+ = (0, v_{0_1}^+, 1)$, has to correspond to $P_{0_2}^+$ (after transformation to chart K_1). Evaluating (27) in Σ_1^+ and noting that $v_{0_1}^+ = -\frac{1}{\sqrt{3}} = v_{0_2}^+$, by (12), we find $\alpha = -\frac{1}{3}$, which completes the construction of Γ_1.

The geometry in chart K_1 is summarized in Figure 2.

3. Existence and uniqueness of Δc

In this section, we prove the existence of a unique function $\Delta c = \Delta c(\varepsilon)$ such that the singular orbit Γ persists, for ε positive and sufficiently small, as a heteroclinic connection between Q_ε^- and Q_ε^+ in (8). Since the persistent heteroclinic corresponds to a front solution of the cut-off Nagumo equation in (4), that equation will support a propagating front for precisely one value of c. The proof is based on a transversality argument that will merely be outlined here; for details, the reader is referred to [10, Proposition 3.1] or to [4, Proposition 3.1], where a similar argument was applied in the context of the cut-off FKPP and Zeldovich equations, respectively.

Proposition 2. *For $\varepsilon \in [0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, there exists a unique $\Delta c(\varepsilon)$ such that there is a heteroclinic connection between Q_ε^- and Q_ε^+ for $c = \Delta c$ in (8). Moreover, $\Delta c(0) = 0$, while $\Delta c(\varepsilon) \gtrsim 0$ (i.e., $\Delta c \sim 0$ as well as $\Delta c > 0$) when $\varepsilon \in (0, \varepsilon_0]$.*

Proof. The discussion in Section 2 implies the existence of a singular heteroclinic connection Γ for $(c, \varepsilon) = (0, 0)$ in (8): $\overline{\Gamma}$ is defined (in blown-up phase space) as the union of the orbits $\overline{\Gamma}^-$ and $\overline{\Gamma}^+$ and of the singularities $\overline{Q}_0^-$, $\overline{P}^s$, and $\overline{Q}_0^+$; see Figure 3. Hence, $\Delta c(0) = 0$, and we only need to consider ε positive here.

Now, given the analyticity of $\mathcal{W}_2^s(\ell_2^+)$, cf. Section 2.2, it follows that the intersection of that manifold with Σ_2^+ can be written as the graph of an analytic function $\psi^+(c, \varepsilon)$, with $\frac{\partial \psi^+}{\partial c} < 0$. Thus, for ε sufficiently small, the intersection of the stable manifold $\mathcal{W}^s(Q_\varepsilon^+)$ of Q_ε^+ with $\{u = \varepsilon\}$ is given by $\Psi^+(c, \varepsilon) = \varepsilon \psi^+(c, \varepsilon)$, after blow-down. Since, moreover, $\psi^+(0, 0) = -\frac{1}{\sqrt{3}}$ (recall the definition of $P_{0_2}^+$ in Section 2.2), we certainly have $\Psi^+(c, \varepsilon) < -\frac{\varepsilon}{3}$ for $c \gtrsim 0$.

Similarly, the intersection of $\mathcal{W}^u(Q_\varepsilon^-)$ (the unstable manifold of Q_ε^-) with $\{u = \varepsilon\}$ can be represented as the graph of an analytic function $\Psi^-(c, \varepsilon)$, with $\frac{\partial \Psi^-}{\partial c} > 0$. Since a standard phase plane argument shows that $\Psi^-(c, \varepsilon)$ is $O(1)$ and positive for $c \gtrsim 0$, it follows from the above that $\Psi^- > \Psi^+$ in that case.

Finally, for $c = 0$, the integrability of (20) implies $\Psi^+(0, \varepsilon) = -\frac{\varepsilon}{\sqrt{3}}\sqrt{1 - \frac{9}{4}\varepsilon + \frac{6}{5}\varepsilon^2}$, whereas $\Psi^-(0, \varepsilon) = \frac{1}{\sqrt{2}}\varepsilon$ $(\varepsilon - 1)$, recall (16). Since one can show that $\Psi^+ > \Psi^-$ then, we conclude that $\mathcal{W}^u(Q_\varepsilon^-)$ and $\mathcal{W}^s(Q_\varepsilon^+)$ must coincide in $\{u = \varepsilon\}$ for some positive value of c, which we denote by $\Delta c(\varepsilon)$. Finally, the uniqueness of Δc follows from $\frac{\partial \Psi^+}{\partial c} < 0$ and $\frac{\partial \Psi^-}{\partial c} > 0$ for $c \gtrsim 0$ and ε sufficiently small. $\square$

4. Leading-order asymptotics of Δc

Given the result of Proposition 2, it remains to determine the leading-order ε-asymptotics of Δc, as stated in Proposition 1. The corresponding proof relies on a detailed analysis of the transition through the intermediate region defined in Section 2.3: specifically, Δc is determined by the (global) condition that P^- (the point of intersection of the unstable manifold $\mathcal{W}^u(Q_\varepsilon^-)$ of Q_ε^- with Σ^-) must necessarily be mapped onto P_2^+ (the point of intersection of the stable manifold $\mathcal{W}_2^s(Q_2^+)$ of Q_2^+ with Σ_2^+), for any $\varepsilon \in (0, \varepsilon_0]$. The argument is performed entirely in chart K_1; correspondingly, for c and ε sufficiently small, we denote the equivalents of P^- and P_2^+ in $(r_1, v_1, \varepsilon_1)$-coordinates by $P_1^- = (\rho, v_1^-, \frac{\varepsilon}{\rho})$ and $P_1^+ = (\varepsilon, v_1^+, 1)$, respectively; cf. (11) and (12).

4.1. Preparatory analysis

In a first step, we translate the point P_1^s to the origin by introducing the new variable $z = v_1 + \frac{1}{\sqrt{2}}$ in (24); moreover, we replace c with Δc, where we recall that, necessarily, $\lim_{\varepsilon \to 0^+} \Delta c = 0$, as in Section 2.3. With these transformations, we find

$$r_1' = -(\tfrac{1}{\sqrt{2}} - z) r_1, \tag{28a}$$

$$z' = (\tfrac{1}{\sqrt{2}} - z)\Delta c + (\sqrt{2} - z) z - \tfrac{3}{2} r_1 + r_1^2, \tag{28b}$$

$$\varepsilon_1' = (\tfrac{1}{\sqrt{2}} - z)\varepsilon_1. \tag{28c}$$

Dividing out a factor of $\frac{1}{\sqrt{2}} - z$ (which is positive) from the right-hand sides in (28), we obtain

$$r_1' = -r_1, \tag{29a}$$

$$z' = \Delta c + 2\frac{1-\frac{z}{\sqrt{2}}}{1-\sqrt{2}z}z - \frac{3}{\sqrt{2}}\frac{1-\frac{2}{3}r_1}{1-\sqrt{2}z}r_1, \tag{29b}$$

$$\varepsilon_1' = \varepsilon_1. \tag{29c}$$

This last transformation corresponds to a rescaling of ξ that leaves the phase portrait of (28) unchanged. (Correspondingly, the prime now denotes differentiation with respect to a new independent variable ζ; without loss of generality, we assume $\zeta^- = 0$ in Σ_1^-.)

Next, we simplify the equations in (29) via a sequence of (near-identity) normal form transformations, removing all but the resonant r_1-dependent terms from (29b):

Proposition 3. *Let $\mathcal{V} = \{(r_1, z, \varepsilon_1) \,|\, (r_1, z, \varepsilon_1) \in [0, \rho] \times [0, z_0] \times [0, 1]\}$, for ρ positive and sufficiently small and $z_0 \in (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}})$. Then, there exists a sequence of $\mathcal{C}^\infty$-smooth coordinate transformations on $\mathcal{V}$, with $(r_1, z, \varepsilon_1) \mapsto (r_1, \hat{z}, \varepsilon_1)$, such that (29) can be written as*

$$r_1' = -r_1, \tag{30a}$$

$$\hat{z}' = \Delta c + 2\frac{1-\frac{\hat{z}}{\sqrt{2}}}{1-\sqrt{2}\hat{z}}\hat{z} + K r_1^2\hat{z}^2[1 + O(r_1^2\hat{z})], \tag{30b}$$

$$\varepsilon_1' = \varepsilon_1. \tag{30c}$$

(Here, $O(r_1^2\hat{z})$ denotes terms that are $\mathcal{C}^\infty$-smooth in $r_1^2\hat{z}$ and powers thereof, and K is a computable constant.)

Proof. We first expand the second term on the right-hand side in (29b), taking into account that v and, hence, v_1 is negative in the regime considered here, which implies $|z| < \frac{1}{\sqrt{2}}$ for ρ sufficiently small:

$$z' = \Delta c + 2\frac{1-\frac{z}{\sqrt{2}}}{1-\sqrt{2}z}z - \frac{3}{\sqrt{2}}(1 - \tfrac{2}{3}r_1)r_1 \times \left[1 + \sqrt{2}z + (\sqrt{2}z)^2 + \dots\right]. \tag{31}$$

The result now follows from standard normal form theory [14]; in particular, all non-resonant terms in (31) can successively be removed via a sequence of $\mathcal{C}^\infty$-smooth near-identity transformations of the form

$$z \mapsto z + h_{10}r_1 \mapsto z + h_{20}r_1^2 \mapsto z + h_{11}r_1 z \mapsto \dots$$

(with Δc-dependent coefficients h_{jk}) that leave r_1 and ε_1 unchanged. The lowest-order term that cannot be eliminated in this manner is the resonant $O(r_1^2 z^2)$-term; in general, any resonant term in (31) has to be of the form $r_1^{2k}z^{k+1}$, with $k \geq 1$. Hence, (29b) can be transformed into (30b), as stated, which completes the proof. □

Remark 9. Since (29b) is independent of ε_1, the sequence of transformations defined in Proposition 3 can only depend on r_1 and z, as well as on the parameter Δc. (The ε_1-dependence in Δc enters through the product $\varepsilon = r_1\varepsilon_1$, which is constant.) In particular, it follows from the above that $\hat{z} = z + O(r_1)$.

Remark 10. The restrictions on z_0 in the statement of Proposition 3 are motivated in part by the definition of $P_{0_2}^+$: since $v_{0_2}^+ = -\frac{1}{\sqrt{3}}$, see Section 2.2, we have $z_{0_2}^+ = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$ for the corresponding z-value; hence, we may assume $z_0 > \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$ for Δc and ε sufficiently small. By contrast, the requirement that $z_0 < \frac{1}{\sqrt{2}}$ is necessitated by the fact that the vector field in (29) becomes undefined as $z \to \frac{1}{\sqrt{2}}^-$; see also the discussion in Section 5 below.

Finally, we denote by $\widehat{P}_1^-$ and $\widehat{P}_1^+$ the two points which correspond to $P_1^- \in \Sigma_1^-$ and $P_1^+ \in \Sigma_1^+$, respectively, after application of the sequence of near-identity transformations from Proposition 3.

4.2. Estimates for $\widehat{P}_1^-$ and $\widehat{P}_1^+$

We now derive estimates for the $\hat{z}$-coordinates of $\widehat{P}_1^-$ and $\widehat{P}_1^+$, as follows:

Lemma 2. *For $\rho \geq \varepsilon$, with $\varepsilon \in (0, \varepsilon_0]$ and Δc sufficiently small, the points $\widehat{P}_1^- = (\rho, \hat{z}^-, \frac{\varepsilon}{\rho})$ and $\widehat{P}_1^+ = (\varepsilon, \hat{z}^+, 1)$ satisfy*

$$\hat{z}^- = \hat{z}^-(\rho, \Delta c) = \nu(\rho, \Delta c)\Delta c, \tag{32}$$

with

$$\nu(\rho, 0) = \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, 0)[1 + O(\rho)] > 0, \tag{33}$$

and

$$\hat{z}^+ = \hat{z}^+(\Delta c, \varepsilon) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \omega(\Delta c, \varepsilon), \tag{34}$$

respectively. Here, ν and ω are $\mathcal{C}^\infty$-smooth functions in their respective arguments; moreover, $\omega(0, 0) = 0$.

Remark 11. Strictly speaking, the $O(\rho)$-terms in (33) are smooth down to $\rho = 0$, whereas ν itself becomes unbounded there. Similarly, the function ω is smooth in a full neighborhood of $(0, 0)$. We refer the reader to [3, Remark 9] for a detailed discussion of these and similar issues.

Proof. The argument follows the proof of [3, Lemma 2.2], to which the reader is referred for details; see also [4, Lemma 3.5].

To obtain the estimate for $\hat{z}^-$, we first evaluate the expansion for $v(u, c)$ from (15) in Σ^-, taking into account that $v^- = v(\rho, 0) = \frac{1}{\sqrt{2}}\rho(\rho - 1)$, by (16). Transforming the result to chart K_1 via $v_1^- = \frac{v^-}{\rho}$, see (11), and recalling the definition of z, we find

$$z^- = v_1^- + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\rho + \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, 0)\Delta c + O[(\Delta c)^2]. \tag{35}$$

(Here, the $O[(\Delta c)^2]$-terms remain $\mathcal{C}^\infty$-smooth as long as ρ is bounded away from zero.) Finally, applying the sequence of $\mathcal{C}^\infty$-smooth normal form transformations from Proposition 3 to (35), we obtain (32), as claimed; in particular, the absence of any $O(\rho)$-terms in $\hat{z}^-$, *i.e.*, the fact that $\hat{z}^- = O(\Delta c)$ for ρ positive, is a consequence of the invariance of $\hat{z}^- = 0$ for $\Delta c = 0$ in (30b).

The estimate for $\hat{z}^+$ is found in a similar fashion: we begin by noting that P_1^+ must necessarily equal P_2^+ (after transformation to chart K_1) for the singular heteroclinic orbit Γ to persist when $\varepsilon \in (0, \varepsilon_0]$. Since $v_2^+ = -\frac{1}{\sqrt{3}} + o(1)$, cf. Section 2.2, and since $v_2^+ = v_1^+$, recall (12), we have $z^+ = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + o(1)$, where $o(1)$ denotes $\mathcal{C}^\infty$-smooth terms of at least order 1 in Δc and ε. Hence, it follows that (34) is satisfied, in $(r_1, \hat{z}, \varepsilon_1)$-coordinates, for some $\mathcal{C}^\infty$-smooth function ω, as claimed, which completes the proof. □

4.3. Normal form approximation

Let $\check{z}(\zeta)$ be defined as the solution of the simplified normal form equation that is obtained by omitting the higher-order (r_1-dependent) terms in (30b):

$$\check{z}' = \Delta c + 2\frac{1 - \frac{\check{z}}{\sqrt{2}}}{1 - \sqrt{2}\check{z}}\check{z}. \tag{36}$$

We now show that the approximation provided by (36) is sufficiently accurate to the order considered here. Let $\hat{z}_-$ and $\check{z}_-$ denote the (unique) solutions to (30b) and (36), respectively, for which $\hat{z}(0) = \hat{z}^- = \check{z}(0)$. Moreover, let $\hat{z}_-^+ = \hat{z}_-(\zeta^+)$ and $\check{z}_-^+ = \check{z}_-(\zeta^+)$, where $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$ is the value of ζ in Σ_1^+. (Here, ζ^+ can *e.g.* be found from $r_1(\zeta) = \rho e^{-\zeta}$, in combination with $r_1(\zeta^+) = \varepsilon$; recall (25) and (30a).)

Proposition 4. *For $\hat{z}_-^+$ and $\check{z}_-^+$ defined as above and $\varepsilon \in (0, \varepsilon_0]$, there holds*

$$|\hat{z}_-^+ - \check{z}_-^+| = O(\varepsilon). \tag{37}$$

Proof. The proof is based on a variant of Gronwall's Lemma; see the proof of [4, Proposition 3.3] for details:

Lemma 3 (Gronwall's Lemma). *Let $\mathcal{U}$ be an open set in $\mathbb{R}$, let $f, g : [0,T] \times \mathcal{U} \to \mathbb{R}$ be continuous, and let $x(t)$ and $y(t)$ be solutions of the initial value problems $x'(t) = f(t, x(t))$, with $x(0) = x_0$, and $y'(t) = g(t, x(t))$, with $y(0) = y_0$, respectively. Assume that there exists $C \geq 0$ such that*

$$|g(t, y_2) - g(t, y_1)| \leq C|y_2 - y_1|; \tag{38}$$

moreover, let $\varphi : [0,T] \to \mathbb{R}^+$ be a continuous function, with

$$|f(t, x(t)) - g(t, x(t))| \leq \varphi(t). \tag{39}$$

Then, there holds

$$|x(t) - y(t)| \leq e^{Ct}|x_0 - y_0| + e^{Ct}\int_0^t e^{-C\tau}\varphi(\tau)\, d\tau \tag{40}$$

for $t \in [0,T]$.

Setting $t \equiv \zeta$, $x \equiv \hat{z}$, and $y \equiv \check{z}$ and denoting the right-hand side in (36) by g, we find

$$\begin{aligned} &|g(\zeta, \check{z}_2) - g(\zeta, \check{z}_1)| \\ &\qquad = |\check{z}_2 - \check{z}_1|\Big|1 + \frac{1}{1 - \sqrt{2}(\check{z}_2 + \check{z}_1) + 2\check{z}_2\check{z}_1}\Big|. \end{aligned} \tag{41}$$

Since the last term in (41) is monotonically increasing in $\check{z}_j$ $(j = 1, 2)$, it follows from (34) that $1 - \sqrt{2}(\check{z}_2 + \check{z}_1) + 2\check{z}_2\check{z}_1 \gtrsim \frac{2}{3}$, for $\check{z}_j \in [\hat{z}^-, \hat{z}^+]$ and Δc and ε sufficiently small. Hence, (38) is certainly satisfied with $C = 3$.

Similarly, writing f for the right-hand side in (30b), we have

$$|f(\zeta, \hat{z}) - g(\zeta, \hat{z})| = \big|K r_1^2 \hat{z}^2[1 + O(r_1^2\hat{z})]\big| \leq 2|K|\frac{|r_1^4\hat{z}^2|}{r_1^2},$$

for $r_1^2\hat{z} \in [\rho^2\hat{z}^-, \varepsilon^2\hat{z}^+]$ sufficiently small. Now, we note that $|r_1^2\hat{z}| = r_1^2\hat{z}$, since $\hat{z}$ is non-negative for $\zeta \in [0, \zeta^+]$, by Lemma 2 and Proposition 2. Therefore,

$$|r_1^2\hat{z}|' = r_1^2\Delta c + \frac{\sqrt{2}\hat{z}}{1 - \sqrt{2}\hat{z}}|r_1^2\hat{z}|[1 + O(|r_1^2\hat{z}|)] \geq 0,$$

and it follows that $|r_1^2\hat{z}| \leq |r_1^2\hat{z}|(\zeta^+) = \varepsilon^2\hat{z}^+ \leq \frac{1}{\sqrt{2}}\varepsilon^2$, see again Lemma 2. In sum, we find that (39) holds with $\varphi(\zeta) = |K|\varepsilon^4[r_1(\zeta)]^{-2} = |K|\varepsilon^4\rho^{-2}e^{2\zeta}$, which, by (40), implies

$$|\hat{z}_-(\zeta) - \check{z}_-(\zeta)| \leq e^{3\zeta}\int_0^\zeta |K|\frac{\varepsilon^4}{\rho^2}e^{-s}\, ds \leq |K|\frac{\varepsilon^4}{\rho^2}e^{3\zeta}; \tag{42}$$

recall the definition of $\hat{z}_-$ and $\check{z}_-$ above. In particular, evaluating (42) at $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$, we obtain (37), as claimed, which concludes the argument. □

Remark 12. The estimate in (37), while sufficiently accurate for our purposes, is most probably not optimal: preliminary analysis suggests that $K = O(\Delta c) (= o(1))$ in (30b), which would contribute an additional factor of Δc in (42) and, hence, in (37).

4.4. End of proof of Proposition 1

Finally, we derive a necessary condition for the existence of a connection between the two points $\widehat{P}_1^-$ and $\widehat{P}_1^+$ (*i.e.*, for the persistence of the singular heteroclinic Γ), for ε positive and sufficiently small. That condition will determine $\Delta c(\varepsilon)$ to lowest order in ε, which will complete the proof of Proposition 1.

Proposition 5. *For the singular heteroclinic orbit Γ to persist when $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, $\Delta c(\varepsilon)$, as defined in Proposition 2, must necessarily satisfy*

$$\Delta c = \frac{1}{\sqrt{2}}\varepsilon^2 + o(\varepsilon^2), \tag{43}$$

to leading order in ε.

Proof. First, we note that the estimate in (37) is uniform in Δc, as well as that the leading-order Δc-dependence of (30b) equals that of (36). Hence, the variation of $\hat{z}_-^+$ with respect to Δc is encapsulated in $\check{z}_-^+$, to lowest order; in other words, it suffices to consider the approximate normal form equation in (36) in order to determine the leading-order asymptotics of Δc.

Now, (36) can be integrated by separation of variables, which gives

$$\zeta - \frac{1}{2}\ln\left|2\check{z}^2 - 2(\sqrt{2}-\Delta c)\check{z} - \sqrt{2}\Delta c\right| - \frac{\Delta c}{\sqrt{2+(\Delta c)^2}}\operatorname{arctanh}\left(\frac{2\check{z}-\sqrt{2}+\Delta c}{\sqrt{2+(\Delta c)^2}}\right) \equiv \text{constant.} \tag{44}$$

For Γ to persist, $\widehat{P}_1^-$ must be mapped onto $\widehat{P}_1^+$ in the transition through the intermediate region; equivalently, we need to impose $\check{z}_-(\zeta^+) = \hat{z}^+$. (Here, $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$, and $\check{z}_-$ denotes the solution to (36) with $\check{z}(0) = \hat{z}^-$, as before.) Substituting the estimates for $\hat{z}^-$ and $\hat{z}^+$ from (32) and (34), respectively, into (44), rewriting the hyperbolic arctangent via $\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$, and expanding the resulting expression in Δc and ε, we find

$$-\ln\frac{\varepsilon}{\rho} - \frac{1}{2}\ln\left|\tfrac{1}{3}+o(1)\right| + \frac{1}{2}\ln\left|\sqrt{2}[1+2\nu(\rho,0)+o(1)]\Delta c\right| - \frac{\Delta c}{2\sqrt{2}}[1+o(1)]\left\{\ln\left|\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}+o(1)\right| - \ln\left|\left[\frac{1+2\nu(\rho,0)}{2\sqrt{2}}+o(1)\right]\Delta c\right|\right\} = 0, \tag{45}$$

where $o(1)$ again denotes terms of at least order 1 in Δc and ε that are $\mathcal{C}^\infty$-smooth for ρ bounded away from zero.

Exponentiating (45) and recalling that both $\nu(\rho,0)$ and Δc are positive, by Lemma 2 and Proposition 2, respectively, we obtain

$$\Delta c = \frac{1}{3\sqrt{2}}\frac{1}{[1+2\nu(\rho,0)]\rho^2}\varepsilon^2[1+o(1)]; \tag{46}$$

here, the $o(1)$-terms are now $\mathcal{C}^\infty$-smooth in Δc, $\Delta c\ln(\Delta c)$, and ε. Since the relation in (46) is clearly satisfied at $(\Delta c,\varepsilon) = (0,0)$ and since, moreover, $3\sqrt{2}[1+2\nu(\rho,0)]\rho^2 > 0$, it follows from the Implicit Function Theorem that (46) has a solution $\Delta c = \Delta c(\varepsilon,\rho)$ for Δc and ε sufficiently small, with $\lim_{\varepsilon\to 0^+}\Delta c = 0 (= c_0)$.

Finally, we recall that Δc gives precisely the c-value for which the singular orbit Γ persists as a heteroclinic connection between Q_ε^- and Q_ε^+ in (8), after blow-down. Hence, Δc has to be independent of ρ, *i.e.*, of the (arbitrary) definition of Σ^- in (19), and we may take the limit as $\rho \to 0^+$ in (46). Since $\rho^2\nu(\rho,0) = \rho\frac{\partial v}{\partial c}(\rho,0) = \frac{1}{6}+O(\rho)$, by (18), we obtain (43), as claimed, which completes the proof. □

We note that the leading-order expansion in (43) implies at least $\mathcal{C}^2$-smoothness of Δc in $\varepsilon \in [0,\varepsilon_0]$, which cannot, however, be deduced from the proof of Proposition 5 given here.

Remark 13. As observed already in [3] for $\gamma \in (0,\frac{1}{2})$ in (1), *i.e.*, in the bistable propagation regime, the leading-order exponent in the ε-asymptotics of Δc is given by the ratio of the two eigenvalues $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$ of the linearization of (24) at P_1^s.

Remark 14. Intuitively, the result of Proposition 5 can be seen by considering the linearized dynamics of (29b): with $r_1(\zeta) = \rho e^{-\zeta}$, one finds $z' = \Delta c + 2z - \frac{3}{\sqrt{2}}\rho e^{-\zeta}$. Solving this equation, with $z(0) = z^-$ as defined in (35), and noting that $z(\zeta^+)$ must equal z^+, one obtains $\Delta c = O(\varepsilon^2)$ (to leading order), as required. However, the corresponding coefficient does not agree with (43): not unexpectedly, the dynamics of (29b) is not captured by its small-z approximation, as z varies by $O(1)$ in the transition through the intermediate region; recall Lemma 2.

5. Discussion

In this article, we have discussed front propagation in the Nagumo equation at a Maxwell point, with $\gamma = \frac{1}{2}$ in (1), in the presence of a linear cut-off at the zero rest state. Since the front propagation speed c_0 that is realized in the absence of a cut-off reduces to zero in that case, the corresponding front solution is stationary; however, the correction to c_0 that is due to the cut-off is positive, cf. Proposition 2. Hence, the cut-off Equation (4) supports front solutions that propagate with positive speed Δc, as observed also in [3] in the classical bistable regime [1], with $\gamma \in (0,\frac{1}{2})$. (The other 'boundary' case, where $\gamma = 0$ in (1), yields the Zeldovich equation [16], which was studied in detail in [4].) Here, we have given a geometric proof for the existence and uniqueness of these solutions, and we have determined the asymptotics of Δc in terms of the cut-off parameter ε, to lowest order; recall Proposition 1. In particular, we have shown how the inherent non-smoothness of the cut-off dynamics can be resolved, and the non-hyperbolic zero state of (4) desingularized, in the framework of geometric singular perturbation theory [17] and 'blow-up' [11].

The calculation of the leading-order coefficient in the expansion for Δc in (7) (and, in particular, of the estimate for $\hat{z}^-$ derived in Lemma 2) required explicit knowledge of the traveling front solution in (3), as well as of the lowest-order variation along the equivalent orbit of (8) with respect to c. We remark that the solution of the associated variational equation in (17) is simpler than the corresponding expression found for $\gamma \in (0,\frac{1}{2})$ in [3, Lemma 2.1]; recall Lemma 1. (Similarly, the solution simplifies substantially when $\gamma = 0$ in (1); see [4, Lemma 4.3].) A more general discussion of these and related issues can be found in [3, 4], where it was shown that a front solution to Equation (1) (in the absence of a cut-off) must necessarily be known for Δc to be computable analytically.

Our analysis of the dynamics of (4) was complicated by the occurrence of resonances between the eigenvalues of the linearized dynamics in the phase-directional chart K_1 that was introduced to describe the blown-up vector field in the intermediate region; see Section 2.3. This resonance phenomenon was also observed for $\gamma = 0$ in (1), cf. [4], which necessitated an approximation of the corresponding normal form equations, in analogy to the one performed in Proposition 4. In both cases, the contribution from the r_1-dependent terms in the normal form was proven to be of higher order when compared to the z-dependent terms alone. Notably, these resonances are destroyed for $\gamma \in (0, \frac{1}{2})$: considering the dynamics of Equation (2) in that regime, as was done in [3], one finds that the eigenvalues of the linearization at the corresponding two equilibrium points $P_1^s = (0, -\frac{1}{\sqrt{2}}, 0)$ and $P_1^u = (0, \sqrt{2}\gamma, 0)$ in chart K_1 are given by $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}(1+2\gamma)$, and $\frac{1}{\sqrt{2}}$; and $\sqrt{2}\gamma$, $-\frac{1}{\sqrt{2}}(1+2\gamma)$, and $-\sqrt{2}\gamma$, respectively. Consequently, the leading-order normal form approximation in (36) becomes exact then, as shown in [3, Proposition 2.1].

While the effects of a cut-off on propagating fronts in reaction-diffusion systems of the type in (1) have traditionally been studied in the context of the Heaviside cut-off, the case considered in this article provides an example of a system in which the leading-order correction to the front propagation speed induced by a linear cut-off can be determined in closed form.

In fact, the analysis presented here remains valid for any choice of cut-off function Θ in (4) for which $\psi^+(0,0)$ (the v_2-coordinate of the point of intersection of the singular orbit Γ_2^+ with Σ_2^+) can be restricted to a closed subset of $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$: one readily verifies that the proof of Proposition 4 carries over verbatim then; similarly, the proof of Proposition 2 can easily be adapted to show the existence and uniqueness of $\Delta c(\varepsilon)$. Even in cases where the limiting equations that are obtained for $r_2 \to 0^+$ in chart K_2 cannot be solved exactly (*i.e.*, even when the portion of Γ that is located in the inner region is not known in closed form), Γ_2^+ can be defined via the zero level curve of the corresponding constant of motion; recall Remark 5. It then follows from regular perturbation theory that v_2^+ (the v_2-coordinate of the point of intersection of $\mathcal{W}_2^s(Q_2^+)$ with Σ_2^+) will lie in $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$, for c and r_2 positive and small. (The singular dynamics that defines Γ_1^+ in the phase-directional chart K_1 remains unchanged, and is still governed by (26), which implies that the unique solution in $\{r_1 = 0\}$ is given by (27), as before.)

However, our analysis does not necessarily carry over to the case where Θ is the Heaviside cut-off considered *e.g.* in [3]: somewhat surprisingly, that case seems to be more complex dynamically than the linear cut-off studied here. First, the portion of Γ that is located in K_2 is now given by a segment of the u_2-axis, with $u_2 \in [0,1]$, which is a line of equilibria for the system of equations that governs the corresponding singular dynamics; in particular, we find $\psi^+(0,0) = 0$ for the point of intersection of Γ_2^+ with Σ_2^+. Still, regular perturbation theory implies that $\mathcal{W}_2^s(Q_2^+)$ can be written as $v_2(u_2) = -cu_2$, for c and r_2 sufficiently small. (The analysis in chart K_1 remains virtually unchanged compared to Section 2.3.)

The proof of Proposition 4 given above, however, certainly breaks down when $\psi^+(0,0) = 0$: as $v_2^+ \to 0$ or, equivalently, as $\hat{z}^+ \to \frac{1}{\sqrt{2}}$, the second term on the right-hand side in (30b) becomes unbounded. Consequently, the bound on (41) becomes non-uniform (in ε) in that limit, and the estimate in (37) cannot be deduced. The leading-order normal form in (36) can, of course, still be solved formally: reasoning along the lines of the proof of Proposition 4, one finds $\Delta c = \frac{3}{\sqrt{2}}\varepsilon^2 + o(\varepsilon^2)$. This asymptotics, which also seems to be supported by numerical evidence (data not shown), agrees with the (formal) limit as $\gamma \to \frac{1}{2}^-$ in [3, Theorem 2.1]; however, a rigorous proof is beyond the scope of this article. In particular, since the leading-order coefficient in the above expansion differs from the one given in (7), we conjecture that this coefficient will typically depend on the choice of cut-off function Θ in (4). The cut-off dependence of Δc was already observed in [3], for $\gamma \in (0, \frac{1}{2})$ in (1), and contrasts with the situation encountered in the study of the cut-off Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation in [5, 10], where the corresponding coefficient was universal.

While we have only determined the leading-order asymptotics of Δc here, we remark that the expansion in (7) can, in principle, be taken to arbitrary order, provided the sequence of normal form transformations defined in Proposition 3 is performed explicitly to the corresponding order. In particular, to find the lowest-order logarithmic (switchback) term that arises in the transition through the intermediate region, one would need to calculate the coefficient K in (30b): a heuristic argument suggests that the corresponding (resonant) $r_1^2\hat{z}^2$-term will translate into a term of the form $\rho^2(\hat{z}^-)^2\zeta e^{2\zeta}$ during that transition. With $\hat{z}^- = O(\Delta c)$, $\Delta c = O(\varepsilon^2)$, and $\zeta^+ = O(-\ln\varepsilon)$, it follows that the resulting term in the asymptotics of $\hat{z}_-^+$ will be of the order $O(\varepsilon^2 \ln\varepsilon)$ and, hence, that the expansion for Δc will, generically, also contain logarithmic terms in ε. For a rigorous proof, one would additionally have to refine the result of Proposition 4: the estimate in (37) was sufficiently accurate for our purposes, as $\hat{z}^+$ had only been estimated to leading order; cf. (34). For $\gamma = 0$ in (1) and Θ the Heaviside cut-off, the corresponding analysis was performed in [4]; in particular, the ε-asymptotics of Δc was determined explicitly up to and including the lowest-order logarithmic term there.

Finally, a prominent characteristic of the propagation regime discussed here is the integrability of Equation (2) for $\gamma = \frac{1}{2}$: the phase portrait of (8) in the absence of a cut-off is symmetric about the u-axis when $c_0 = 0$, in that the eigenvalues of the linearization at the two saddle equilibria at $Q_0^\mp$ are identical, while the third equilibrium point Q_0° is a center; recall Remark 3. (The symmetry is broken for $\gamma \in (0, \frac{1}{2})$; in particular, $c_0 > 0$ in that case, and the integrability is lost.) This integrable structure is also evident in the associated cut-off Equation (4); it manifests itself after blow-up, as the resulting systems in (20) and (24) that are obtained in charts K_2 and K_1, respectively, are both integrable for $c = 0$. (The dynamics in K_2 is Hamiltonian irrespective of r_2, as Θ does not introduce any v_2-dependence in (20b), while the integrability of (24) follows trivially whenever Equation (2) is integrable, since $\Theta \equiv 1$ in K_1.) Hence, it might be feasible (and, indeed, natural) to complement the approach developed here with techniques from the well-developed theory of integrable systems: variants of the classical Melnikov technique, for instance, have previously been applied successfully in the framework of geometric singular perturbation theory; see *e.g.* [13, 21] for details and references.

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