

# A GEOMETRIC ANALYSIS OF FRONT PROPAGATION IN A FAMILY OF DEGENERATE REACTION-DIFFUSION EQUATIONS WITH CUT-OFF

NIKOLA POPOVIĆ

ABSTRACT. We investigate the effects of a Heaviside cut-off on the dynamics of traveling fronts in a family of scalar reaction-diffusion equations with degenerate polynomial potential that includes the classical Zeldovich equation. We prove the existence and uniqueness of front solutions in the presence of the cut-off, and we derive the leading-order asymptotics of the corresponding propagation speed in terms of the cut-off parameter. For the Zeldovich equation, an explicit solution to the equation without cut-off is known, which allows us to calculate higher-order terms in the resulting expansion for the front speed; in particular, we prove the occurrence of logarithmic (switchback) terms in that case. Our analysis relies on geometric methods from dynamical systems theory and, in particular, on the desingularization technique known as ‘blow-up.’

## 1. INTRODUCTION

In this article, we are concerned with front propagation in the family of ‘cut-off’ scalar reaction-diffusion equations

$$(1.1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u)\Theta(u - \varepsilon),$$

where the potential  $f_m$  is given by  $f_m(u) = 2u^m(1 - u)$  for  $0 \leq u \leq 1$  and integer-valued  $m \geq 2$  and where the Heaviside cut-off function  $\Theta$  introduced at the zero rest state satisfies

$$(1.2) \quad \Theta(u - \varepsilon) \equiv 0 \quad \text{if } u < \varepsilon \quad \text{and} \quad \Theta(u - \varepsilon) \equiv 1 \quad \text{if } u > \varepsilon,$$

for  $\varepsilon > 0$  small. (In other words,  $\Theta$  deactivates the reaction terms in (1.1) in a neighborhood of  $u = 0$ .) Traveling front solutions that propagate between the two rest states at 1 and 0 in (1.1) are naturally studied by reverting to a co-moving frame: introducing the new variable  $\xi = x - ct$  and writing  $U(\xi) = u(t, x)$ , we find

$$(1.3) \quad U'' + cU' + f_m(U)\Theta(U - \varepsilon) = 0$$

for the traveling front equation corresponding to (1.1), where the front  $U$  has to satisfy

$$(1.4) \quad \lim_{\xi \rightarrow -\infty} U(\xi) = 1 \quad \text{and} \quad \lim_{\xi \rightarrow \infty} U(\xi) = 0.$$

Cut-offs were introduced by Brunet and Derrida in [12] to model fluctuations in propagating fronts that arise in the large-scale (or mean-field) limit of discrete  $N$ -particle systems, with  $\varepsilon = N^{-1}$ . A more detailed exposition of front propagation in reaction-diffusion equations in the presence of a cut-off can be found in [32] as well as in [37]; see also [17] and the references therein. Front propagation in the context of Equation (1.1) without cut-off, *i.e.*, of

$$(1.5) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u)$$

---

*Date:* March 7, 2011.

*2000 Mathematics Subject Classification.* Primary 35K57, 34E15; Secondary 34E05.

*Key words and phrases.* Reaction-diffusion, front propagation, cut-off, front speed, geometric desingularization, asymptotics.

and the corresponding traveling front equation

$$(1.6) \quad U'' + cU' + f_m(U) = 0,$$

is well-understood [4, 10, 11]; a comprehensive and more recent account is given in [21]. In particular, the regimes where  $m$  is close to 1 and 2 have been studied in detail, using geometric singular perturbation theory [16] and matched asymptotics [9, 30, 40], while the limit of  $m \rightarrow \infty$  in (1.5) (first considered in [31] as well as in [40] via the method of matched asymptotic expansions) was analyzed rigorously in [16]. Most importantly, it has been shown that, for any  $m \geq 1$ , there exists a ‘critical’ propagation speed  $c_{\text{crit}} > 0$  such that traveling front dynamics is observed for  $c \geq c_{\text{crit}}$  in (1.6). When  $m \geq 2$ ,  $c_{\text{crit}}$  characterizes the traveling front with the strongest (exponential in  $\xi$ ) decay at the zero rest state in (1.6), whereas fronts corresponding to  $c > c_{\text{crit}}$  decay at a weaker (algebraic) rate. Moreover, the propagation speed is selected by a global bifurcation in that case. For  $m = 1$ , (1.5) reduces to the classical Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation [20, 26]; see also [35] and the references therein. The relevant critical speed  $c_{\text{crit}} = 2\sqrt{2}$  is determined by a local transition condition that is obtained by linearization about the zero rest state; the decay is now merely algebro-exponential for  $c = c_{\text{crit}}$ , whereas it is exponential when  $c > c_{\text{crit}}$ .

Front propagation in the corresponding cut-off equation was first studied by Brunet and Derrida [12]; they found, in particular, that the leading-order correction to  $c_{\text{crit}}$  depends on the inverse square of the logarithm of the cut-off parameter  $\varepsilon$ , and they conjectured that the (negative) coefficient of that correction is independent of the choice of cut-off function in (1.1). Subsequently, Dumortier *et al.* [17] gave a rigorous proof of these findings, using a combination of standard phase space arguments and the geometric desingularization technique known as ‘blow-up’ [14, 27]. In addition to proving the existence and uniqueness of traveling front solutions for  $m = 1$  in (1.1), they also explained the asymptotic structure of the corresponding propagation speed, and they showed that this structure is universal within a very general family of cut-off functions. The case we consider, with  $m \geq 2$ , is more degenerate; in particular, since  $\frac{d^k f_m}{du^k}(0) = 0$  for  $k = 1, \dots, m - 1$  now, the classical existence analysis of Aronson and Weinberger [4] does not apply to (1.5). In that sense, our results complement those obtained in [17] for  $m = 1$ , where the geometric approach was pioneered in this context of critical front propagation in the presence of a cut-off.

Our analysis of (1.1) is partly motivated by results of Benguria *et al.* [8], who studied the effects of a cut-off on propagating fronts in the family of equations

$$(1.7) \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( u^{m-1} \frac{\partial u}{\partial x} \right) + f_1(u)$$

when  $m \geq 2$ ; see also [5, 6] and the references therein. We remark that the family in (1.7) represents one of the simplest ways of incorporating density-dependent diffusivities, while still allowing for a fairly explicit analysis of the resulting dynamics. General results on the existence and uniqueness of propagating fronts in equations of the type in (1.7) have *e.g.* been obtained in [25] and [38]. Although the front speed  $c$  cannot be determined from linear considerations in that case, one can show that the traveling front equation

$$(U^{m-1}U')' + cU' + f_1(U) = 0$$

corresponding to (1.7) is equivalent to (1.6) under the transformation  $\xi \mapsto \int_0^\xi [U(s)]^{1-m} ds$  [8]; cf. also [21, Theorem 3.1] or [35, Remark 12]. Since that equivalence extends to the respective cut-off equations, our analysis remains valid even when the diffusion coefficient in (1.1) is not constant, but obeys a power law (in  $u$ ).

We remark that the restriction to the Heaviside cut-off  $\Theta$  in (1.1) is made for the sake of exposition; in particular, since Equation (1.1) only permits traveling front solutions when  $c$  equals the

critical propagation speed in that case, as we will show in Section 2.4 below, we will for simplicity write  $c(\varepsilon)$  instead of  $c_{\text{crit}}(\varepsilon)$  throughout. Other choices of cut-off function can be analyzed in a similar fashion; cf. [17] for details. (However, we note that the leading-order asymptotics of the correction to the front propagation speed that is due to a cut-off will be cut-off dependent when  $m \geq 2$  in (1.1), in contrast to the universality of that asymptotics for  $m = 1$ , *i.e.*, in the cut-off FKPP equation [12, 17]; cf. also Section 3.2 below.) Furthermore, we will only consider integer-valued  $m \geq 2$  in the following; see Section 5 for an indication of how our approach can potentially be extended to cover non-integer exponents in (1.1).

The following theorem is the main result of this article:

**Theorem 1.1.** *Let  $m \geq 2$  in (1.1), with  $m$  integer, and let  $\varepsilon \in [0, \varepsilon_0]$ , for  $\varepsilon_0 > 0$  sufficiently small. Then, there exists a unique  $c(\varepsilon)$  such that Equation (1.3), subject to (1.4), supports a unique traveling front solution  $U(\xi)$  that propagates with speed  $c(\varepsilon)$ . Moreover,  $c(\varepsilon)$  satisfies  $c(\varepsilon) = c(0) + \Delta c(\varepsilon)$ , where  $c(0) > 0$  denotes the front propagation speed in (1.6) (the corresponding equation without cut-off) and where*

$$(1.8) \quad \Delta c(\varepsilon) = \gamma_m \varepsilon^m + o(\varepsilon^m),$$

with  $\gamma_m$  a negative constant.

The leading-order  $\varepsilon$ -asymptotics of  $\Delta c$  in (1.8) agrees with results obtained by Benguria *et al.* [8], in the context of (1.7). Moreover, it is argued in [8] that the corresponding coefficient – *i.e.*, the constant  $\gamma_m$ , in our notation – in the asymptotic expansion for  $c(\varepsilon)$  is computable for all values of  $m$  in (1.6) (the traveling front equation without cut-off) for which the function that maximizes a certain variational functional can be found in closed form. We will discuss that claim from a geometric point of view at the end of Section 3, where we will confirm that the leading-order coefficient  $\gamma_m$  in (1.8) cannot, in general, be evaluated analytically; see also the discussion in Section 5. However, the only  $m$ -value for which an exact solution exists seems to be 2, in which case (1.5) is also known as the Zeldovich equation; see *e.g.* [21] and the references therein. Moreover,  $c(0) = 1$  in that case, and the result of Theorem 1.1 can be refined, in that the expansion in (1.8) can be taken to higher order in  $\varepsilon$ :

**Theorem 1.2.** *For  $m = 2$  in (1.3), the propagation speed  $c(\varepsilon)$ , as defined in Theorem 1.1, satisfies*

$$(1.9) \quad c(\varepsilon) = 1 - 3\varepsilon^2 + 6\varepsilon^3 - 6\varepsilon^4 \ln \varepsilon - 21\varepsilon^4 + o(\varepsilon^4).$$

To the best of our knowledge, only the lowest-order correction in (1.9) had been determined explicitly before. We remark that geometric singular perturbation theory and blow-up have been applied previously in the derivation of rigorous asymptotic expansions in a variety of settings; see [22] as well as [35] or [36] and the references therein for details.

**Remark 1.** Our scaling of the reaction terms in (1.1) differs from that used in [8, 29] by a factor of 2. Retracing the proof of Theorem 1.2 in Section 4 below, one finds that the expansion for  $c(\varepsilon)$  can be obtained by dividing (1.9) by a factor of  $\sqrt{2}$  in that case:  $c(\varepsilon) = \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}\varepsilon^2 + O(\varepsilon^3)$  (in our notation); cf. also [35, Lemma 7]. While this scaling is consistent with the findings in [29, Section VI], it does not agree with [8, Equation (16)], where the leading-order correction to  $c(0)$  is given as  $-6\sqrt{2}\varepsilon^2$ . Based on our own results (both analytical and numerical) and those of [29], we believe [8, Equation (16)] to be in error.  $\square$

**Remark 2.** The exceptional nature of the Zeldovich equation was also recognized in [35], where it was shown that, when perturbing  $m$  to  $m + \varepsilon$  in (1.6), the leading-order coefficient in the resulting expansion for  $c_{\text{crit}}(\varepsilon)$  is computable if a closed-form solution is known for  $\varepsilon = 0$ .  $\square$

This article is organized as follows. In Section 2, we establish a geometric framework for the analysis of (1.3). In Section 3, we build on that framework to prove Theorem 1.1. In Section 4, we refine the estimates obtained previously, proving Theorem 1.2. Finally, in Section 5, we summarize our findings, and we discuss possible topics for future study.

## 2. GEOMETRIC FRAMEWORK FOR (1.3)

In this section, we introduce the geometric framework that underlies our study of (1.1) or, rather, of the associated traveling front problem in (1.3). However, instead of the nonlinear second-order Equation (1.3), we will in the following consider the equivalent first-order system

$$\begin{aligned} (2.1a) \quad & U' = V, \\ (2.1b) \quad & V' = -cV - 2U^m(1-U)\Theta(U-\varepsilon), \\ (2.1c) \quad & \varepsilon' = 0. \end{aligned}$$

Here, we have appended the trivial  $\varepsilon$ -dynamics, and the prime denotes differentiation with respect to  $\xi$ . Traveling front solutions propagating between the rest states at 1 and 0 in (1.1) now correspond to heteroclinic orbits connecting the two equilibrium points  $Q_\varepsilon^- := (1, 0, \varepsilon)$  and  $Q_\varepsilon^+ := (0, 0, \varepsilon)$  of (2.1), with  $\varepsilon \in [0, \varepsilon_0]$  and  $\varepsilon_0 > 0$  small. We note that, for  $U < \varepsilon$ , any point  $(U, 0, \varepsilon)$  is an equilibrium of (2.1), which is due to the fact that the Heaviside cut-off  $\Theta$  deactivates the reaction terms in (2.1) in an  $\varepsilon$ -neighborhood of  $Q_\varepsilon^+$ ; cf. (1.2). However, only the equilibrium at  $Q_\varepsilon^+$  is of interest to us here, as it corresponds to the original zero rest state in (1.1).

The following lemma is obtained by straightforward calculation:

**Lemma 2.1.** *For any  $m \geq 2$  and  $\varepsilon \in [0, \varepsilon_0]$  fixed, the point  $Q_\varepsilon^-$  is a semi-hyperbolic equilibrium of (2.1), with eigenvalues  $\lambda_\pm := -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 2}$  and 0; the corresponding eigenspaces are spanned by  $\{(1, \lambda_\pm, 0)\}$  and  $\{(0, 0, 1)\}$ , respectively. The eigenvalues of the linearization at  $Q_\varepsilon^+$  are given by  $-c$  and 0 (double); the  $-c$ -eigenspace is spanned by  $\{(1, -c, 0)\}$ , while the 0-eigenspace is given by  $\text{span}\{(1, 0, 0), (0, 0, 1)\}$ .*

We remark that the partial hyperbolicity of  $Q_\varepsilon^-$  is solely due to the inclusion of Equation (2.1c), which provides one zero eigendirection. The point  $Q_\varepsilon^+$  is more degenerate, and non-hyperbolic even in  $(U, V)$ -space, regardless of whether the  $\varepsilon$ -dynamics is included; this degeneracy is again caused by the cut-off  $\Theta$ .

The proof of Theorems 1.1 and 1.2 is based on a (geometric) phase space analysis of the equations in (2.1), in combination with the blow-up transformation

$$(2.2) \quad U = \bar{r}\bar{u}, \quad V = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon},$$

which maps the (degenerate) equilibrium  $Q_0^+ = (0, 0, 0)$  of (2.1) to the 2-sphere  $\mathbb{S}^2 = \{(\bar{u}, \bar{v}, \bar{\varepsilon}) \mid \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1\}$  in  $\mathbb{R}^3$ ; here,  $\bar{r} \in [0, r_0]$ , with  $r_0 > 0$  sufficiently small. (For our purposes, it will suffice to consider the quarter-sphere  $\mathbb{S}_+^2$  that is defined by restricting  $\mathbb{S}^2$  to  $\bar{u} \geq 0$  and  $\bar{\varepsilon} \geq 0$ .) The flow of the corresponding blown-up vector field will be studied in two charts,  $K_1$  and  $K_2$ , which are defined by  $\bar{u} = 1$  and  $\bar{\varepsilon} = 1$  in (2.2), respectively.

As will become clear in the following, these charts correspond to the ‘unmodified’ and the cut-off regimes in (2.1), respectively: while the dynamics in the inner region, *i.e.*, near the degenerate equilibrium at  $Q_\varepsilon^+$ , will be analyzed in the ‘rescaling’ chart  $K_2$ , the ‘phase-directional’ chart  $K_1$  will cover both the ‘outer’ region (the neighborhood of the equilibrium at  $Q_\varepsilon^-$ ) and the transition between the outer and inner regions (called the ‘intermediate’ region below). The dynamics obtained separately in these three regions will then be combined to construct a singular heteroclinic orbit  $\Gamma$ , *i.e.*, a connection between  $Q_0^-$  and  $Q_0^+$  for  $\varepsilon = 0$  in (2.1).

Given any object  $\square$  in  $(U, V, \varepsilon)$ -space, we will denote the corresponding blown-up object by  $\bar{\square}$ ; in chart  $K_i$ ,  $i \in \{1, 2\}$ , that same object will generally be denoted by  $\square_i$ . Correspondingly, we write  $(r_1, v_1, \varepsilon_1)$  and  $(u_2, v_2, r_2)$  for the respective coordinates in the two charts; in particular, setting  $\bar{u} = 1$  and  $\bar{\varepsilon} = 1$ , respectively, we find that the blow-up transformation defined in (2.2) takes the form

$$(2.3) \quad U = r_1, \quad V = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1$$

in the phase-directional chart  $K_1$ , whereas it is given by

$$(2.4) \quad U = r_2 u_2, \quad V = r_2 v_2, \quad \text{and} \quad \varepsilon = r_2$$

in the rescaling chart  $K_2$ . Finally, the relationship between the coordinates in these charts on their domain of overlap is described as follows:

**Lemma 2.2.** *The change of coordinates  $\kappa_{12} : K_1 \rightarrow K_2$  is given by*

$$u_2 = \varepsilon_1^{-1}, \quad v_2 = v_1 \varepsilon_1^{-1}, \quad \text{and} \quad r_2 = r_1 \varepsilon_1,$$

while the inverse change  $\kappa_{21} : K_2 \rightarrow K_1$  satisfies

$$r_1 = r_2 u_2, \quad v_1 = v_2 u_2^{-1}, \quad \text{and} \quad \varepsilon_1 = u_2^{-1}.$$

For details on the blow-up technique (geometric desingularization) in general and for its application in this context of front propagation in the presence of a cut-off in particular, the reader is again referred to [17] and the references therein.

**2.1. ‘Outer’ region.** In the ‘outer’ region, where  $U = O(1)$ , the cut-off  $\Theta$  has no impact on the dynamics of (1.1), as  $\Theta \equiv 1$  for  $U > \varepsilon$ , by (1.2). Hence, (1.3) reduces to the traveling front equation without cut-off in (1.6) there. In terms of the equivalent first-order system (2.1), we have

$$(2.5a) \quad U' = V,$$

$$(2.5b) \quad V' = -cV - 2U^m(1 - U),$$

$$(2.5c) \quad \varepsilon' = 0.$$

The existence of a heteroclinic connection between  $Q_0^-$  and  $Q_0^+$ , *i.e.*, for  $\varepsilon = 0$  in (2.5), is well-known [10, 11]; we will denote the corresponding value of  $c$  by  $c_0$ . We recall that, in the context of Equation (1.5), that connection corresponds to a traveling front solution with propagation speed  $c_0$ ; moreover, we remark that this solution is unique (up to a translation in  $\xi$ ), as shown *e.g.* in [30].

From Lemma 2.1, it follows that, for  $\varepsilon$  positive and fixed, the point  $Q_\varepsilon^-$  has an unstable manifold  $\mathcal{W}^u(Q_\varepsilon^-)$  that is analytic in the state variables  $U$  and  $V$  and in the parameters  $c$  and  $\varepsilon$  as long as  $U > \varepsilon$ . (Since the vector field in (2.1) has a discontinuity at  $U = \varepsilon$ , the smoothness of  $\mathcal{W}^u(Q_\varepsilon^-)$  clearly does not extend beyond that point.) However, the structure of (2.5) implies that  $V$  is independent of  $\varepsilon$  as long as  $c$  is, since any  $\varepsilon$ -dependence in (2.5b) can only enter through  $c$ . In other words, the unstable manifold of the line  $\ell^- := \bigcup_{\varepsilon \in [0, \varepsilon_0]} Q_\varepsilon^- = \{(1, 0, \varepsilon) \mid \varepsilon \in [0, \varepsilon_0]\}$ , which is a foliation in  $\varepsilon$  with fibers  $\mathcal{W}^u(Q_\varepsilon^-)$ , depends on  $\varepsilon$  in a trivial fashion, since these fibers lie in hyperplanes in  $(U, V, \varepsilon)$ -space, with  $\varepsilon$  constant. Hence, for  $\varepsilon \in [0, \varepsilon_0]$ , with  $\varepsilon_0 > 0$  sufficiently small, we write  $c = c_0 + \Delta c(\varepsilon)$ , where  $\Delta c(\varepsilon) := c - c_0$  is  $o(1)$  and as yet undetermined. We can then expand  $V$  in terms of  $U$  and  $c$  as

$$(2.6) \quad V(U, c) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j V}{\partial c^j}(U, c_0) (\Delta c)^j,$$

where the derivatives  $\frac{\partial^j V}{\partial c^j}(U, c_0)$  are smooth functions of  $U \in (\varepsilon, 1)$  for  $\varepsilon \in (0, \varepsilon_0)$  and  $j = 0, 1, 2, \dots$ . In particular,  $V(U, c_0)$  corresponds precisely to the heteroclinic connection between  $Q_0^-$  and  $Q_0^+$  that is realized in the singular limit as  $\varepsilon \rightarrow 0^+$  in (2.5).

Finally, we introduce a section for the flow of (2.5), as follows: for  $\rho \geq \varepsilon_0$  positive and small, we denote the hyperplane  $\{U = \rho\}$  by  $\Sigma^-$ ; specifically, we define

$$(2.7) \quad \Sigma^- := \{(\rho, V, \varepsilon) \mid (V, \varepsilon) \in [-V_0, 0] \times [0, \varepsilon_0]\};$$

here,  $V_0$  is some fixed, positive constant. We note that  $\Sigma^-$  naturally separates the outer region from the adjacent intermediate region that is characterized by  $U < O(1)$ , as discussed in Section 2.3 below. For  $c$  fixed and  $\varepsilon$  positive, we write  $P^- = (\rho, V^-, \varepsilon)$  for the point of intersection of  $\mathcal{W}^u(Q_\varepsilon^-)$  with  $\Sigma^-$ , *i.e.*, we suppress the dependence of this ‘entry’ point on  $c$  and  $\varepsilon$ , for convenience of notation. The point  $\mathcal{W}^u(Q_0^-) \cap \Sigma^-$  that is obtained in the singular limit will be denoted by  $P_0^-$ ; the restriction of  $\mathcal{W}^u(Q_0^-)$  to  $\{U \geq \rho\}$  yields precisely the portion of the sought-after singular orbit  $\Gamma$  that is located in the outer region.

**2.2. ‘Inner’ region (chart  $K_2$ ).** When  $U < \varepsilon$ , the Heaviside cut-off satisfies  $\Theta \equiv 0$ ; see again (1.2). The dynamics of (2.1) in this ‘inner’ region is naturally studied in the rescaling chart  $K_2$ : the resulting cut-off equations in terms of the rescaled  $(u_2, v_2, r_2)$ -coordinates are given by

$$(2.8a) \quad u_2' = v_2,$$

$$(2.8b) \quad v_2' = -cv_2,$$

$$(2.8c) \quad r_2' = 0;$$

recall (2.4). Here, we have taken into account that the reaction terms are set to zero by  $\Theta$ ; in other words, the cut-off simplifies the dynamics in this inner region substantially.

For  $r_2 \in [0, r_0]$  fixed, any point in  $\{v_2 = 0\}$  is an equilibrium of (2.8); however, since only points on the line  $\ell_2^+ = \{(0, 0, r_2) \mid r_2 \in [0, r_0]\}$  can correspond to  $Q_\varepsilon^+$  after blow-down (*i.e.*, after transformation to the original coordinates  $U, V$ , and  $\varepsilon$ ), we will focus on those points here, and we will collectively denote them by  $Q_2^+ = (0, 0, r_2)$ .

Dividing (2.8b) by (2.8a) (formally) to obtain an equation for  $v_2 = v_2(u_2)$ , we find

$$(2.9) \quad \frac{dv_2}{du_2} = -c \quad \text{with } v_2(0) = 0.$$

In the singular limit, *i.e.*, for  $r_2 = 0$ , the unique solution of (2.9) is given by  $v_2(u_2) = -c_0 u_2$ . The corresponding orbit, which is the stable manifold of  $Q_{0_2}^+ := (0, 0, 0) \in \ell_2^+$ , gives precisely the portion  $\Gamma_2^+$  of the singular heteroclinic connection  $\Gamma$  that lies in  $K_2$ . For  $r_2 > 0$  small and  $(u_2, v_2)$  bounded,  $\Gamma_2^+$  will perturb, in a smooth and regular fashion, to the stable manifold  $\mathcal{W}_2^s(\ell_2^+)$  of  $\ell_2^+$ , with  $v_2(u_2) = -cu_2$ . The resulting geometry in chart  $K_2$  is illustrated in Figure 1.

By (2.4), the condition that  $U < \varepsilon$  translates into  $u_2 < 1$  in chart  $K_2$ . Moreover, by regular perturbation theory, it follows from (2.9) that  $v_2(1) = -c_0[1 + o(1)]$  for  $r_2 > 0$  small and  $c = c_0[1 + o(1)]$ , as the only  $r_2$ -dependence in (2.8) enters through  $c$ . (Here,  $o(1)$  denotes higher-order terms in  $r_2$ .) Hence, we introduce a section  $\Sigma_2^+$  for the flow of (2.8) via

$$(2.10) \quad \Sigma_2^+ := \{(1, v_2, r_2) \mid (v_2, r_2) \in [-v_0, 0] \times [0, \rho]\},$$

where  $v_0$  is some appropriately defined constant, with  $v_0 > c_0$ ; cf. again Figure 1. We note that  $\Sigma_2^+$  naturally separates the inner region, in which the reaction terms in (2.1b) are cut off, from the adjacent ‘intermediate’ region, where the cut-off has no effect on the dynamics. Given  $r_2 (= \varepsilon) > 0$  and  $c$  fixed, we will denote the point of intersection of the stable manifold  $\mathcal{W}_2^s(Q_2^+)$  of  $Q_2^+$  with  $\Sigma_2^+$  by  $P_2^+ = (1, v_2^+, \varepsilon)$ , *i.e.*, the dependence of this ‘exit’ point on  $c$  and  $\varepsilon$  is again encoded implicitly

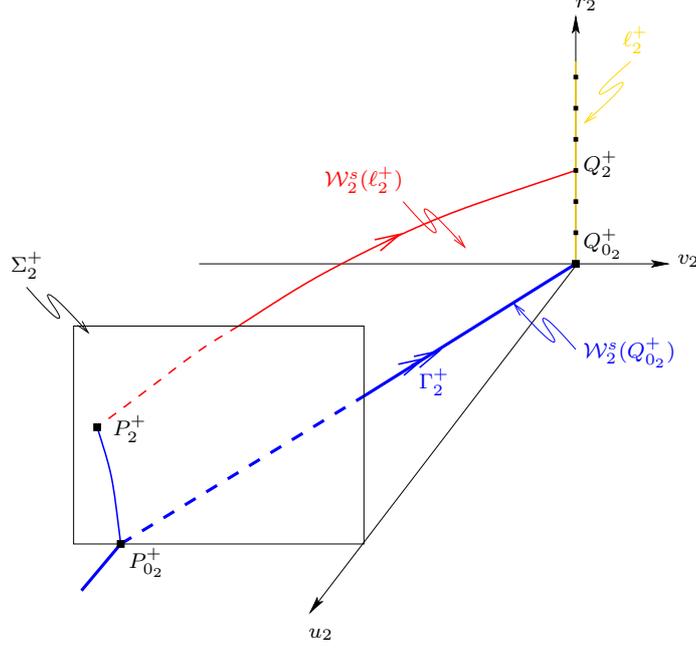


FIGURE 1. The geometry in chart  $K_2$ .

in the notation; recall the definition of  $P^-$  in Section 2.1. Finally, we note that  $P_2^+$  reduces to  $P_{0_2}^+ := (1, -c_0, 0)$  in the singular limit, since  $v_2^+ = -c_0$  at  $(c, \varepsilon) = (c_0, 0)$ ; see again Figure 1.

**Remark 3.** Strictly speaking, the definition of  $\Theta$  in (1.2) does not extend to  $\{U = \varepsilon\}$ ; hence, in chart  $K_2$ ,  $\Theta \equiv 0$  is *a priori* only satisfied on  $\{u_2 < 1\}$ . Correspondingly,  $\Theta$  has to be extended continuously (by 0) to  $\{u_2 = 1\}$  in order for the vector field in (2.8) to be defined in  $\Sigma_2^+$ ; see [17, Section 2.1] for details.  $\square$

**2.3. ‘Intermediate’ region (chart  $K_1$ ).** The ‘intermediate’ region is characterized by  $\varepsilon < U < O(1)$ , and is conveniently analyzed in the phase-directional chart  $K_1$ . Substituting (2.3) into (2.1) and recalling that again  $\Theta \equiv 1$ , as in Section 2.1, we obtain

$$(2.11a) \quad r_1' = r_1 v_1,$$

$$(2.11b) \quad v_1' = -c v_1 - v_1^2 - 2r_1^{m-1}(1 - r_1),$$

$$(2.11c) \quad \varepsilon_1' = -\varepsilon_1 v_1,$$

which is precisely the first-order system (2.5), after transformation to chart  $K_1$ . The principal equilibrium of (2.11) in this intermediate region lies at  $P_1 = (0, -c_0, 0)$ ; since  $\frac{U'}{U} = \frac{V}{U} = v_1$ , see (2.3),  $P_1$  corresponds to the stable eigendirection of the linearization at  $Q_0^+$  of (2.5). (Other equilibria of (2.11) are located on the  $\varepsilon_1$ -axis; these correspond to equilibria previously found on the  $u_2$ -axis in chart  $K_2$  and are thus of no interest to us.) In other words, the blow-up transformation in (2.2) teases apart the asymptotics close to  $Q_\varepsilon^+$  and, hence, desingularizes the dynamics there down to  $\varepsilon = 0$ .

Recalling that  $c_0 > 0$  must hold in (2.5) [10, 11], we have the following result:

**Lemma 2.3.** *For any  $m \geq 2$ , the point  $P_1$  is a hyperbolic saddle equilibrium of (2.11), with eigenvalues  $-c_0$  and  $c_0$  (double). The  $c_0$ -eigenspace is given by  $\text{span}\{(0, 1, 0), (0, 0, 1)\}$ , while the  $-c_0$ -eigenspace is spanned by  $\{(1, 0, 0)\}$  for  $m \geq 3$  and by  $\{(c_0, 1, 0)\}$  when  $m = 2$ .*

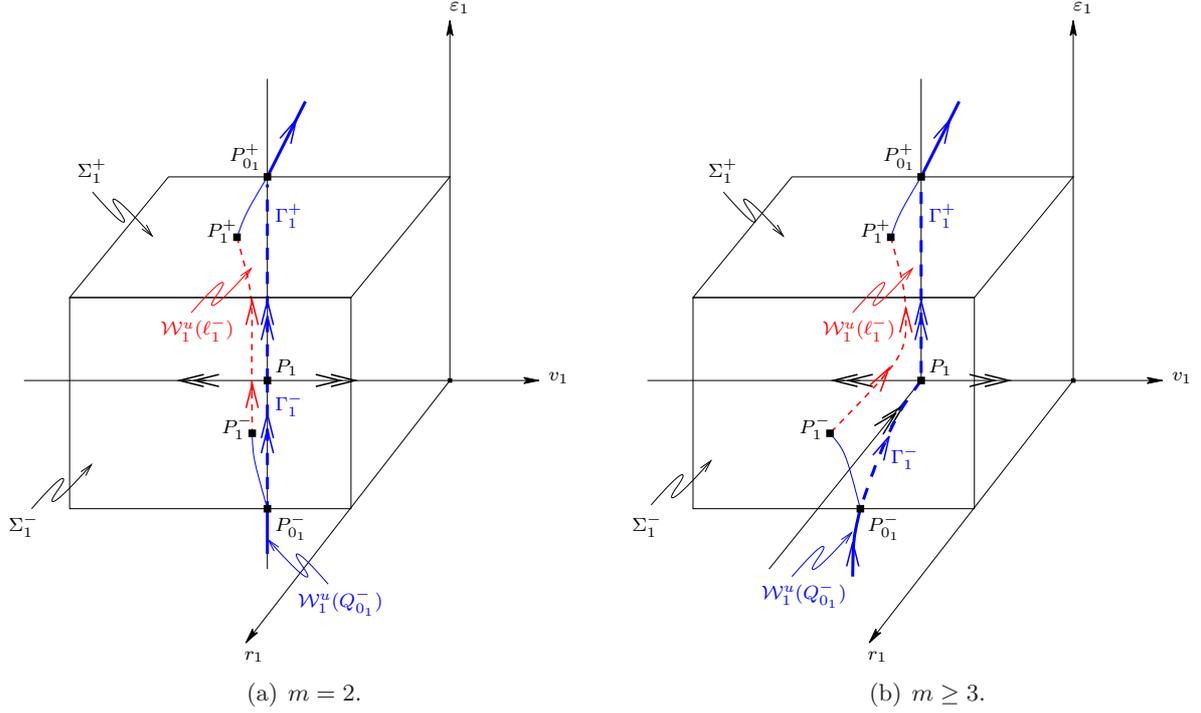


FIGURE 2. The geometry in chart  $K_1$ .

As in Section 2.2 above, it is convenient to introduce sections for the flow of (2.11) (or, rather, of the continuous extension of that system to  $\{\varepsilon_1 = 1\}$ , which is again given by (2.11)): for  $\rho$  and  $v_0$  as in (2.7) and (2.10), respectively, we define

(2.12)

$$\Sigma_1^- = \{(\rho, v_1, \varepsilon_1) \mid (v_1, \varepsilon_1) \in [-v_0, 0] \times [0, 1]\} \quad \text{and} \quad \Sigma_1^+ = \{(r_1, v_1, 1) \mid (r_1, v_1) \in [0, \rho] \times [-v_0, 0]\}.$$

We note that  $\Sigma_1^+$  separates the inner region from the intermediate region, and that it corresponds to the section  $\Sigma_2^+$  under the change of coordinates  $\kappa_{12}$  since, by definition,  $u_2 = 1$  in  $\Sigma_2^+$ ; see Lemma 2.2. To state it differently, trajectories leaving the phase-directional chart  $K_1$  under the flow of (2.11) have to traverse  $\Sigma_1^+$  before entering the rescaling chart  $K_2$ . Similarly, the intermediate region is naturally separated from the outer region by the section  $\Sigma_1^-$ , which corresponds precisely to  $\Sigma^-$ , as defined in (2.7), after blow-down.

Given any orbit of (2.11), for  $\varepsilon$  positive and small, we will denote the points of intersection of that orbit with  $\Sigma_1^-$  and  $\Sigma_1^+$  by  $P_1^- := (\rho, v_1^-, \varepsilon \rho^{-1})$  and  $P_1^+ := (\varepsilon, v_1^+, 1)$ , respectively. (Here, we have taken into account that  $r_1 \varepsilon_1 = \varepsilon$  is constant; moreover, we have suppressed any parameter dependence in the notation, as before.) Correspondingly, for  $(c, \varepsilon) = (c_0, 0)$ , these points will be labeled  $P_{0_1}^-$  and  $P_{0_1}^+$ , respectively.

Finally, the singular dynamics in the intermediate region is described by the limiting systems that are obtained by setting  $r_1 = 0$  or  $\varepsilon_1 = 0$  in (2.11), as both limits are realized for  $\varepsilon = 0$  in (2.1) (before the blow-up). We denote the corresponding singular orbits by  $\Gamma_1^+$  and  $\Gamma_1^-$ , and we note that these orbits lie in the hyperplanes  $\{r_1 = 0\}$  and  $\{\varepsilon_1 = 0\}$ , respectively, which are invariant for (2.11). (The union of  $\Gamma_1^+$  and  $\Gamma_1^-$  will yield precisely the portion of  $\Gamma$  that is found in the intermediate region, *i.e.*, in chart  $K_1$ .) We remark that  $\kappa_{12}(P_{0_1}^+) = P_{0_2}^+$ , since  $v_1^+ = v_2^+$ , by Lemma 2.2 and, hence, that  $P_{0_1}^+ = (0, -c_0, 1)$  in the singular limit. The orbit  $\Gamma_1^+$ , with  $\Gamma_1^+ \rightarrow P_{0_1}^+$

as  $\varepsilon_1 \rightarrow 1$ , is then easily found by solving (2.11) for  $r_1 = 0$ : since the unique solution to

$$(2.13) \quad \frac{dv_1}{d\varepsilon_1} = \frac{c_0 + v_1}{\varepsilon_1} \quad \text{with } v_1(1) = -c_0$$

is given by  $v_1(\varepsilon_1) \equiv -c_0$ ,  $\Gamma_1^+$  is a straight line segment connecting  $P_1$  and  $P_{0_1}^+$ . We note that, in general, no corresponding expression is available for  $\Gamma_1^-$ , since (2.11) can only be solved explicitly for  $\varepsilon_1 = 0$  if the traveling front equation in (1.6) is solvable in closed form. (The portion of  $\Gamma_1^-$  lying in the intermediate region corresponds precisely to the ‘tail’ of that traveling front or, equivalently, to the manifold  $\mathcal{W}^u(Q_0^-)$ , restricted to  $\{U \leq \rho\}$ , in  $(U, V, \varepsilon)$ -space.) The geometry in chart  $K_1$  is summarized in Figure 2, where we have to distinguish between  $m = 2$  and  $m \geq 3$  in (1.1); cf. Lemma 2.3.

**2.4. Construction of  $\Gamma$ .** We now combine the results obtained in Sections 2.1 through 2.3 above to demonstrate the existence of the singular heteroclinic connection  $\Gamma$  introduced at the beginning of Section 2.

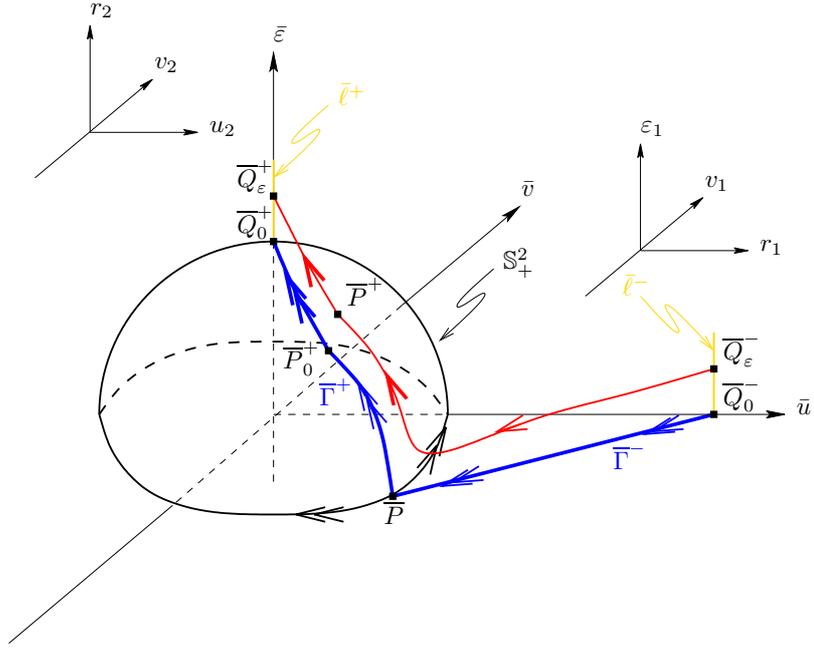
First, we note that the blown-up locus, which is defined by  $\bar{r} = 0$  in (2.2), corresponds to the hyperplanes  $\{r_2 = 0\}$  and  $\{r_1 = 0\}$  in the respective coordinate charts. Then, (2.8) and (2.11) reduce to (2.9) and (2.13), respectively. The union of the corresponding orbits  $\Gamma_2^+$  and  $\Gamma_1^+$  provides the desired singular connection from  $P_1$  via  $P_{0_1}^+$  (or, equivalently, via  $P_{0_2}^+$ ) to  $Q_{0_2}^+$ ; recall Figures 1 and 2. (Clearly, that connection is only  $\mathcal{C}^0$ -smooth in  $\Sigma_1^+$  or, alternatively, in  $\Sigma_2^+$ .) Finally, in  $\{\varepsilon_1 = 0\}$ , the orbit  $\Gamma_1^-$  connecting  $Q_{0_1}^-$  and  $P_1$  (via  $P_1^-$ ) is given by the unstable manifold  $\mathcal{W}_1^u(Q_{0_1}^-)$  of  $Q_{0_1}^-$ , cf. again Figure 2; that manifold corresponds to the unique traveling front solution of (1.6) with propagation speed  $c_0$ , after blow-up and transformation to chart  $K_1$ . (Here,  $Q_{0_1}^- := (1, 0, 0) \in \ell_1^- = \{(1, 0, \varepsilon_1) \mid \varepsilon_1 \in [0, \varepsilon_0]\}$  denotes the equivalent of  $Q_0^-$  in  $K_1$ .) In sum, the desired heteroclinic connection  $\Gamma$  – or, rather, the corresponding orbit  $\bar{\Gamma}$  in blown-up phase space – is obtained as the union of the orbits  $\bar{\Gamma}^-$  and  $\bar{\Gamma}^+$  as well as of the singularities  $\bar{Q}_0^-$ ,  $\bar{P}$ , and  $\bar{Q}_0^+$ ; see Figure 3 for a schematic illustration of the global geometry of the blown-up vector field that is induced by (2.5).

**Remark 4.** Since we are restricting ourselves to the Heaviside cut-off  $\Theta$  in (1.1),  $\Gamma_2^+$  is the unique orbit that is asymptotic to  $Q_{0_2}^+$  in  $K_2$ . It then follows as in [17, Section 2.3] that the cut-off traveling front equation in (1.3) has a solution for precisely one value of  $c$ . (As was shown in [17], that statement is generally true for any choice of cut-off function that has an accumulation point of positive zeros at 0.)  $\square$

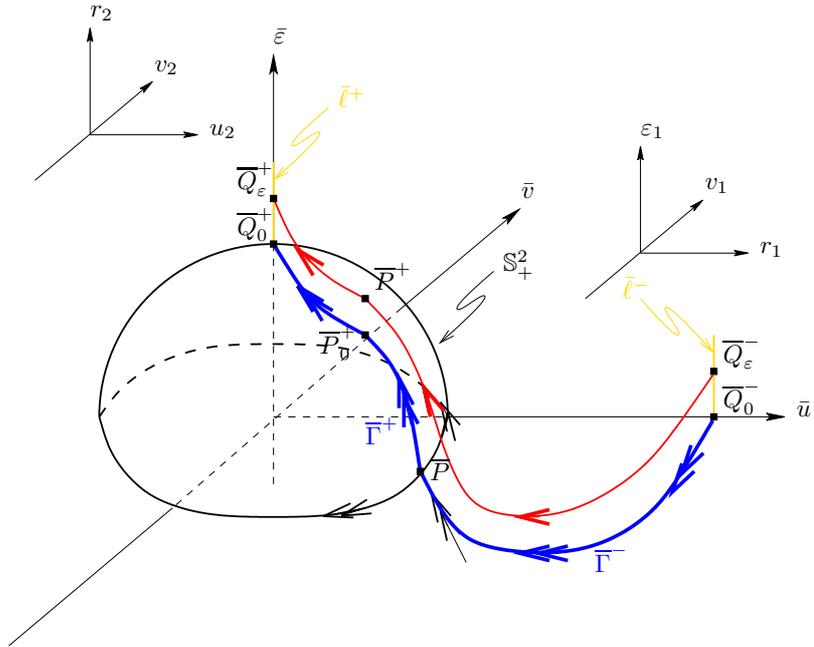
**Remark 5.** For  $m = 2$ , the transformation in (2.3) is equivalent to the desingularization performed in [38, Section 2], based on results by Aronson [3]: while [38] is concerned with equations of the type of (1.7) in which the diffusivity  $D$  is density-dependent, [38, Equation (24)] corresponds to (2.11) for  $D(u) = u$  (in our notation). However, their argumentation then proceeds via a center manifold reduction about the (partially hyperbolic) origin, whereas our analysis focuses on the dynamics near  $P_1$ , in the intermediate region. The two approaches do not seem to be related for higher values of  $m$ ; see the discussion in [38, Section 3].  $\square$

### 3. PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1. We will first show that, for  $\varepsilon > 0$  sufficiently small in (2.1), the singular heteroclinic connection  $\Gamma$  constructed in Section 2 persists for a unique value of  $c$ , denoted  $c(\varepsilon)$ ; the persistent heteroclinic will lie in the intersection of the unstable manifold  $\mathcal{W}^u(\ell^-)$  of  $\ell^-$  with the stable manifold  $\mathcal{W}^s(\ell^+)$  of  $\ell^+ := \bigcup_{\varepsilon \in [0, \varepsilon_0]} Q_\varepsilon^+ = \{(0, 0, \varepsilon) \mid \varepsilon \in [0, \varepsilon_0]\}$ .



(a)  $m = 2$ .



(b)  $m \geq 3$ .

FIGURE 3. The global geometry of the blown-up vector field.

(The latter corresponds to the manifold  $\mathcal{W}_2^s(\ell_2^+)$  in chart  $K_2$ , as defined in Section 2.2, after blow-down.) To state it differently, the manifolds  $\mathcal{W}^u(Q_\varepsilon^-)$  and  $\mathcal{W}^s(Q_\varepsilon^+)$  will coincide for  $c = c(\varepsilon)$ , which will prove the existence of a traveling front solution with propagation speed  $c(\varepsilon)$  in the presence of the (Heaviside) cut-off  $\Theta$ ; see again Figure 3 for an illustration of the small- $\varepsilon$  dynamics of (2.1) in blown-up phase space. Then, in a second step, we will derive a necessary condition

for the persistence of  $\Gamma$ ; that condition, in turn, will determine the leading-order  $\varepsilon$ -asymptotics of  $c(\varepsilon)$ , completing the proof of Theorem 1.1. The required analysis is summarized in the following subsections; see also [17, Section 3], where a similar strategy was applied in the study of the cut-off Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation.

**3.1. Existence and uniqueness of  $c(\varepsilon)$ .** We set out by proving that, for  $\varepsilon$  sufficiently small,  $\mathcal{W}^u(Q_\varepsilon^-)$  and  $\mathcal{W}^s(Q_\varepsilon^+)$  intersect for a unique value of  $c(\varepsilon)$  in (2.1). Furthermore, we show that  $c(\varepsilon) < c(0)$ , where  $c(0)$  equals  $c_0$ , the front speed corresponding to  $\varepsilon = 0$  in (2.5).

**Proposition 3.1.** [17, Proposition 3.1] *For  $\varepsilon \in [0, \varepsilon_0]$  with  $\varepsilon_0 > 0$  sufficiently small, and any  $m \geq 2$ , there exists a unique  $c(\varepsilon)$  such that there is a heteroclinic orbit connecting  $Q_\varepsilon^-$  and  $Q_\varepsilon^+$  for  $c = c(\varepsilon)$  in (2.1). Moreover, there holds  $c(\varepsilon) \lesssim c(0)$ , i.e.,  $c(\varepsilon) \approx c(0)$  as well as  $c(\varepsilon) < c(0)$ .*

*Proof.* The existence of a singular heteroclinic orbit  $\Gamma$  already follows from the discussion in Section 2.4. Hence,  $c(\varepsilon)$  reduces to  $c_0$  in the singular limit as  $\varepsilon \rightarrow 0^+$ , and we only need to consider  $\varepsilon > 0$  in (2.1) here.

To prove the statement in that case, we investigate how the singular heteroclinic connection corresponding to  $c(0)$  will perturb for  $\varepsilon$  positive and small. The argumentation is based on the proof of [17, Proposition 3.1], to which the reader is referred for details.

Recalling the definition of  $\Sigma_2^+$  in (2.10) as well as the fact that the manifold  $\mathcal{W}_2^s(\ell_2^+)$  is analytic in  $c$  and  $r_2(= \varepsilon)$ , one sees that the intersection of  $\mathcal{W}_2^s(\ell_2^+)$  with  $\Sigma_2^+$  can be written as the graph of an analytic function  $\phi^+$ , with  $\phi^+(c, \varepsilon) = -c$ ; in particular, there certainly holds  $\frac{\partial \phi^+}{\partial c} = -1 < 0$ . (Here, we remark that, for  $(c, \varepsilon)$  fixed,  $\phi^+$  equals  $v_2^+$ , the  $v_2$ -coordinate of  $P_2^+$ , as defined at the end of Section 2.2.) It then follows from regular perturbation theory that the intersection of the stable manifold  $\mathcal{W}^s(Q_\varepsilon^+)$  of  $Q_\varepsilon^+$  with the hyperplane  $\{U = \varepsilon\}$ , which is given by  $\Phi^+(c, \varepsilon) = \varepsilon \phi^+(c, \varepsilon)$  after blow-down, satisfies  $-\frac{3}{2}\varepsilon c(0) < \Phi^+(c, \varepsilon) < -\frac{1}{2}\varepsilon c(0)$  for  $\varepsilon$  sufficiently small and  $c \approx c(0)$ .

On  $\{U \geq \varepsilon\}$ , the unstable manifold  $\mathcal{W}^u(Q_\varepsilon^-)$  of  $Q_\varepsilon^-$  can be estimated by considering (2.5) and by noting that the intersection of  $\mathcal{W}^u(Q_\varepsilon^-)$  with  $\{U = \varepsilon\}$  can be represented as the graph of an analytic function  $\Psi^+(c, \varepsilon)$ , with  $\frac{\partial \Psi^+}{\partial c} > 0$ . For fixed  $c < c(0)$ , a standard phase plane argument shows that

$$\lim_{\varepsilon \rightarrow 0^+} \Psi^+(c, \varepsilon) = \mathcal{W}^u(Q_0^-) \cap \{U = 0\}$$

is well-defined, strictly  $O(1)$ , and negative, which implies that the same must be true of  $\Psi^+(c, \varepsilon)$ , for  $\varepsilon$  positive and small. Since  $\Phi^+(c, \varepsilon) > -\frac{3}{2}\varepsilon c(0)$ , we conclude that  $\Phi^+ > \Psi^+$  for  $c \lesssim c(0)$ .

Finally, when  $c = c(0)$ , we first note that  $\Phi^+(c(0), \varepsilon) = -c(0)\varepsilon$ , by the above. Since one can show as in the proof of [17, Lemma 2.5] that the flow of (2.5) is trapped in the wedge bounded by the curves  $\{V = 0\}$  and  $\{V = -c(0)U + O(U^{\frac{m+1}{2}})\}$  and since the  $O(U^{\frac{m+1}{2}})$ -coefficient is positive, it follows that  $\Psi^+(c(0), \varepsilon) > -c(0)\varepsilon$  must hold in  $\{U = \varepsilon\}$ . Therefore,  $\Phi^+ < \Psi^+$  for  $c = c(0)$  in (2.1).

In sum, we find that, for any  $\varepsilon \in [0, \varepsilon_0]$ ,  $\mathcal{W}^u(Q_\varepsilon^-)$  and  $\mathcal{W}^s(Q_\varepsilon^+)$  must connect in  $\{U = \varepsilon\}$  for some value of  $c$ , which we denote by  $c(\varepsilon)$ . Moreover, the above argument implies  $c(\varepsilon) < c(0)$ . The uniqueness of  $c(\varepsilon)$  follows from  $\frac{\partial \Phi^+}{\partial c} < 0$  and  $\frac{\partial \Psi^+}{\partial c} > 0$  for  $c \approx c(0)$  and  $\varepsilon > 0$  small, which completes the proof.  $\square$

For future reference, we note that, for fixed  $\varepsilon \in [0, \varepsilon_0]$  and  $V \in [-V_0, 0]$ , with  $V_0$  as in (2.7), the hyperplane  $\{U = \varepsilon\}$  is contained in  $\Sigma^+$ , which is the section corresponding to  $\Sigma_1^+$  (or, equivalently, to  $\Sigma_2^+$ ) in the original  $(U, V, \varepsilon)$ -coordinates.

**3.2. Asymptotics of  $c(\varepsilon)$ .** In this section, we derive a necessary condition that  $c(\varepsilon)$ , as defined in Proposition 3.1, has to satisfy in order for the singular orbit  $\Gamma$  to persist, for  $\varepsilon$  sufficiently small, as a heteroclinic connection between  $Q_\varepsilon^-$  and  $Q_\varepsilon^+$  in (2.1).

3.2.1. *Transition map*  $\Pi_1 : \Sigma_1^- \rightarrow \Sigma_1^+$ . Let  $\Pi_1$  denote the transition map which is defined as a mapping, under the flow of (2.11), between the two sections  $\Sigma_1^-$  and  $\Sigma_1^+$  introduced in (2.12). To obtain an approximation for  $\Pi_1$  that is sufficiently accurate to the order considered here, we formulate a normal form system for (2.11) that can be solved explicitly to leading order. The corresponding analysis is carried out entirely in the intermediate region, *i.e.*, in chart  $K_1$ .

We begin by casting (2.11) into a form that is more convenient for the following discussion: replacing  $c$  with  $c(\varepsilon)$  in (2.11b), where  $c(\varepsilon)$  is defined as in Proposition 3.1, and writing  $c(\varepsilon) = c(0) + [c(\varepsilon) - c(0)] = c(0) + \Delta c(\varepsilon)$  now, we translate the point  $P_1$  into the origin by introducing the new variable  $z = v_1 + c(0)$ . Under these transformations, the equations in (2.11) become

$$\begin{aligned} (3.1a) \quad & r_1' = -[c(0) - z]r_1, \\ (3.1b) \quad & z' = [c(0) - z](\Delta c + z) - 2r_1^{m-1}(1 - r_1), \\ (3.1c) \quad & \varepsilon_1' = [c(0) - z]\varepsilon_1. \end{aligned}$$

Next, we divide out the (positive) factor of  $c(0) - z$  from the right-hand sides in (3.1), which corresponds to a rescaling of time that leaves the phase portrait unchanged:

$$\begin{aligned} (3.2a) \quad & r_1' = -r_1, \\ (3.2b) \quad & z' = \Delta c + z - \frac{2r_1^{m-1}(1 - r_1)}{c(0) - z}, \\ (3.2c) \quad & \varepsilon_1' = \varepsilon_1. \end{aligned}$$

(Here, the prime now denotes differentiation with respect to a new independent variable  $\zeta$ ; writing  $\zeta^-$  for the value of  $\zeta$  in  $\Sigma_1^-$ , we may, without loss of generality, assume  $\zeta^- = 0$ .)

Finally, we simplify the equations in (3.2) via a sequence of coordinate transformations that eliminate all but the resonant terms from (3.2b):

**Proposition 3.2.** *Let  $m \geq 2$  in (3.2), and let  $\mathcal{V} = \{(r_1, z, \varepsilon_1) \mid (r_1, z, \varepsilon_1) \in [0, \rho] \times [-z_0, 0] \times [0, 1]\}$ , for  $\rho$  positive and sufficiently small and  $z_0 = v_0 + c_0$ , with  $v_0$  as in (2.10). Then, there exists a sequence of  $C^\infty$ -smooth coordinate transformations on  $\mathcal{V}$ , with  $(r_1, z, \varepsilon_1) \mapsto (r_1, \hat{z}, \varepsilon_1)$  and  $\hat{z} = z + O(r_1^{m-1}, \Delta c)$ , such that (3.2) can be written as*

$$\begin{aligned} (3.3a) \quad & r_1' = -r_1, \\ (3.3b) \quad & \hat{z}' = \hat{z} - \frac{2}{c(\varepsilon)^{m+1}} r_1^{m-1} \hat{z}^m [1 + O(r_1 \hat{z})], \\ (3.3c) \quad & \varepsilon_1' = \varepsilon_1, \end{aligned}$$

where  $O(r_1 \hat{z})$  denotes a smooth function of  $r_1 \hat{z}$ .

*Proof.* We first remove the constant ( $\Delta c$ -)term from (3.2b) by defining the new variable  $\tilde{z} := z + \Delta c (= v_1 + c(\varepsilon))$ ; then, we expand the right-hand side of the resulting equation to find

$$(3.4) \quad \tilde{z}' = \tilde{z} - \frac{2r_1^{m-1}(1 - r_1)}{c(0) + \Delta c - \tilde{z}} = \tilde{z} - \frac{2}{c(\varepsilon)} r_1^{m-1}(1 - r_1) \left[ 1 + \frac{\tilde{z}}{c(\varepsilon)} + \left(\frac{\tilde{z}}{c(\varepsilon)}\right)^2 + \cdots + \left(\frac{\tilde{z}}{c(\varepsilon)}\right)^m + \cdots \right],$$

where we note that  $|\frac{\tilde{z}}{c(\varepsilon)}| = |\frac{v_1}{c(\varepsilon)} + 1| < 1$  can be made as small as required by taking  $\varepsilon$  sufficiently small. Next, we transform  $\tilde{z}$  via

$$\tilde{z} \mapsto \tilde{z} - \frac{1}{m} \frac{2}{c(\varepsilon)} r_1^{m-1},$$

to eliminate the  $O(r_1^{m-1})$ -terms from (3.4):

$$(3.5) \quad \tilde{z}' = \tilde{z} + \frac{2}{c(\varepsilon)} r_1^m - \frac{2}{c(\varepsilon)^2} r_1^{m-1} \tilde{z} + \cdots - \frac{2}{c(\varepsilon)^{m+1}} r_1^{m-1} \tilde{z}^m + \dots$$

The result then follows from standard normal form theory [23]: applying a sequence of near-identity transformations to  $\tilde{z}$  which leave  $r_1$  and  $\varepsilon_1$  unchanged, we successively remove all non-resonant terms of order  $m$  and higher, jointly in  $r_1$  and  $\tilde{z}$ , from (3.5). (Due to the structure of (3.5), the non-identity part in these transformations will in fact only involve terms of at least order  $m$ .) The lowest-order term that cannot be removed via such a transformation is the resonant  $O(r_1^{m-1} \tilde{z}^m)$ -term. Since the coefficient of that term is not affected by any of the preceding coordinate changes, one obtains the leading-order normal form in (3.3b), as claimed.

Similarly, one finds that any term of order  $2m$  or higher can be eliminated from (3.5), with the exception of higher-order resonant terms. Clearly, these terms are of the general form  $r_1^k \tilde{z}^{k+1}$ , with  $k \geq m$ , which gives (3.3).

Finally, Equations (3.2b) and (3.2c) are decoupled, *i.e.*, the  $\Delta c$ -term in (3.2b) only depends on the product  $r_1 \varepsilon_1 (= \varepsilon)$ , which is constant. (Correspondingly, the definition of  $\mathcal{V}$  contains no restriction on  $\varepsilon_1$ , apart from the trivial requirement that  $\varepsilon_1 \in [0, 1]$ , which is the equivalent, in chart  $K_1$ , of  $U > \varepsilon$ .) Hence, the sequence of normal form transformations defined above is independent of  $\varepsilon_1$  to all orders, which implies  $\hat{z} = z + O(r_1^{m-1}, \Delta c)$ , completing the proof.  $\square$

We denote by  $\hat{P}_1^-$  and  $\hat{P}_1^+$  the points obtained from the entry and exit points  $P_1^- \in \Sigma_1^-$  and  $P_1^+ \in \Sigma_1^+$ , respectively, after application of the sequence of normal form transformations from Proposition 3.2; for simplicity of notation, we will still write  $\Sigma_1^-$  and  $\Sigma_1^+$ , respectively, for the corresponding sections in  $(r_1, \hat{z}, \varepsilon_1)$ -space. Finally, we note that the point  $P_1$  will stay at the origin under these transformations.

Given the normal form system in (3.3), we can now approximate the transition map  $\Pi_1$  to the order required here:

**Proposition 3.3.** *For any  $m \geq 2$  and  $|\rho \hat{z}^-|$  sufficiently small, the map  $\Pi_1 : \Sigma_1^- \rightarrow \Sigma_1^+$  satisfies  $(\rho, \hat{z}^-, \varepsilon \rho^{-1}) \mapsto (\varepsilon, \hat{z}^+, 1)$ , with*

$$(3.6) \quad \hat{z}^+ = \frac{\rho \hat{z}^-}{\varepsilon} \{1 + O[(\rho \hat{z}^-)^{m-1} \ln \varepsilon]\}.$$

*Proof.* We recall the fact that the equations for  $r_1$  and  $\varepsilon_1$  in (3.3) decouple; the corresponding explicit solutions are given by  $r_1(\zeta) = \rho e^{-\zeta}$  and  $\varepsilon_1(\zeta) = \frac{\varepsilon}{\rho} e^{\zeta}$ , respectively, where again  $\zeta^- = 0$  in  $\Sigma_1^-$ . Since, moreover,  $r_1 = \varepsilon$  in  $\Sigma_1^+$ , see again (2.12), the transition ‘time’ from  $\Sigma_1^-$  to  $\Sigma_1^+$  under the flow of (3.3) is given by  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$ .

To prove the estimate for  $\hat{z}^+$  in (3.6), we will first approximate  $\Pi_1$  by solving the leading-order normal form system obtained by retaining only the lowest-order resonant terms in (3.3b). Then, we will show that the error incurred in this approximation is negligible to the order considered here. Substituting  $r_1(\zeta)$  into (3.3b) and omitting terms of order  $2m + 1$  and higher, we find that the resulting approximate equation

$$(3.7) \quad \tilde{z}' = \tilde{z} - \frac{2}{c(\varepsilon)^{m+1}} r_1^{m-1} \tilde{z}^m$$

is a first-order Bernoulli equation [1]. The corresponding closed-form solution, which we denote by  $\check{z}(\zeta)$ , is given by

$$(3.8) \quad \check{z}(\zeta) = \left[ \frac{c(\varepsilon)^{m+1} e^{(m-1)\zeta}}{2(m-1)\rho^{m-1}\zeta + C_m c(\varepsilon)^{m+1}} \right]^{\frac{1}{m-1}}.$$

Here,  $C_m$  is a constant of integration that can be fixed by imposing the requirement that  $\tilde{z}(\zeta^-) = \hat{z}^-$ : recalling  $\zeta^- = 0$ , we find  $C_m = (\hat{z}^-)^{-(m-1)}$ , which we substitute back into (3.8). Expanding the result for  $|\rho\hat{z}^-|$  small, we have

$$(3.9) \quad \tilde{z}(\zeta) = \hat{z}^- e^\zeta \{1 + O[(\rho\hat{z}^-)^{m-1}\zeta]\}$$

(after some simplification). Evaluating (3.9) at  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$ , we obtain (3.6), as claimed.

It remains to show that, for any  $\zeta \in [0, \zeta^+]$ , the error  $|\hat{z}(\zeta) - \tilde{z}(\zeta)|$  that results from replacing  $\hat{z}$  with  $\tilde{z}$  is irrelevant to leading order. To that end, we make use of a version of Gronwall's Lemma, as stated in Appendix A: setting  $t \equiv \zeta$ ,  $x \equiv \hat{z}$ , and  $y \equiv \tilde{z}$  in Lemma A.1 and denoting the corresponding right-hand sides in (3.3b) and (3.7) by  $f$  and  $g$ , respectively, we find that (A.1) becomes

$$|g(\zeta, \tilde{z}_2) - g(\zeta, \tilde{z}_1)| = \left| (\tilde{z}_2 - \tilde{z}_1) \left( 1 - \frac{2}{c(\varepsilon)^{m+1}} r_1(\zeta)^{m-1} \sum_{k=0}^{m-1} \tilde{z}_2^k \tilde{z}_1^{m-1-k} \right) \right|.$$

Now, it follows from (3.3a) and (3.7) that  $(r_1\tilde{z})' = -\frac{2}{c(\varepsilon)^{m+1}}(r_1\tilde{z})^m$ , which, in combination with  $(r_1\tilde{z})(0) = \rho\hat{z}^-$ , implies  $(r_1\tilde{z})(\zeta) = \rho\hat{z}^- [1 + o(1)]$  for  $|\rho\hat{z}^-|$  sufficiently small. Hence, (A.2) is satisfied with (for instance)  $C = 1 + \frac{4m}{c(\varepsilon)^{m+1}} |\rho\hat{z}^-|^{m-1}$ . Similarly, we can estimate

$$|f(\zeta, \hat{z}) - g(\zeta, \hat{z})| = \left| \frac{2}{c(\varepsilon)^{m+1}} r_1^m \hat{z}^{m+1} [1 + O(r_1\hat{z})] \right| \leq \frac{4}{c(\varepsilon)^{m+1}} \frac{|\rho\hat{z}^-|^{m+1}}{r_1},$$

which shows (A.3), with  $\varphi(\zeta) = \frac{4}{c(\varepsilon)^{m+1}} \frac{|\rho\hat{z}^-|^{m+1}}{\rho} e^\zeta$ . Now, making again use of  $\hat{z}(0) = \hat{z}^- = \tilde{z}(0)$ , we find that (A.4) reduces to

$$(3.10) \quad |\hat{z}(\zeta) - \tilde{z}(\zeta)| \leq \frac{1}{m} \frac{(\rho\hat{z}^-)^2}{\rho} e^\zeta \left[ e^{\frac{4m}{c(\varepsilon)^{m+1}} |\rho\hat{z}^-|^{m-1}\zeta} - 1 \right] = e^\zeta \frac{(\rho\hat{z}^-)^2}{\rho} O[(\rho\hat{z}^-)^{m-1}\zeta].$$

In particular, evaluating (3.10) at  $\zeta^+$ , we have  $|\hat{z}^+ - \tilde{z}^+| = \frac{\rho\hat{z}^-}{\varepsilon} O[(\rho\hat{z}^-)^m \ln \varepsilon]$ , which is of higher order when compared to (3.6). This completes the proof of Proposition 3.3.  $\square$

We remark that the resonant terms in (3.3b) will typically give rise to logarithmic terms (in  $\varepsilon$ ) in the transition through chart  $K_1$ , which can be seen intuitively as follows. Approximating the solution to (3.3b) iteratively, we find the lowest-order approximation  $\hat{z}^{(0)}(\zeta) = \hat{z}(0)e^\zeta$  for  $\hat{z}$ . Recalling  $r_1(\zeta) = \rho e^{-\zeta}$  and substituting  $\hat{z}^{(0)}$  back into (3.3b), we obtain

$$\hat{z}^{(1)'} = \hat{z}^{(1)} - \frac{2}{c(0)^{m+1}} \rho^{m-1} (\Delta c)^m e^\zeta,$$

which has the solution

$$(3.11) \quad \hat{z}^{(1)}(\zeta) = \hat{z}(0)e^\zeta - \frac{2}{c(0)^{m+1}} \rho^{m-1} (\Delta c)^m \zeta e^\zeta.$$

Evaluating  $\hat{z}^{(1)}$  in  $\Sigma_1^+$ , *i.e.*, at  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$ , we find that the  $(\Delta c)^m \zeta e^\zeta$ -term in (3.11) will result in a  $(\Delta c)^m \frac{\ln \varepsilon}{\varepsilon}$ -contribution in  $\hat{z}^{(1)}(\zeta^+)$ . (The fact that  $\Delta c = O(\varepsilon^m)$ , as shown in Proposition 3.6 below, implies that the actual order of that term will be  $O(\varepsilon^{m^2-1} \ln \varepsilon)$ .) Continuing this iteration, one can derive the asymptotics of  $\hat{z}(\zeta^+)$  to arbitrary order. In particular, higher-order resonant terms in (3.4), *i.e.*, terms of the form  $r_1^k \hat{z}^{k+1}$  with  $k \geq m$ , will generate higher-order logarithmic contributions. For a more general discussion of these so-called switchback terms, the reader is referred to [28]; a detailed analysis from a geometric point of view can be found in [33].

Finally, we note that the normal form system in (3.3) – and, specifically, Equation (3.3b) – depends on the thus far unspecified correction  $\Delta c(\varepsilon)$  to  $c(0)$ . That correction can be determined by imposing the condition that  $\hat{P}_1^-$  is mapped to  $\hat{P}_1^+$  by  $\Pi_1$ , which is equivalent to the persistence

of the singular orbit  $\Gamma$  for  $\varepsilon > 0$  sufficiently small. Hence, we need to approximate the points  $\widehat{P}_1^-$  and  $\widehat{P}_1^+$  or, rather, the corresponding  $\widehat{z}$ -coordinates  $\widehat{z}^-$  and  $\widehat{z}^+$ , to the order considered here. In particular, it will follow that  $\rho\widehat{z}^- = o(1)$  for  $\rho$  sufficiently small, in accordance with the *a priori* assumption made in the statement of Proposition 3.3.

**Remark 6.** Alternatively, the planar  $(r_1, z)$ -subsystem in (3.2) can be transformed, via a  $C^\infty$ -smooth sequence of coordinate transformations, into a three-term normal form that coincides with the leading-order truncation in (3.7) up to an additional higher-order term [13]; see also [15] and the references therein. However, we will not pursue this approach here.  $\square$

**3.2.2. Estimates for  $\widehat{P}_1^-$  and  $\widehat{P}_1^+$ .** We recall the definition of the manifold  $\mathcal{W}_2^s(Q_2^+)$  from Section 2.3, as well as the fact that the point of intersection  $P_2^+$  of  $\mathcal{W}_2^s(Q_2^+)$  with  $\Sigma_2^+$  is a regular perturbation of  $P_{0_2}^+$ , i.e., of  $\Gamma_2^+ \cap \Sigma_2^+$ , for  $r_2(=\varepsilon)$  fixed and sufficiently small. Since  $\widehat{P}_1^+ = \kappa_{21}(\widehat{P}_2^+)$  must hold when  $c = c(\varepsilon)$  for the singular heteroclinic connection  $\Gamma$  to persist, an estimate for  $\widehat{z}^+$  can be obtained from  $v_2^+$ , after transformation to chart  $K_1$ , by applying the sequence of normal form transformations defined in the proof of Proposition 3.2. Similarly, to approximate  $\widehat{P}_1^-$ , we will estimate the point of intersection  $P^-$  of the manifold  $\mathcal{W}^u(Q_\varepsilon^-)$  with  $\Sigma^-$ , see (2.7), which we will then transform to chart  $K_1$  to find an estimate for  $z^-$  and, consequently, for  $\widehat{z}^-$ .

The required approximation for  $\mathcal{W}^u(Q_\varepsilon^-)$  is naturally obtained from the expansion for  $V(U, c(\varepsilon))$  in (2.6). To the accuracy considered here, it suffices to retain the linear term in  $\Delta c$  in that expansion – the coefficient of which is given by  $\frac{\partial V}{\partial c}(U, c(0))$  – in addition to the leading-order term  $V(U, c(0))$ . However, due to the fact that no exact traveling front solution to (1.1) is known for  $m \geq 3$ , no closed-form expressions are available, in general, for either  $V(U, c(0))$  or its derivatives with respect to  $c$ . Therefore, we will derive a small- $U$  approximation that is valid in the overlap domain between the outer and intermediate regions. Here, we will argue in the original  $(U, V, \varepsilon)$ -coordinates instead of in chart  $K_1$ , as the argument is algebraically simpler. (Conceptually,  $V(U, c(0))$  and its derivatives could of course be approximated in  $K_1$ , as that chart covers the entire phase space of (2.1) for  $U \geq \varepsilon$ , down to and including the limit as  $\varepsilon \rightarrow 0^+$ ; cf. the proof of Proposition 3.1.)

**Lemma 3.4.** *For any  $m \geq 2$  and  $U \in [0, U_0]$ , with  $U_0 > 0$  sufficiently small, there holds*

$$(3.12) \quad V(U, c(0)) = -c(0)U + \frac{1}{m} \frac{2}{c(0)} U^m + O(U^{m+1})$$

and

$$(3.13) \quad \frac{\partial V}{\partial c}(U, c(0)) = \begin{cases} \nu_2 + \left(\frac{2}{c(0)^2} \nu_2 - 1\right)U + O(U^2) & \text{for } m = 2, \\ \nu_m - U + O(U^{m-1}) & \text{for } m \geq 3 \end{cases}$$

in (2.6), where  $\nu_m = \frac{\partial V}{\partial c}(0, c(0))$  is a positive constant.

*Proof.* Dividing (2.5b) formally by (2.5a), we find

$$(3.14) \quad V \frac{dV}{dU} = -cV - 2U^m(1 - U).$$

We note that  $V(U, c(0))$  is smooth in  $U$ ; in fact, since  $\mathcal{W}^u(Q_0^-)$  connects to  $\mathcal{W}^s(Q_0^+)$  for  $c = c(0)$  in (2.5),  $V(0, c(0)) = 0$  must hold. Hence, we may make the Ansatz  $V(U, c(0)) = \sum_{j=1}^{\infty} v_j U^j$ , which we then substitute into (3.14) to determine the coefficients  $v_j$  recursively. In sum, we have  $v_1 = -c(0)$ ,  $v_j = 0$  for  $j \in \{2, \dots, m-1\}$ , as well as  $v_m = \frac{1}{m} \frac{2}{c(0)}$ , which gives (3.12).

Next, we differentiate (3.14) with respect to  $c$  and evaluate the resulting variational equation at  $(U, c(0))$  to obtain

$$(3.15) \quad V(U, c(0)) \frac{\partial}{\partial U} \left( \frac{\partial V}{\partial c}(U, c(0)) \right) = -V(U, c(0)) - \left[ c(0) + \frac{\partial V}{\partial U}(U, c(0)) \right] \frac{\partial V}{\partial c}(U, c(0)).$$

Making use of (3.12) in (3.15) and rearranging, we find

$$\frac{\partial}{\partial U} \left( \frac{\partial V}{\partial c}(U, c(0)) \right) = \frac{2}{c(0)^2} U^{m-2} [1 + O(U^{m-1})] \frac{\partial V}{\partial c}(U, c(0)) - 1.$$

Assuming a series expansion for  $\frac{\partial V}{\partial c}(U, c(0))$ , as before, we obtain (3.13), as claimed, where the leading-order term  $\frac{\partial V}{\partial c}(0, c(0))$  in the series, *i.e.*, the constant  $\nu_m$ , has to remain undetermined. However, as in the proof of Proposition 3.1, a standard phase plane argument (performed by evaluating (3.14) in  $\{U = 0\}$ ) shows that  $\nu_m > 0$ , as claimed.  $\square$

Given the equivalence (by definition) of (3.14) and (2.5),  $V(U, c(0))$  again corresponds to the portion of the singular heteroclinic connection  $\Gamma$  that is located in the outer region, *i.e.*, in  $\{U \geq \rho\}$ ; cf. the discussion in Section 2.4. Moreover, we remark that, while Equation (3.14) is *a priori* singular at  $V = 0$ , the proof of Lemma 3.4 implies that this singularity is removable, as  $V(U, c(0)) \propto U$ ; similarly,  $\frac{\partial V}{\partial c}(U, c(0))$  is  $C^\infty$ -smooth for  $U \in (0, 1)$ . However, the smoothness of  $\frac{\partial^j V}{\partial c^j}$ , with  $j \geq 2$ , cannot be guaranteed: thus, for  $m = 2$ ,  $\frac{\partial^2 V}{\partial c^2}(U, c(0))$  becomes unbounded as  $U \rightarrow 0^+$ ; see Lemma 4.3 below.

Next, we substitute the result of Lemma 3.4 into the expansion for  $V(U, c(\varepsilon))$  in (2.6) to obtain the required estimates for  $\hat{z}^-$  and  $\hat{z}^+$ :

**Lemma 3.5.** *For any  $m \geq 2$  and  $\rho \in (\varepsilon, 1)$ , with  $\varepsilon \in (0, \varepsilon_0]$  and  $\Delta c$  sufficiently small, the points  $\hat{P}_1^- = (\rho, \hat{z}^-, \varepsilon \rho^{-1})$  and  $\hat{P}_1^+ = (\varepsilon, \hat{z}^+, 1)$  satisfy*

$$(3.16) \quad \hat{z}^- = \hat{z}^-(\rho, \Delta c) = \frac{\nu_m}{\rho} \Delta c [1 + o(1)]$$

and

$$(3.17) \quad \hat{z}^+ = \hat{z}^+(\Delta c, \varepsilon) = -\frac{1}{m} \frac{2}{c(0)} \varepsilon^{m-1} [1 + o(1)],$$

respectively, where  $\nu_m$  is defined as in Lemma 3.4 and  $o(1)$  denotes higher-order terms that are  $C^\infty$ -smooth in  $\rho$  and  $\Delta c$  and in  $\Delta c$  and  $\varepsilon$ , respectively.

*Proof.* Recalling the definition of  $\Sigma^-$  in (2.7), as well as of  $P^- = (\rho, V^-, \varepsilon)$ , we first evaluate (3.12) in  $\Sigma^-$  and then substitute the result into (2.6) to find

$$\begin{aligned} V^- &= V(\rho, c(\varepsilon)) = V(\rho, c(0)) + \frac{\partial V}{\partial c}(\rho, c(0)) \Delta c + O[(\Delta c)^2] \\ &= -c(0) \rho \left[ 1 - \frac{1}{m} \frac{2}{c(0)^2} \rho^{m-1} + O(\rho^m) \right] + \nu_m [1 + O(\rho)] \Delta c + O[(\Delta c)^2], \end{aligned}$$

by Lemma 3.4. (Here, the  $O[(\Delta c)^2]$ -terms are  $C^\infty$ -smooth, uniformly in  $\Delta c$ , if  $\rho$  is restricted to compact subsets of  $(0, 1)$ .) Then, we translate that estimate into chart  $K_1$ : since  $V^- = \rho v_1^-$ , cf. (2.3), we obtain  $v_1^- = V^- \rho^{-1}$  for the  $v_1$ -coordinate of  $P_1^-$ , which gives

$$(3.18) \quad z^- = v_1^- + c(0) = \frac{1}{m} \frac{2}{c(0)} \rho^{m-1} [1 + O(\rho)] + \frac{\nu_m}{\rho} [1 + O(\rho)] \Delta c + O[(\Delta c)^2]$$

in  $\Sigma_1^-$ . Next, applying the sequence of normal form transformations defined in the proof of Proposition 3.2 to (3.18), we find that the  $O(\rho^{m-1})$ -terms cancel in  $\tilde{z}^- = z^- + \Delta c - \frac{1}{m} \frac{2}{c(0)} \rho^{m-1}$ . Similarly, the  $O(\rho^m)$ -terms in (3.18) are removed after transformation to  $(r_1, \hat{z}, \varepsilon_1)$ -coordinates, as  $\{\hat{z} = 0\}$  corresponds to the rectified stable manifold  $\mathcal{W}_1^s(\hat{P}_1)$  of the origin  $\hat{P}_1$  when  $\Delta c = 0$ : since that manifold is invariant for (3.3), it follows that  $\hat{z}^-(\rho, 0) = 0$  must hold. Finally, since all the above transformations are near-identity, we find that  $\hat{z}^-$  has to satisfy (3.16), as stated. In particular,

since  $V^- = V(\rho, c(\varepsilon))$  is smooth in  $\rho$  and  $\Delta c$ , by Section 2.1, and since the sequence of transformations that takes  $V^-$  to  $\hat{z}$  is smooth,  $\hat{z}^-$  must be smooth in  $\rho$  and  $\Delta c$  (at least as long as  $\rho$  is positive).

To estimate  $\hat{z}^+$ , we recall that, necessarily,  $P_1^+ = \kappa_{21}(P_2^+)$  in the transition through the intermediate region for the singular heteroclinic orbit  $\Gamma$  to persist when  $c = c(\varepsilon)$  in (2.1). Correspondingly, we will first consider the point  $P_2^+$ , which we will then transform to chart  $K_1$ . Now, our discussion of the  $K_2$ -dynamics in Section 2.2 implies that  $v_2^+ = -c(\varepsilon)$ . Since, moreover,  $v_1^+ = v_2^+$ , cf. Section 2.3, it follows that  $z^+ = -\Delta c$ , which, together with  $r_1 = \varepsilon$  in  $\Sigma_1^+$ , yields  $\hat{z}^+ = -\frac{1}{m} \frac{2}{c(\varepsilon)} \varepsilon^{m-1}$ . Performing once again the sequence of near-identity transformations from Proposition 3.2 and recalling  $\Delta c = o(1)$ , we find (3.17) (to leading order in  $\Delta c$  and  $\varepsilon$ ), as claimed. Finally, the smoothness of  $\hat{z}^+$  is due to the fact that both  $v_1^+ = v_2^+$  and the coordinate transformations defined in the proof of Proposition 3.2 are smooth in  $\Delta c$  and  $\varepsilon$ .  $\square$

We note that the estimates in (3.16) and (3.17), in combination with  $\nu_m > 0$  and  $\Delta c < 0$  – recall Lemma 3.4 and Proposition 3.1, respectively – imply that both  $\hat{z}^-$  and  $\hat{z}^+$  are negative.

**Remark 7.** For  $m = 2$ , the expansions in (3.12) truncate, *i.e.*, the leading-order approximations  $V(U, c(0)) = -U + U^2$  and  $\frac{\partial V}{\partial c}(U, c(0)) = \nu_2 + \left(\frac{2}{c(0)^2} \nu_2 - 1\right)U$  are exact in that case; see Lemma 4.3, where we will show in particular that  $\nu_2 = \frac{1}{3}$ .  $\square$

**Remark 8.** The estimate for  $\hat{z}^+$  in (3.17) could be refined by calculating the sequence of normal form transformations in Proposition 3.2 explicitly to higher order. However, as will become clear in the following, the accuracy provided by Lemma 3.5 is sufficient to the order considered here; in particular, explicit knowledge of the leading-order  $O(\varepsilon^{m-1})$ -term in (3.17) is crucial for the proof of Theorem 1.1.  $\square$

**3.2.3. End of proof of Theorem 1.1.** Finally, we make use of Lemma 3.5 to solve the approximate normal form system that is obtained from (3.3) by neglecting the  $O(2m + 1)$ -terms in (3.3b). The corresponding solution will yield the desired condition that  $c(\varepsilon)$  has to satisfy for  $\hat{P}_1^-$  to be mapped to  $\hat{P}_1^+$  under the flow of (3.3), *i.e.*, for  $\Pi_1(\hat{P}_1^-) = \hat{P}_1^+$  to hold, to leading order. That condition will fix the leading-order  $\varepsilon$ -asymptotics of  $\Delta c$ , as stated in (1.8), thus completing the proof of Theorem 1.1.

**Proposition 3.6.** *For  $\Gamma$  to persist when  $\varepsilon \in (0, \varepsilon_0]$  in (2.1),  $\Delta c$  must necessarily satisfy*

$$(3.19) \quad \Delta c(\varepsilon) = \gamma_m \varepsilon^m + o(\varepsilon^m),$$

with  $\gamma_m$  a negative constant.

*Proof.* Given the approximation for the transition map  $\Pi_1$  derived in Proposition 3.3, it only remains to impose the requirement that  $\hat{z}(\zeta^+) = \hat{z}^+$  in (3.6), which will allow us to determine the leading-order  $\varepsilon$ -asymptotics of  $\Delta c$ .

Substituting the estimates for  $\hat{z}^-$  and  $\hat{z}^+$  from Lemma 3.5 into (3.6) and recalling  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$ , we find the following necessary condition for the existence of a connecting orbit between  $\hat{P}_1^-$  and  $\hat{P}_1^+$ :

$$(3.20) \quad -\frac{1}{m} \frac{2}{c(0)} \varepsilon^{m-1} [1 + o(1)] = \hat{z}^+(\Delta c, \varepsilon) = \frac{\rho \hat{z}^-(\rho, \Delta c)}{\varepsilon} \{1 + O[(\rho \hat{z}^-(\rho, \Delta c))^{m-1} \ln \varepsilon]\} \\ = \frac{\nu_m \Delta c}{\varepsilon} [1 + o(1)].$$

(Here, the  $o(1)$ -terms in (3.20) are smooth in  $\Delta c$  and  $\varepsilon$ , by Lemma 3.5, for  $\rho$  in compact subsets of  $(0, 1)$ .) The as yet undetermined parameter  $\Delta c$  is implicitly defined by the relation in (3.20); since

that relation is trivially satisfied at  $(\Delta c, \varepsilon) = (0, 0)$  and since  $\nu_m > 0$ , by Lemma 3.4, it follows from the Implicit Function Theorem that a solution  $\Delta c = \Delta c(\varepsilon)$  exists for  $(\Delta c, \varepsilon)$  small. Although this solution might *a priori* be  $\rho$ -dependent, no such dependence is possible conceptually, as  $c(\varepsilon)$  cannot depend on the (arbitrary) definition of the section  $\Sigma_1^-$  in (2.12). In particular, since (3.20) is valid for *any*  $\rho$  positive and small, we may pass to the limit of  $\rho \rightarrow 0^+$  to obtain

$$(3.21) \quad \frac{\Delta c}{\varepsilon^m} = -\frac{1}{m} \frac{2}{c(0)} \frac{1}{\nu_m} \equiv \text{constant}$$

to leading order, where the constant, which we denote by  $\gamma_m \equiv \gamma(m, c(0), \nu_m)$ , is  $O(1)$ . Finally,  $\nu_m > 0$  implies that  $\gamma_m$  is negative (as stated also in Proposition 3.1), which completes the proof.  $\square$

A detailed analysis of the condition in (3.20) suggests that the  $o(\varepsilon^m)$ -term in (3.19) will actually be of the order  $O(\varepsilon^{m+1})$ , as well as that the lowest-order logarithmic term in  $\Delta c$  will enter at  $O(\varepsilon^{m^2} \ln \varepsilon)$ . (This intuition will be confirmed explicitly in Section 4 for  $m = 2$ .) Similarly, expanding (3.8) in terms of  $\varepsilon$  and making use of  $\zeta^+ = -\ln \varepsilon + O(1)$  as well as of  $\Delta c(\varepsilon) = O(\varepsilon^m)$ , one confirms that the lowest-order logarithmic term in  $\hat{z}^+$  is of the order  $O(\varepsilon^{m^2-1} \ln \varepsilon)$ , as postulated already in Section 3.2.1. Finally, we conjecture that  $\Delta c$  is a  $C^\infty$ -smooth function of  $\varepsilon$  and  $\ln \varepsilon$ : since, by standard results on differentiability with respect to initial conditions and parameters [2], the solution  $\hat{z}(\zeta)$  of (3.3b) is smooth in  $\hat{z}^-$  as well as in  $\Delta c$ , and since  $\hat{z}^-$  and  $\hat{z}^+$  are smooth in their arguments, by Lemma 3.5, we expect the relation in (3.20) to be smooth in  $\varepsilon$  and  $\ln \varepsilon$ . However, a fully rigorous discussion of these questions for general  $m \geq 3$  in (1.1) has to be left for future study.

We remark that the negativity of  $\gamma_m$  in (3.19) agrees with the more general argumentation given in [8, 29], which implies that a cut-off slows down ‘pulled’ and ‘pushed’ fronts, but that it speeds up ‘bistable’ fronts. By that classification, the decrease in propagation speed that is caused by the cut-off in (1.1) is due to  $f_m(u) > 0$  for  $u < \varepsilon$ , as is generally the case for fronts propagating into unstable states. However, while the presence of a cut-off induces a logarithmic correction in the propagation speed for  $m = 1$  in (1.1), as shown *e.g.* in [12, 17], the leading-order asymptotics of  $\Delta c$  obeys a power law in the degenerate case discussed here, with  $m \geq 2$ , as was also observed in [8, 29] in the pushed and bistable propagation regimes. Moreover, while that asymptotics is universal in the pulled regime, its counterpart in (3.19) is cut-off dependent, as the estimate for  $\hat{z}^+$  in (3.17) depends crucially on the choice of (Heaviside) cut-off  $\Theta$  in (1.1); recall the proof of Lemma 3.5. A systematic analysis of the bistable propagation regime, from a geometric point of view, can be found in [18], while the pushed regime will be studied in the upcoming article [19].

**Remark 9.** Intuitively, the result of Proposition 3.6 can be seen as follows: retaining only the leading-order terms in (3.2b), we find

$$z' = \Delta c + z - \frac{2}{c(0)} r_1^{m-1},$$

which, together with  $r_1(\zeta) = \rho e^{-\zeta}$  and  $z(0) = \frac{1}{m} \frac{2}{c(0)} \rho^{m-1}$  (to lowest order), gives the leading-order solution

$$z(\zeta) = \Delta c(e^\zeta - 1) + \frac{2}{c(0)} \frac{\rho^{m-1}}{m} e^{-(m-1)\zeta}.$$

Then, the requirement that  $z^+ = z(\zeta^+)$ , in combination with  $\zeta^+ = -\ln \varepsilon + O(1)$  and  $z^+ = -\Delta c$ , yields a necessary condition on  $\Delta c$ . Solving that condition, we have  $\Delta c = O(\varepsilon^m)$  to leading order, as claimed.  $\square$

**3.3. Computability of  $\Delta c(\varepsilon)$ .** Finally, we investigate the question of whether the correction  $\Delta c = c(\varepsilon) - c(0)$  can be evaluated exactly; in particular, we discuss the computability of the leading-order coefficient  $\gamma_m$  in (3.19). First, we make use of Proposition 3.6, *i.e.*, of the fact that the asymptotics

of  $\Delta c(\varepsilon)$  is now known to lowest order, to rewrite the expansion for  $\mathcal{W}^u(Q_\varepsilon^-)$  in (2.6) in terms of  $U$  and  $\varepsilon$ :

$$(3.22) \quad V(U, c(\varepsilon)) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j V}{\partial c^j}(U, c(0)) [\gamma_m \varepsilon^m + o(\varepsilon^m)]^j =: \sum_{j=0}^m V_{j0}(U) \varepsilon^j + o(\varepsilon^m).$$

(The notation in (3.22), with  $V_{jk}$  denoting the coefficient of  $\varepsilon^j (\ln \varepsilon)^k$  in the expansion, is chosen for consistency with Section 4 below, where the lowest-order logarithmic term corresponding to  $k = 1$  will be evaluated explicitly.) Although the coefficient functions  $V_{j0}$  cannot be determined exactly in general, one can derive their small- $U$  asymptotics as follows:

**Proposition 3.7.** *Let  $m \geq 2$ , let  $(U, \varepsilon) \in [0, U_0] \times [0, \varepsilon_0]$ , with  $U_0$  and  $\varepsilon_0$  positive and small, and let  $c(\varepsilon)$  be defined as in Proposition 3.1. Then, the coefficient functions  $V_{j0}$  in (3.22) satisfy*

$$(3.23) \quad V_{00}(U) = -c(0)U + \frac{1}{m} \frac{2}{c(0)} U^m + O(U^{m+1}),$$

$V_{j0}(U) \equiv 0$  for  $j \in \{1, \dots, m-1\}$ , and

$$(3.24) \quad V_{m0}(U) = \begin{cases} [\nu_2 + (\frac{2}{c(0)^2} \nu_2 - 1)U + O(U^2)] \gamma_2 & \text{for } m = 2, \\ [\nu_m - U + O(U^{m-1})] \gamma_m & \text{for } m \geq 3, \end{cases}$$

where  $\nu_m$  and  $\gamma_m$  are defined as in Lemma 3.4 and Proposition 3.6, respectively.

*Proof.* Since  $V_{00}(U) = V(U, c(0))$ , Lemma 3.4 implies that  $V_{00}$  is regular, and that its small- $U$  asymptotics is as given in (3.23); recall (3.12). Moreover, it follows from (3.22) by inspection that  $V_{j0}$  is identically zero for  $j \in \{1, \dots, m-1\}$ .

Hence, the only higher-order term making a contribution in (3.22), to the accuracy considered here, is the  $V_{m0}$ -term, and it remains to show (3.24). To that end, we note that

$$V_{m0}(U) = \frac{1}{1!} \frac{\partial V}{\partial c}(U, c(0)) \cdot \gamma_m,$$

see (3.22). Given the (regular)  $U$ -asymptotics of  $\frac{\partial V}{\partial c}(U, c(0))$ , as stated in (3.13), we immediately find (3.24), which completes the proof.  $\square$

While Proposition 3.7 yields an approximation for  $V(U, c(\varepsilon))$  that is accurate up to an  $o(\varepsilon^m)$ -error, it does not automatically imply that the  $O(\varepsilon^m)$ -coefficient  $\gamma_m$  in  $\Delta c$  can be evaluated: the expansion for  $V_{m0}$  in (3.24) depends on the as yet undetermined constant  $\nu_m$ . However, by (3.21),  $\gamma_m$  is computable only if  $\nu_m$  is.

By contrast, expanding  $V(U, c(\varepsilon))$  about  $U = 1$ , *i.e.*, considering  $\mathcal{W}^u(Q_\varepsilon^-)$  close to the equilibrium point  $Q_\varepsilon^-$ , we find  $V(W, c(0)) = -\lambda_+ W + O(W^2)$ , with  $W = 1 - U$ . (Here and in the following,  $\lambda_\pm = -\frac{c(0)}{2} \pm \sqrt{\frac{c(0)^2}{4} + 2}$ ; see Lemma 2.1.) As in the proof of Lemma 3.4, it now follows that  $\frac{\partial V}{\partial c}(W, c(0))$  satisfies the variational equation

$$(3.25) \quad \frac{\partial}{\partial W} \left( \frac{\partial V}{\partial c}(W, c(0)) \right) = \frac{\lambda_-}{\lambda_+} \frac{1}{W} [1 + O(W)] \frac{\partial V}{\partial c}(W, c(0)) + 1,$$

which has the leading-order solution

$$(3.26) \quad \frac{\partial V}{\partial c}(W, c(0)) = \frac{\lambda_+}{\lambda_+ - \lambda_-} W + \omega W^{\frac{\lambda_-}{\lambda_+}},$$

with  $\omega$  a constant of integration. Since  $\frac{\partial V}{\partial c}(W, c(0))$  must remain bounded (and, indeed, go to zero) as  $W \rightarrow 0^+$ , it follows from  $\frac{\lambda_-}{\lambda_+} < 0$  that the only admissible solution is obtained for  $\omega = 0$

in (3.26). Reverting to the original  $(U, V)$ -coordinates, we find  $\frac{\partial V}{\partial c}(U, c(0)) = -\frac{\lambda_+}{\lambda_+ - \lambda_-}(U - 1)$  for  $U \approx 1$ , where the coefficient  $\frac{\lambda_+}{\lambda_+ - \lambda_-}$  is computable if  $c(0)$  is known.

However, knowledge of that coefficient does not necessarily allow one to calculate the constant  $\nu_m$ , since both (3.26) and its small- $U$  analogue in (3.13) are only valid locally, *i.e.*, in a neighborhood about 1 and 0, respectively. In other words, there seems to be no way of relating the two expansions in general, which implies that  $\gamma_m$  is certainly not computable if no explicit solution to the traveling front equation without cut-off in (1.6) is available.

To see why  $\gamma_m$  can, in principle, be evaluated exactly in cases where such a solution can be found, we recall the proof of Lemma 3.4 and, in particular, the variational equation in (3.15), which determines  $\frac{\partial V}{\partial c}(U, c(0))$ . Here, it is important to note that (3.15) is *a priori* exact, and that an error is introduced only by the fact that, generally,  $V(U, c(0))$  has to be approximated, as was done in Lemma 3.4. Making use of (3.14) (with  $c$  replaced by  $c(0)$ ) and rearranging, we may rewrite (3.15) as

$$(3.27) \quad \frac{\partial}{\partial U} \left( \frac{\partial V}{\partial c}(U, c(0)) \right) = \frac{2U^m(1-U)}{V(U, c(0))^2} \cdot \frac{\partial V}{\partial c}(U, c(0)) - 1,$$

where we additionally require  $\frac{\partial V}{\partial c}(1, c(0)) = 0$ , as before. For values of  $m$  for which  $V(U, c(0))$  is known explicitly, (3.27) can be integrated using the variation-of-constants formula, *i.e.*, standard results on differentiability with respect to parameters [2] imply that the variational equation for (3.14) certainly has a solution. As a consequence,  $\frac{\partial V}{\partial c}(U, c(0))$  can be found in closed form provided the integrals that arise in solving (3.27) can be evaluated analytically. This observation seems to agree with the findings of Benguria *et al.* in [8], where it is claimed that the leading-order coefficient in  $\Delta c$  is computable for *all* values of  $m$  for which the function that maximizes a certain variational functional can be found; see also the discussion in Section 5 below. Even in cases where an explicit solution to Equation (1.6) – and, consequently,  $V(U, c(0))$  – is available, there seems to be no guarantee that (3.27) can also be integrated in closed form, *i.e.*, that  $\frac{\partial V}{\partial c}(U, c(0))$  can be determined exactly and evaluated at  $U = 0$  to give  $\nu_m$ . Whether this restriction is intrinsic, or merely of a methodological nature, must remain open here.

To the best of our knowledge, the only  $m$ -value for which an explicit traveling front solution to (1.5) is known is, in fact,  $m = 2$ . We note that the leading-order approximation in (3.26) is exact, and that it agrees with (3.13) in that case, as  $\frac{\partial V}{\partial c}(U, 1) = -\frac{1}{3}(U - 1)$ ; cf. Proposition 4.6 below.

**Remark 10.** As stated in the proof of Proposition 3.6, the  $\varepsilon$ -asymptotics of  $\Delta c$  in (3.19) has to be independent of  $\rho$ ; correspondingly, it was derived here by taking the limit of  $\rho \rightarrow 0^+$  in the definition of  $\Sigma_1^-$ . Alternatively, (3.19) can also be obtained by concatenating the dynamics in the outer and intermediate regions for  $\rho$  small, but fixed. Specifically, taking the sequence of normal form transformations defined in Proposition 3.2 to a sufficient degree of accuracy, one can eliminate explicitly any  $\rho$ -dependence from  $\widehat{P}_1^-$ , with the notable exception of terms that involve  $\ln \rho$  (and powers thereof); however, those terms can be removed by refining the approximation for the transition map  $\Pi_1$  in Proposition 3.3 accordingly and by retaining the corresponding  $\ln \varepsilon$ -dependent terms in the expansion for  $V(U, c(\varepsilon))$  in (3.22). (For general  $m \geq 3$ , the lowest-order such term will most probably correspond to  $V_{m^2_1}$ ; recall the discussion in Section 3.2.3.) This approach, though conceptually equivalent to the one adopted here, is more involved algebraically and will therefore not be pursued further.  $\square$

#### 4. PROOF OF THEOREM 1.2

In this section, we discuss in detail the case where  $m = 2$  in (1.1). That case is of particular interest, as it corresponds to the only value of  $m$  for which both the critical front speed and the

corresponding front solution to the traveling front equation without cut-off in (1.6) are known explicitly [35], with

$$(4.1) \quad U(\xi) = \frac{1}{1 + e^\xi} \quad \text{for } c(0) = 1$$

in (1.3). (One checks easily that (4.1) satisfies  $\lim_{\xi \rightarrow -\infty} U(\xi) = 1$  and  $\lim_{\xi \rightarrow \infty} U(\xi) = 0$ , as required in (1.4).) Explicit knowledge of both the front and its propagation speed will allow us to refine the result of Theorem 1.1 and to obtain the  $\varepsilon$ -asymptotics of  $\Delta c$  up to and including terms of the order  $O(\varepsilon^4)$ , as stated in Theorem 1.2.

We begin by recalling some of the relevant equations from Section 2 for convenience here: when  $m = 2$ , the cut-off first-order system in (2.1) reads

$$(4.2a) \quad U' = V,$$

$$(4.2b) \quad V' = -cV - 2U^2(1 - U)\Theta(U - \varepsilon),$$

$$(4.2c) \quad \varepsilon' = 0.$$

For future reference, we note that the singular heteroclinic orbit  $\Gamma$  corresponding to (4.1) – *i.e.*, the connection between the two equilibria  $Q_0^-$  and  $Q_0^+$ , for  $\varepsilon = 0$  in (4.2) – is known explicitly in this case, with  $V(U) = U(U - 1)$  [35] for  $0 \leq U \leq 1$ , as can also easily be seen from (4.1).

Applying the blow-up transformation in (2.2) to (4.2), we find

$$(4.3a) \quad u_2' = v_2,$$

$$(4.3b) \quad v_2' = -cv_2,$$

$$(4.3c) \quad r_2' = 0$$

for the relevant cut-off equations in chart  $K_2$ , as given in (2.8). Similarly, the dynamics in  $K_1$  is described by

$$(4.4a) \quad r_1' = r_1 v_1,$$

$$(4.4b) \quad v_1' = -cv_1 - v_1^2 - 2r_1(1 - r_1),$$

$$(4.4c) \quad \varepsilon_1' = -\varepsilon_1 v_1;$$

recall (2.11).

**4.1. Asymptotics of  $c(\varepsilon)$ .** The existence of  $c(\varepsilon)$ , as defined in Theorem 1.1, has already been shown in Section 3.1, since the proof of Proposition 3.1 is equally valid for  $m = 2$ . Hence, we only need to prove the corresponding  $\varepsilon$ -asymptotics of  $\Delta c$ , as stated in (1.9). To that end, we retrace the argumentation from Section 3.2, which we adapt as required; in particular, given the closed-form solution to the traveling front problem without cut-off in (4.1), many estimates become more explicit now. To avoid unnecessary repetition, we will omit some of the details in the following.

4.1.1. *Transition map  $\Pi_1 : \Sigma_1^- \rightarrow \Sigma_1^+$ .* For  $m = 2$ , the transformed equations in (3.2), with  $c(\varepsilon) = 1 + \Delta c(\varepsilon)$  and  $z = v_1 + 1$ , are given by

$$(4.5a) \quad r_1' = -r_1,$$

$$(4.5b) \quad z' = \Delta c + z - \frac{2r_1(1 - r_1)}{1 - z},$$

$$(4.5c) \quad \varepsilon_1' = \varepsilon_1.$$

In analogy to Proposition 3.2, we now have the following result on the normal form system corresponding to (4.5):

**Proposition 4.1.** *Let  $\mathcal{V} = \{(r_1, z, \varepsilon_1) \mid (r_1, z, \varepsilon_1) \in [0, \rho] \times [-z_0, 0] \times [0, 1]\}$ , for  $\rho$  positive and sufficiently small and  $z_0$  defined as before. Then, there exists a sequence of  $\mathcal{C}^\infty$ -smooth coordinate transformations on  $\mathcal{V}$ , with  $(r_1, z, \varepsilon_1) \mapsto (r_1, \hat{z}, \varepsilon_1)$  and  $\hat{z} = z + O(r_1, \Delta c)$ , such that (4.5) can be written as*

$$(4.6a) \quad r'_1 = -r_1,$$

$$(4.6b) \quad \hat{z}' = \hat{z} - \frac{2}{(1 + \Delta c)^3} r_1 \hat{z}^2 [1 + O(r_1 \hat{z})],$$

$$(4.6c) \quad \varepsilon'_1 = \varepsilon_1,$$

where  $O(r_1 \hat{z})$  denotes a smooth function of  $r_1 \hat{z}$ .

*Proof.* The proof is similar to that of Proposition 3.2; however, since we are interested in deriving the higher-order asymptotics of  $\Delta c$  now, we calculate the required sequence of normal form transformations explicitly to higher order than was done in Section 3.

In particular, noting that the lowest-order resonance in (4.5) is found at  $O(r_1 z^2)$ , after expansion of the third term in (4.5b), one can eliminate all non-resonant terms up to and including order 3 via the following sequence of coordinate transformations:

$$(4.7) \quad \begin{aligned} z \mapsto \tilde{z} := z + \Delta c \mapsto \tilde{z} - \frac{1}{1 + \Delta c} r_1 \mapsto \tilde{z} + \frac{2}{3} \frac{1}{1 + \Delta c} \left[ 1 - \frac{1}{(1 + \Delta c)^2} \right] r_1^2 - \frac{2}{(1 + \Delta c)^2} r_1 \tilde{z} \\ \mapsto \tilde{z} - \frac{1}{2} \frac{1}{(1 + \Delta c)^3} \left[ 1 - \frac{1}{(1 + \Delta c)^2} \right] r_1^3 + \frac{1}{(1 + \Delta c)^2} r_1^2 \tilde{z}. \end{aligned}$$

Applying a sequence of near-identity transformations (as defined in Proposition 3.2 for general  $m \geq 3$ ) to the resulting  $\tilde{z}$ -equation

$$(4.8) \quad \tilde{z}' = \tilde{z} - \frac{2}{(1 + \Delta c)^3} r_1 \tilde{z}^2 + O(4),$$

one obtains the normal form in (4.6b), as claimed. (As in the proof of Proposition 3.2, we note that the non-identity part in these transformations is actually of at least order 4, jointly in  $r_1$  and  $\tilde{z}$ .)  $\square$

Finally, to approximate the transition map  $\Pi_1 : \Sigma_1^- \rightarrow \Sigma_1^+$  to the order considered here, we require the following (refined) analogue of Proposition 3.3:

**Proposition 4.2.** *For  $m = 2$  and  $|\rho \hat{z}^-|$  sufficiently small, the map  $\Pi_1 : \Sigma_1^- \rightarrow \Sigma_1^+$  satisfies  $(\rho, \hat{z}^-, \varepsilon \rho^{-1}) \mapsto (\varepsilon, \hat{z}^+, 1)$ , with*

$$(4.9) \quad \hat{z}^+ = \frac{\rho \hat{z}^-}{\varepsilon} \left\{ 1 + \frac{2}{(1 + \Delta c)^3} \rho \hat{z}^- \ln \frac{\varepsilon}{\rho} + O[(\rho \hat{z}^-)^2 (\ln \varepsilon)^2] \right\}.$$

*Proof.* As in the proof of Proposition 3.3, we solve the leading-order normal form system that is obtained by retaining only the lowest-order resonant terms in (4.6b), which gives

$$(4.10) \quad \check{z}(\zeta) = \frac{c(\varepsilon)^3 e^\zeta}{2\rho\zeta + C_2 c(\varepsilon)^3};$$

cf. (3.8). From  $\zeta^- = 0$  and  $\check{z}(0) = \hat{z}^-$ , it follows as before that the constant  $C_2$  in (4.10) satisfies  $C_2 = (\hat{z}^-)^{-1}$ . Expanding (4.10) for  $|\rho \hat{z}^-|$  small, we find

$$\check{z}(\zeta) = \hat{z}^- e^\zeta \left\{ 1 - \frac{2}{(1 + \Delta c)^3} \rho \hat{z}^- \zeta + O[(\rho \hat{z}^-)^2 \zeta^2] \right\}.$$

Evaluating  $\tilde{z}(\zeta)$  at  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$  and noting that the estimate for the error incurred when replacing  $\hat{z}$  with  $\tilde{z}$  from Proposition 3.3 remains valid when  $m = 2$ , since  $|\hat{z}^+ - \tilde{z}^+| = \frac{\rho \hat{z}^-}{\varepsilon} O[(\rho \hat{z}^-)^2 \ln \varepsilon]$  in that case, we obtain (4.9), which completes the proof.  $\square$

4.1.2. *Estimates for  $\hat{P}_1^-$  and  $\hat{P}_1^+$ .* Next, we recall the expansion for  $V(U, c(\varepsilon))$  in (2.6), as well as the fact that the coefficients in that expansion are determined by successive derivatives of  $V$  with respect to  $c$ , evaluated at  $c(0)$ . In contrast to the general case discussed in Section 3 above, the argument for  $m = 2$  is simplified by the fact that these derivatives can now be found in closed form, at least to the order considered here. In particular, given  $c(0) = 1$  and  $V(U, 1) = U(U - 1)$ , the result of Lemma 3.4 can be refined as follows:

**Lemma 4.3.** *For  $m = 2$ , there holds  $V(U, 1) = U(U - 1)$ ,*

$$(4.11) \quad \frac{\partial V}{\partial c}(U, 1) = -\frac{1}{3}(U - 1),$$

and

$$(4.12) \quad \frac{\partial^2 V}{\partial c^2}(U, 1) = \frac{2U^2 - 4U + 3 + 2 \ln U}{9(U - 1)^2}$$

in (2.6).

*Proof.* Proceeding as in the proof of Lemma 3.4, we first differentiate

$$(4.13) \quad V \frac{dV}{dU} = -cV - 2U^2(1 - U)$$

once with respect to  $c$ . Evaluating the result at  $(U, 1)$ , substituting in  $V(U, 1) = U(U - 1)$ , and rearranging, we find the variational equation

$$(4.14) \quad \frac{\partial}{\partial U} \left( \frac{\partial V}{\partial c}(U, 1) \right) = -\frac{2}{U - 1} \cdot \frac{\partial V}{\partial c}(U, 1) - 1$$

for  $\frac{\partial V}{\partial c}(U, 1)$ ; cf. (3.15). The general solution of (4.14) is given by

$$(4.15) \quad \frac{\partial V}{\partial c}(U, 1) = \frac{\theta_1}{(U - 1)^2} - \frac{1}{3}(U - 1),$$

where  $\theta_1$  is a constant of integration. Hence, for  $\frac{\partial V}{\partial c}(U, 1)$  to remain bounded (and, indeed, go to zero) as  $U \rightarrow 1^-$ , we require  $\theta_1 = 0$ , which yields (4.11), as stated.

Similarly, differentiating (4.13) twice with respect to  $c$  and substituting once again gives

$$\frac{\partial}{\partial U} \left( \frac{\partial^2 V}{\partial c^2}(U, 1) \right) = -\frac{2}{U - 1} \frac{\partial^2 V}{\partial c^2}(U, 1) + \frac{4}{9U},$$

which has the solution

$$(4.16) \quad \frac{\partial^2 V}{\partial c^2}(U, 1) = \frac{\theta_2}{(U - 1)^2} + \frac{2U^2 - 4U + 2 \ln U}{9(U - 1)^2}$$

for some constant  $\theta_2$ . Although (4.16) is *a priori* singular as  $U \rightarrow 1^-$ , since the second term will become unbounded, that singularity can be removed, and the boundary condition at  $U = 1$  satisfied, by a suitable choice of the constant  $\theta_2$ : for  $\theta_2 = \frac{2}{3}$  in (4.16), we find with l'Hôpital's Rule that  $\lim_{U \rightarrow 1^-} \frac{\partial^2 V}{\partial c^2}(U, 1) = 0$ , as required. This verifies (4.12), which completes the proof.  $\square$

Based on the result of Lemma 4.3, we can improve the estimates for the transformed entry and exit points  $\hat{P}_1^-$  and  $\hat{P}_1^+$  that were obtained in Lemma 3.5:

**Lemma 4.4.** For  $m = 2$  and  $\rho \in (\varepsilon, 1)$ , with  $\varepsilon \in (0, \varepsilon_0]$  and  $\Delta c$  sufficiently small, the points  $\widehat{P}_1^- = (\rho, \widehat{z}^-, \varepsilon\rho^{-1})$  and  $\widehat{P}_1^+ = (\varepsilon, \widehat{z}^+, 1)$  satisfy

$$(4.17) \quad \widehat{z}^- = \widehat{z}^-(\rho, \Delta c) = \frac{\Delta c}{3\rho} \left\{ 1 + \left( 1 + \frac{2}{3} \ln \rho \right) \Delta c + O[(\Delta c)^2] \right\}$$

and

$$(4.18) \quad \widehat{z}^+ = \widehat{z}^+(\Delta c, \varepsilon) = -\varepsilon[1 - 2\varepsilon + \varepsilon^2 + O(\varepsilon^3)] + \varepsilon \left[ 1 - \frac{14}{3}\varepsilon + 2\varepsilon^2 + O(\varepsilon^3) \right] \Delta c + O[(\Delta c)^2],$$

respectively, where the corresponding higher-order terms are  $\mathcal{C}^\infty$ -smooth in  $\rho$  and  $\Delta c$  and in  $\Delta c$  and  $\varepsilon$ , respectively.

*Proof.* The proof is similar to that of Lemma 3.5; however, we now consider terms up to and including  $O[(\Delta c)^2]$  in (2.6) in the expansion for  $V^- = V(\rho, c(\varepsilon))$ , which gives

$$\begin{aligned} V^- &= V(\rho, 1) + \frac{\partial V}{\partial c}(\rho, 1)\Delta c + \frac{1}{2} \frac{\partial^2 V}{\partial c^2}(\rho, 1)(\Delta c)^2 + O[(\Delta c)^3] \\ &= \rho(\rho - 1) - \frac{1}{3}(\rho - 1)\Delta c + \frac{1}{9} \frac{\rho^2 - 4\rho + 3 + 2 \ln \rho}{(\rho - 1)^2} (\Delta c)^2 + O[(\Delta c)^3]. \end{aligned}$$

(Here and in the following, the error terms are again  $\mathcal{C}^\infty$ -smooth in  $\Delta c$  as long as  $\rho$  is positive, uniformly in compact subsets of  $(0, 1)$ .) Recalling  $v_1^- = V^- \rho^{-1}$  in  $\Sigma_1^-$  and expanding the result in  $\rho$ , we have

$$(4.19) \quad z^- = \rho - \frac{1}{3\rho}(\rho - 1)\Delta c + \frac{1}{9\rho} [2 \ln \rho + 3 + 4\rho \ln \rho + 2\rho + O(\rho^2 \ln \rho)] (\Delta c)^2 + O[(\Delta c)^3],$$

after transformation to  $z$ . Now, applying first the sequence of coordinate transformations from (4.7) and, subsequently, the sequence of near-identity transformations defined in the proof of Proposition 4.1 to (4.19), we find that the corresponding value of  $\widehat{z}$  is given by (4.17).

The point  $\widehat{P}_1^+$ , on the other hand, only depends on the dynamics in the inner region and is hence not affected by the result of Lemma 4.3. Applying (4.7) again, we find that  $\widehat{z}^+$  and, therefore, also  $\widehat{z}^+$  satisfies (4.18), as claimed.  $\square$

**4.1.3. End of proof of Theorem 1.2.** Given the estimates for  $\widehat{z}^-$  and  $\widehat{z}^+$  found in Lemma 4.4, we can now complete the proof of Theorem 1.2. In contrast to the general case considered in Section 3, however, we can determine both the  $\varepsilon$ -asymptotics of  $\Delta c$  and the numerical values of the corresponding coefficients up to and including  $O(\varepsilon^4)$  here, which is due to the fact that the leading-order normal form transformation in (4.7) is available explicitly to the corresponding order:

**Proposition 4.5.** For  $\Gamma$  to persist when  $\varepsilon \in (0, \varepsilon_0]$  in (4.2),  $\Delta c$  must necessarily satisfy

$$(4.20) \quad \Delta c(\varepsilon) = \gamma_{20}\varepsilon^2 + \gamma_{30}\varepsilon^3 + \gamma_{41}\varepsilon^4 \ln \varepsilon + \gamma_{40}\varepsilon^4 + o(\varepsilon^4),$$

where  $\gamma_{20} = -3$ ,  $\gamma_{30} = 6$ ,  $\gamma_{41} = -6$ , and  $\gamma_{40} = -21$ .

*Proof.* The proof is similar to that of Proposition 3.6: we recall  $\zeta^+ = -\ln \frac{\varepsilon}{\rho}$  and the estimates for  $\widehat{z}^-$  and  $\widehat{z}^+$  from (4.17) and (4.18), respectively. Then, the requirement that  $\widehat{z}^+ = \widehat{z}(\zeta^+)$  in (4.9) yields the following analogue of Equation (3.20) for the implicit condition that  $\Delta c$  has to satisfy in

this case:

$$\begin{aligned}
(4.21) \quad & -\varepsilon[1 - 2\varepsilon + \varepsilon^2 + O(\varepsilon^3)] + \varepsilon\left[1 - \frac{14}{3}\varepsilon + 2\varepsilon^2 + O(\varepsilon^3)\right]\Delta c + O[(\Delta c)^2] \\
& = \hat{z}^+(\Delta c, \varepsilon) = \frac{\rho \hat{z}^-(\rho, \Delta c)}{\varepsilon} \left\{ 1 + \frac{2}{(1 + \Delta c)^3} \rho \hat{z}^-(\rho, \Delta c) \ln \frac{\varepsilon}{\rho} + O[(\rho \hat{z}^-(\rho, \Delta c))^2 \ln \varepsilon] \right\} \\
& = \frac{\Delta c}{3\varepsilon} \left\{ 1 + \left(1 + \frac{2}{3} \ln \varepsilon\right) \Delta c + O[(\Delta c)^2] \right\},
\end{aligned}$$

where the  $O[(\Delta c)^2]$ -terms in (4.21) can additionally depend on (powers of)  $\varepsilon$  and  $\ln \varepsilon$ . We remark that a trivial solution to (4.21) is given by  $(\Delta c, \varepsilon) = (0, 0)$ , as well as that  $\Delta c(\varepsilon)$  is again well-defined for  $\varepsilon > 0$  sufficiently small, by the Implicit Function Theorem. Taking into account that the relation in (4.21) has to be independent of  $\rho$ , as in the proof of Proposition 3.6, we may now pass to the limit as  $\rho \rightarrow 0^+$ . (In particular, we note that (4.21) cannot contain  $\ln \rho$ -dependent terms, on principle, which is due to the fact that any such terms are canceled by the corresponding terms in (4.19), irrespective of how the coefficients  $\gamma_{jk}$  in (4.20) are chosen; similarly, higher-order terms in  $\rho$  must cancel due to the  $\rho$ -independence of  $\Delta c$ .)

Finally, the  $\varepsilon$ -asymptotics of  $\Delta c$ , as stated in (4.20), can be determined recursively so that the condition in (4.21) is satisfied to the order considered here. In particular, we have the lowest-order approximation  $\Delta c = -3\varepsilon^2[1 + o(1)]$ , *i.e.*,  $\gamma_{20} = -3$ , which is in agreement with the general expression for  $\gamma_m$  in (3.21), evaluated at  $m = 2$ . Similarly, defining the next-order correction to  $\Delta c$  as  $\Delta^{(2)}c := \Delta c - (-3\varepsilon^2)$ , substituting into (4.21), and rearranging, we find that the  $O(\varepsilon^2)$ -terms cancel by construction; then, collecting powers of  $\varepsilon$  and noting that  $\Delta^{(2)}c = o(\varepsilon^2)$  must hold, we obtain

$$(4.22) \quad 2\varepsilon^3 - 4\varepsilon^4[1 + o(1)] = \frac{\Delta^{(2)}c}{3} + (3 + 2 \ln \varepsilon)\varepsilon^4[1 + o(1)],$$

which implies that  $\Delta^{(2)}c$  has to satisfy  $2\varepsilon^3 = \frac{\Delta^{(2)}c}{3}$ , to leading order. Hence,  $\Delta^{(2)}c = 6\varepsilon^3[1 + o(1)]$ , and it follows that  $\gamma_{30} = 6$ . Finally, introducing  $\Delta^{(3)}c = \Delta^{(2)}c - 6\varepsilon^3$ , substituting, and canceling terms in  $\varepsilon^3$ , we see that  $\Delta^{(3)}c$  must be chosen so that the relation  $-4\varepsilon^4 = \frac{\Delta^{(3)}c}{3} + (3 + 2 \ln \varepsilon)\varepsilon^4$  balances. (Here, we note that  $\Delta^{(3)}c$  has to cancel the  $O(\varepsilon^4 \ln \varepsilon)$ -term in (4.22), as there is no corresponding term on the left-hand side of that relation.) Therefore,  $\Delta^{(3)}c = -6\varepsilon^4 \ln \varepsilon - 21\varepsilon^4$ , which gives (4.20), as claimed.  $\square$

This completes the proof of Theorem 1.2.

**Remark 11.** The fact that the coefficient  $\gamma_{20}$  in (4.20) must be negative already follows from the proof of Proposition 3.6. Moreover, while (4.20) seems to indicate that, in general, the sign of  $\gamma_{jk}$  alternates with  $j$  irrespective of  $k$ , no rigorous results to that effect are available for  $j \geq 3$ . Finally, we conjecture that the condition in (4.21) can be refined to show that the  $o(\varepsilon^4)$ -term in (4.20) will actually be of the order  $O(\varepsilon^5 \ln \varepsilon)$ , as well as that  $\Delta c$  will again be  $\mathcal{C}^\infty$ -smooth in  $\varepsilon$  and  $\ln \varepsilon$ ; cf. Section 3.2.3. A proof is, however, beyond the scope of this article.  $\square$

**4.2. Regularity of  $\mathcal{W}^u(Q_\varepsilon^-)$ .** As in the general case discussed in Section 3.3 above, we now investigate how the  $\varepsilon$ -asymptotics of  $\Delta c$ , as given in Proposition 4.5, determines the regularity of the unstable manifold  $\mathcal{W}^u(Q_\varepsilon^-)$  of  $Q_\varepsilon^-$ . In particular, we show that  $\mathcal{W}^u(Q_\varepsilon^-)$ , while smooth in the parameters  $c$  and  $\varepsilon$ , is not smooth when considered as depending on  $\varepsilon$  alone. This loss of smoothness is due to the presence of logarithmic (switchback) terms in (4.20), which give rise to logarithmic terms in the corresponding expansion for  $V(U, c(\varepsilon))$  in (2.6). Making use of (4.20) to rewrite that

expansion in terms of  $U$  and  $\varepsilon$ , we find

$$(4.23) \quad \begin{aligned} V(U, c(\varepsilon)) &= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j V}{\partial c^j}(U, 1) \left[ \sum_{k=2}^4 \gamma_{k0} \varepsilon^k + \gamma_{41} \varepsilon^4 \ln \varepsilon + o(\varepsilon^4) \right]^j \\ &=: \sum_{j=0}^4 V_{j0}(U) \varepsilon^j + V_{41}(U) \varepsilon^4 \ln \varepsilon + o(\varepsilon^4). \end{aligned}$$

In analogy to Proposition 3.7, we have the following result:

**Proposition 4.6.** *Let  $c(\varepsilon)$  be defined as in Proposition 3.1. Then, the coefficient functions  $V_{jk}$  in (4.23) satisfy  $V_{00}(U) = U(U-1)$ ,  $V_{10}(U) \equiv 0$ ,*

$$(4.24) \quad V_{jk}(U) = -\frac{\gamma_{jk}}{3}(U-1)$$

for  $j \in \{2, 3\}$  and  $k = 0$  or  $j = 4$  and  $k = 1$ , and

$$(4.25) \quad V_{40}(U) = -\frac{\gamma_{40}}{3}(U-1) + \frac{\gamma_{20}^2}{9} \frac{U^2 - 4U + 3 + 2 \ln U}{(U-1)^2},$$

where the constants  $\gamma_{jk}$  are defined as in (4.20).

*Proof.* Rewriting (4.23) as

$$\begin{aligned} V(U, c(\varepsilon)) &= V(U, 1) + \frac{\partial V}{\partial c}(U, 1) [\gamma_{20} \varepsilon^2 + \gamma_{30} \varepsilon^3 + \gamma_{40} \varepsilon^4 + \gamma_{41} \varepsilon^4 \ln \varepsilon + o(\varepsilon^4)] \\ &\quad + \frac{1}{2} \frac{\partial^2 V}{\partial c^2}(U, 1) [\gamma_{20} \varepsilon^2 + \gamma_{30} \varepsilon^3 + \gamma_{40} \varepsilon^4 + \gamma_{41} \varepsilon^4 \ln \varepsilon + o(\varepsilon^4)]^2 + o(\varepsilon^4) \\ &= V(U, 1) + \gamma_{20} \frac{\partial V}{\partial c}(U, 1) \varepsilon^2 + \gamma_{30} \frac{\partial V}{\partial c}(U, 1) \varepsilon^3 \\ &\quad + \gamma_{41} \frac{\partial V}{\partial c}(U, 1) \varepsilon^4 \ln \varepsilon + \left[ \gamma_{40} \frac{\partial V}{\partial c}(U, 1) + \frac{1}{2} \gamma_{20}^2 \frac{\partial^2 V}{\partial c^2}(U, 1) \right] \varepsilon^4 + o(\varepsilon^4), \end{aligned}$$

one sees immediately that  $V_{00}(U) = V(U, 1) = U(U-1)$ , as before, as well as that  $V_{10}(U) \equiv 0$ . Moreover,

$$(4.26) \quad V_{jk}(U) = \gamma_{jk} \frac{\partial V}{\partial c}(U, 1)$$

for  $j \in \{2, 3\}$  and  $k = 0$  or  $j = 4$  and  $k = 1$ , while

$$(4.27) \quad V_{40}(U) = \gamma_{40} \frac{\partial V}{\partial c}(U, 1) + \frac{1}{2} \gamma_{20}^2 \frac{\partial^2 V}{\partial c^2}(U, 1).$$

Replacing  $\frac{\partial V}{\partial c}(U, 1)$  and  $\frac{\partial^2 V}{\partial c^2}(U, 1)$  in (4.26) and (4.27) with the corresponding expressions found in Lemma 4.3, one obtains (4.24) and (4.25), which completes the proof.  $\square$

In particular, combining the results of Propositions 4.5 and 4.6, *i.e.*, substituting the numerical values for  $\gamma_{jk}$  obtained in the former into the expressions for  $V_{jk}$  found in the latter, we can write (4.23) explicitly as

$$(4.28) \quad \begin{aligned} V(U, c(\varepsilon)) &= U(U-1) + (U-1)\varepsilon^2 - 2(U-1)\varepsilon^3 + 2(U-1)\varepsilon^4 \ln \varepsilon \\ &\quad + \left[ 7(U-1) + \frac{U^2 - 4U + 3 + 2 \ln U}{(U-1)^2} \right] \varepsilon^4 + o(\varepsilon^4). \end{aligned}$$

The presence of the  $O(\varepsilon^4 \ln \varepsilon)$ -term in (4.28) implies that  $V(U, c(\varepsilon))$  is only  $\mathcal{C}^3$ -smooth in  $\varepsilon$  as  $\varepsilon \rightarrow 0^+$  when considered as a function of  $(U, \varepsilon) \in (0, 1) \times (0, \varepsilon_0)$ . (In fact, one can check that the  $O(\varepsilon^4)$ -coefficient  $V_{40}$  is in fact  $\mathcal{C}^\infty$ -smooth in  $U$  away from  $U = 0$ .) However, irrespective of the

value of  $m$ , the smoothness of  $\mathcal{W}^u(Q_\varepsilon^-)$  in  $U$  can only extend up to  $\{U = \varepsilon\}$ , as the vector field in (2.1) has a discontinuity there. Correspondingly, for  $m = 2$ , (4.25) shows that  $V_{40}(U)$  becomes unbounded as  $U \rightarrow 0^+$ , which is also evident from Lemma 4.3; cf. (4.12). Finally, we note that the approximation for  $V(U, c(\varepsilon))$  in (4.28) can in principle be refined to arbitrary order once the corresponding terms in the expansion for  $\Delta c(\varepsilon)$  in (4.20) have been determined.

**Remark 12.** Equivalently, the coefficient functions  $V_{jk}$  can be derived as follows: substituting the expansion for  $V(U, c(\varepsilon))$  in (4.23) into (4.13), making use of  $\frac{dV}{dU} = \sum_{j=0}^4 \frac{dV_{j0}}{dU} \varepsilon^j + \frac{dV_{41}}{dU} \varepsilon^4 \ln \varepsilon + o(\varepsilon^4)$  and of the expansion for  $\Delta c(\varepsilon)$  in (4.20), and comparing terms of like powers of  $\varepsilon$ , one obtains a recursive sequence of differential equations for  $V_{jk}$ :

$$(4.29a) \quad \sum_{i=0}^j V_{i0} \frac{dV_{j-i,0}}{dU} = -V_{j0} - \sum_{i=2}^j \gamma_{i0} V_{j-i,0} \quad \text{for } j \in \{2, 3, 4\},$$

$$(4.29b) \quad V_{00} \frac{dV_{41}}{dU} + V_{41} \frac{dV_{00}}{dU} = -V_{41} - \gamma_{41} V_{00},$$

where we additionally impose the boundary conditions  $V_{jk}(1) = 0$  throughout.

Given  $V_{00}$  and  $V_{10}$ , as before, Equation (4.29a) reads

$$(4.30) \quad \frac{dV_{20}}{dU} = -\frac{2}{U-1} V_{20} - \gamma_{20}$$

for  $j = 2$ , which has the unique non-singular solution  $V_{20}(U) = -\frac{\gamma_{20}}{3}(U-1)$ . Similarly, one finds that (4.29a) again reduces to (4.30) (with  $V_{20}$  replaced by  $V_{30}$ ) when  $j = 3$ , which shows  $V_{30}(U) = -\frac{\gamma_{30}}{3}(U-1)$ . Applying the same reasoning to (4.29b), one deduces  $V_{41}(U) = -\frac{\gamma_{41}}{3}(U-1)$ , which proves (4.24). Finally, given  $V_{j0}$  for  $j \in \{0, \dots, 3\}$ , (4.29a) simplifies to

$$\frac{dV_{40}}{dU} = -\frac{2}{U-1} V_{40} - \gamma_{40} + \frac{2}{9} \frac{\gamma_{20}^2}{U}$$

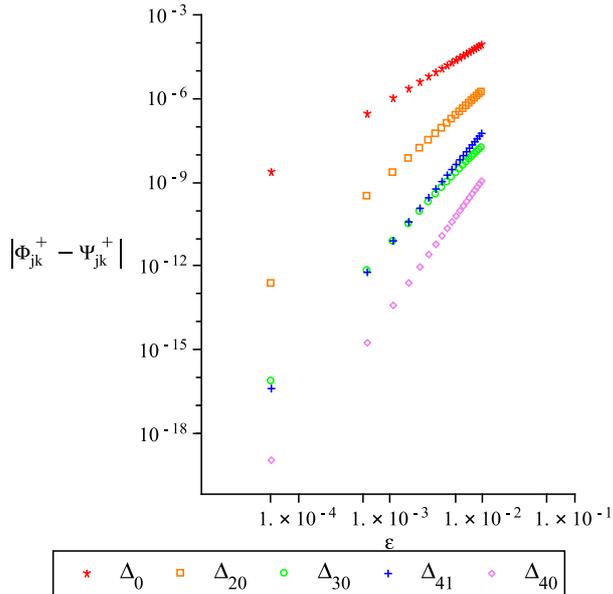
when  $j = 4$ ; the unique non-singular solution is given by

$$V_{40}(U) = -\frac{\gamma_{40}}{3}(U-1) + \frac{\gamma_{20}^2}{9} \frac{U^2 - 4U + 3 + 2 \ln U}{(U-1)^2},$$

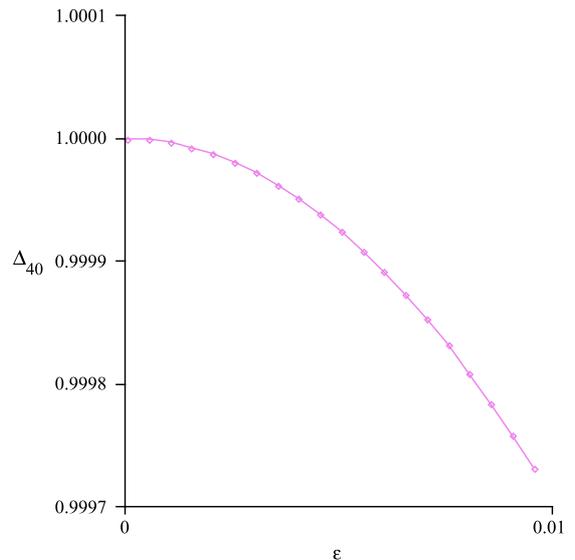
as stated in (4.25). □

**4.3. Numerical verification.** To illustrate the analytical results obtained in this section for  $m = 2$  in (1.1) and, in particular, to verify the  $\varepsilon$ -asymptotics of  $c(\varepsilon)$  derived in Proposition 4.5, we calculated numerically the error incurred in approximating  $c(\varepsilon)$  by successive truncations of the asymptotic expansion in (4.20).

For  $j \in \{2, 3, 4\}$  and  $k \in \{0, 1\}$ , let  $\Delta_{jk}(\varepsilon)$  denote the approximation for  $\Delta c(\varepsilon)$  obtained by retaining all terms up to and including  $O(\varepsilon^j \ln \varepsilon^k)$  in that expansion. Moreover, recall the definition of the section  $\Sigma^+$  which, for  $\varepsilon$  fixed, corresponds to the hyperplane  $\{U = \varepsilon\}$  in  $(U, V, \varepsilon)$ -space, as well as of the functions  $\Phi^+$  and  $\Psi^+$ , which describe the intersection of the manifolds  $\mathcal{W}^s(\ell^+)$  and  $\mathcal{W}^u(\ell^-)$  with  $\Sigma^+$ ; cf. Section 3.1 and, in particular, the proof of Proposition 3.1. Then, the corresponding approximation error was estimated by evaluating the difference, in  $\Sigma^+$ , between the approximate value for  $\Phi^+$  that is given by  $\Phi_{jk}^+(\varepsilon) := -\varepsilon \Delta_{jk}(\varepsilon)$  and  $\Psi_{jk}^+(\varepsilon)$ , which is the approximation for  $\Psi^+$  that is found from a straightforward numerical integration of (4.13), with  $c(\varepsilon)$  replaced by  $\Delta_{jk}(\varepsilon)$ . The results are illustrated in Figure 4, where we have plotted the absolute values of the error  $|\Phi_{jk}^+ - \Psi_{jk}^+|$  on a doubly logarithmic scale, with  $\varepsilon$  between  $10^{-4}$  and  $10^{-2}$ . Clearly, each additional  $O(\varepsilon^j)$ -term in the expansion for  $\Delta c$  reduces the approximation error by about an order of magnitude (in  $\varepsilon$ ), as expected. However, Figure 4(a) shows that including the  $O(\varepsilon^4 \ln \varepsilon)$ -terms alone yields no improvement over the  $O(\varepsilon^3)$ -truncation  $\Delta_{30}$ ; rather, it actually seems to increase



(a) Error  $|\Phi_{jk}^+ - \Psi_{jk}^+|$  for  $\varepsilon = 10^{-4}, \dots, 10^{-2}$ .



(b) Truncation  $\Delta_{40}(\varepsilon)$  (circles) and interpolation (solid line).

FIGURE 4. Numerical results for  $m = 2$ .

the error slightly, at least for larger values of  $\varepsilon$ . Only by taking into account the  $O(\varepsilon^4)$ -correction in (4.20), as well, does one find the anticipated reduction of the error by an additional order of magnitude. Conversely, omitting the logarithmic terms altogether eliminates the difference in error between  $\Delta_{30}$  and  $\Delta_{40}$ , *i.e.*, the  $O(\varepsilon^4)$ -truncation does not improve on the approximation by  $\Delta_{30}$  in that case (data not shown). Finally, in Figure 4(b), we have plotted  $\Delta_{40}$ , where  $\varepsilon$  varies again as above. One sees that, for  $\varepsilon > 0$ ,  $\Delta_{40}(\varepsilon) < c(0) = \Delta_{00}$  is satisfied throughout, as demonstrated analytically for  $c(\varepsilon)$  in Proposition 3.1.

**Remark 13.** Our computations were performed in double-precision MAPLE arithmetic. However, we remark that we did not systematically consider  $\varepsilon$  smaller than  $10^{-4}$ , as shown here; due to the inherent ‘stiffness’ of (4.13), we obtained numerically spurious results already for  $\varepsilon = O(10^{-5})$ .  $\square$

## 5. DISCUSSION

In this section, we summarize our findings, and we discuss open questions that have to be left for future study.

**5.1. Summary.** In this article, we have provided a rigorous geometric proof for the existence and uniqueness of traveling front solutions in the degenerate family of ‘cut-off’ reaction-diffusion equations in (1.1), with integer-valued  $m \geq 2$ . Moreover, we have derived the leading-order  $\varepsilon$ -asymptotics of the corresponding front propagation speed  $c(\varepsilon)$ .

For  $m = 2$  (in which case (1.1) is the exactly solvable Zeldovich equation), we have proven the occurrence of logarithmic (‘switchback’) terms in  $\varepsilon$  in the asymptotic expansion for  $c(\varepsilon)$ , and we have calculated explicitly the lowest-order such term as well as its coefficient. Our analysis shows that this switchback phenomenon is caused by resonances between eigenvalues in one of the coordinate charts used to describe the dynamics in blown-up phase space, as observed already in [27]. A more general discussion of logarithmic switchback can be found in [28]; see also [33]. On

a conceptual level, these logarithmic terms are a consequence of the loss of smoothness incurred during the (resonant) transition through the intermediate region: although the manifolds  $\mathcal{W}^u(Q_\varepsilon^-)$  and  $\mathcal{W}^s(Q_\varepsilon^+)$  are analytic in both  $c$  and  $\varepsilon$  when restricted to the outer and inner regions, respectively, that transition introduces a logarithmic dependence of  $c$  on  $\varepsilon$ , which then translates into a loss of smoothness of the above manifolds when considered as depending on  $\varepsilon$  alone.

In particular, Lemma 4.4 implies that the  $\varepsilon$ -asymptotics of both  $\hat{z}^-$  and  $\hat{z}^+$  will contain logarithmic terms, which is due to the fact that the estimates in (4.17) and (4.18) depend on  $\Delta c(\varepsilon) = c(\varepsilon) - c(0)$ . It then follows from the proof of Proposition 4.5 that these terms feed back into the  $\varepsilon$ -asymptotics of  $c(\varepsilon)$  which, in turn, gives rise to higher-order logarithmic terms in the expansion for  $V(U, c(\varepsilon))$  in (2.6). The resulting non-smoothness of  $\mathcal{W}^u(Q_\varepsilon^-)$  in  $\varepsilon$  was made explicit in Proposition 4.6: substituting the leading-order  $\varepsilon$ -asymptotics of  $c(\varepsilon)$  into (2.6) and rewriting the result as an expansion in terms of  $\varepsilon$ , cf. (4.23), we determined the coefficient  $V_{41}$  of the  $O(\varepsilon^4 \ln \varepsilon)$ -term in closed form. Thus, we found that although  $\mathcal{W}^u(Q_\varepsilon^-)$  is analytic in  $U$ ,  $V$ ,  $c$ , and  $\varepsilon$  (at least as long as  $U > \varepsilon$ ), the  $\ln \varepsilon$ -dependence of  $c(\varepsilon)$  implies that it cannot be analytic (or even  $\mathcal{C}^\infty$ -smooth) in  $U$ ,  $V$ , and  $\varepsilon$  alone.

We note that the proof of Theorem 1.2 could easily be extended to calculate higher-order terms in the asymptotic expansion for  $c(\varepsilon)$  in (4.20). (In fact, by refining explicitly the sequence of normal form transformations in (4.7), that expansion could in principle be taken to arbitrary order; the accuracy provided by the leading-order normal form in (4.6) is exhausted by the order to which  $c(\varepsilon)$  is approximated in Theorem 1.2.) We conjecture that, by taking the expansion in (2.6) to higher order in  $\Delta c$ , one would find logarithmic terms of the general form  $\varepsilon^j (\ln \varepsilon)^k$ , with  $j \geq 4$  and  $k \leq j$ . Similarly, for any (integer-valued)  $m \geq 3$ , one could refine the argumentation in Section 3.2 to describe the structure of the expansion for  $c(\varepsilon)$  in more detail than is provided in Proposition 3.6. Thus, for instance, we postulate that, for given  $m$ , the lowest-order logarithmic term in the  $\varepsilon$ -asymptotics of  $c(\varepsilon)$  will be  $O(\varepsilon^{m^2} \ln \varepsilon)$  (see also Section 3.2.3), as well as that the expansion in (3.19) will contain logarithmic switchback terms of the form  $\varepsilon^j (\ln \varepsilon)^k$ , where  $j \geq m^2$  and  $k \leq j$ . However, in general, it does not seem possible to calculate the corresponding coefficient  $\gamma_{m^2 1}$  in that expansion explicitly.

In fact, the discussion in Section 3.3 implies that explicit knowledge of a solution to the corresponding problem without cut-off is a necessary condition for the correction  $\Delta c$  that is induced by the (Heaviside) cut-off  $\Theta$  to be computable. The only value of  $m$  in (1.6) for which such a solution is known seems to be 2. In the context of the first-order system (2.1), the non-computability of the leading-order coefficient  $\gamma_m$  (or, indeed, of any of the coefficients in the expansion in (3.19)) for  $m \geq 3$  is evident due to the fact that the derivatives  $\frac{\partial^j V}{\partial c^j}(U, c(0))$  of  $V(U, c(0))$  with respect to  $c$  or, equivalently, the coefficient functions  $V_{jk}$  in the expansion for  $V(U, c(\varepsilon))$  in (3.22), cannot be evaluated in closed form in that case. Rather, they have to be approximated locally about either  $Q_\varepsilon^-$  or  $Q_\varepsilon^+$ . The resulting two expansions are not equivalent, in that they cannot be transformed into one another; however,  $\frac{\partial V}{\partial c}(0, c(0))$  – the constant  $\nu_m$  in (3.24) – cannot be determined unless the boundary condition  $\frac{\partial V}{\partial c}(1, c(0)) = 0$  that is imposed at  $Q_\varepsilon^-$  is taken into account; see the discussion in Section 3.3. Therefore, the coefficient  $\gamma_m$  is not computable in that case; cf. (3.21). By contrast, for  $m = 2$ , the derivatives  $\frac{\partial^j V}{\partial c^j}(U, 1)$  (and, consequently, the coefficient functions  $V_{jk}$  in (4.23)) can be found in closed form to the order considered here, *i.e.*, no expansion is required, which implies in particular that  $\nu_2 = \frac{1}{3}$  is known explicitly.

A crucial step in the proof of Theorem 1.1 consisted in determining the appropriate normal form system (3.3) that describes the transition through the intermediate region in the phase-directional chart  $K_1$ . The corresponding leading-order normal form in (3.8) (in which only the lowest-order resonant terms have been retained) is of Bernoulli type [1] and can be solved exactly. Moreover,

knowledge of the resulting approximate solution  $\tilde{z}$  is sufficient to determine the leading-order  $\varepsilon$ -asymptotics of  $\Delta c(\varepsilon)$ ; recall the proof of Proposition 3.6.

However, the leading-order analysis that led to the expansion for  $\Delta c$  does not allow one to calculate the coefficient  $\gamma_m$  in (3.19) analytically, irrespective of the value of  $m$ . From (3.21), it is evident that both  $c(0)$  (the propagation speed in the absence of a cut-off) and  $\frac{\partial V}{\partial c}(U, c(0))$  (the variation of the corresponding front solution with respect to  $c$ ) have to be known for  $\gamma_m$  to be computable. By contrast, in the study of the Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation in [17], *i.e.*, for  $m = 1$  in (1.1), knowledge of  $c(0)$  was sufficient to approximate the transition through the phase-directional chart  $K_1$  to lowest order and, hence, to determine the (universal) leading-order coefficient in  $\Delta c$ ; see in particular [17, Proposition 3.2]. This distinction is due to differences in the corresponding normal forms that characterize the transition through chart  $K_1$ : the  $\hat{z}$ -direction in (3.3) is strongly repelling, whereas it was linearly neutral in [17, Equation (34)]. In other words, both propagation regimes require  $\Delta c$  to be fixed in precisely the right manner for a heteroclinic connection to exist between  $Q_\varepsilon^-$  and  $Q_\varepsilon^+$ , as  $\mathcal{W}^u(Q_\varepsilon^-)$  will connect to  $\mathcal{W}^s(Q_\varepsilon^+)$  for a unique value of  $c(\varepsilon)$ , but veer off otherwise; however, the phenomenon seems even more delicate here than it was in [17].

**5.2. Outlook.** A question that arises naturally, given the result of Theorem 1.1, is whether our analysis can be extended to cover the case where the exponent  $m$  in (1.1) is non-integer. This question concerns in particular the family of equations with variable diffusivity in (1.7), see [38] and the references therein for details and applications.

One potential approach that suggests itself here can be outlined as follows: given  $m \geq 2$  real, one first sets  $m = n + p$ , where  $n = [m] \geq 2$  is the integer part of  $m$  and  $p = m - [m] \in [0, 1)$  denotes the remainder. The equivalent first-order system in (2.1) can then be rewritten as

$$(5.1a) \quad U' = V,$$

$$(5.1b) \quad V' = -cV - 2U^n W(1 - U)\Theta(U - \varepsilon),$$

$$(5.1c) \quad W' = pW \frac{V}{U},$$

$$(5.1d) \quad \varepsilon' = 0,$$

where the new (artificial) variable  $W = U^p$  is introduced to remove the loss of smoothness as  $U \rightarrow 0^+$  in  $U^p = e^{p \ln U}$ . To study the dynamics of (5.1), one can proceed as in Section 3, *i.e.*, one can include the additional variable  $W = \bar{w}$  in the blow-up transformation in (2.2) and then define the same two coordinate charts as before. (Alternatively, one could projectivize the equations by introducing  $\frac{U'}{U} = \frac{V}{U}$  as a new variable; the reduced model that is found by restricting (5.1) to the strongly attracting center manifold given by  $\{U = 0\}$  could then potentially be analyzed as in [35]. In particular, the system obtained in chart  $K_1$ , after blow-up, is equivalent to that projectivization of (5.1); cf. [35] for details.) Due to the fact that the reaction terms in (5.1) are still set to zero by the cut-off  $\Theta$  when  $U < \varepsilon$ , see (1.2), the dynamics in the inner region, *i.e.*, in chart  $K_2$ , should remain largely unchanged. The transition through the intermediate region, where  $\varepsilon < U < O(1)$ , is naturally described in chart  $K_1$ , as before; recall Section 3.2. Moreover, preliminary analysis suggests that no resonance occurs in this case, which would substantially simplify the argument. However, in the outer region, complications seem to arise due to the non-smoothness of the manifold  $\mathcal{W}^u(Q_\varepsilon^-)$ , as well as of the variation of that manifold with respect to  $c$ , at  $(U, c) = (0, c(0))$ ; recall the proof of Lemma 3.4. Still, we conjecture that the approach developed in Section 3 can be adapted to show that the correction  $\Delta c$  induced by the cut-off  $\Theta$  will again satisfy  $\Delta c(\varepsilon) = \gamma_m \varepsilon^m [1 + o(1)]$ , as for integer-valued  $m$ , as well as that  $\Delta c$  will be a smooth function of  $\varepsilon$  and  $\varepsilon^p$ , provided that a traveling front solution to (1.5) exists for some unique value of  $c$  even when the potential  $f_m(u)$  is non-smooth. A rigorous proof of this conjecture is deferred to future work.

A related question concerns the asymptotics of the critical front speed in the two limiting cases where either  $m \rightarrow 1^+$  or  $m \rightarrow \infty$  in (1.5). The dynamics of the corresponding equations without cut-off was analyzed geometrically in [35] and [16], respectively. The question of how the results obtained there would be affected by the presence of a cut-off is left for future study. In particular, it was argued in [29, Section VI] that the reaction terms in (1.5) are negligible in a neighborhood of the zero rest state. In other words, the exponent  $m$  induces an ‘internal’ cut-off in that case, whereas the cut-offs discussed here would be classified as ‘external.’ The question of how these two types of cut-off are related remains to be clarified.

Rigorous results on critical front propagation in the presence of a cut-off are also available in work by Benguria, Depassier, and collaborators [7, 8], as well as by Kessler *et al.* [24] and Méndez and colleagues [29]. Their approach is based on a variational principle which, in the context of Equation (1.5), yields the front propagation speed as the supremum of an appropriately defined variational functional:

$$(5.2) \quad c^2 = \sup_g \left[ -2 \frac{\int_0^1 f_m(U)g(U) dU}{\int_0^1 \frac{g^2(U)}{g'(U)} dU} \right],$$

taken over all positive and decreasing functions  $g$  on  $(0, 1)$  such that the integrals in (5.2) exist [6]. (The supremum in (5.2) is in fact a maximum if the corresponding front is of ‘pushed’ or ‘bistable’ type, whereas it is not attained for fronts of ‘pulled’ type; see [8] and the references therein for details.) The very general expression in (5.2) was adapted to accommodate cut-offs in the form of an ‘improved variational principle’ in [29]; moreover, it was extended by Benguria *et al.* in [8], where they also calculated the leading-order coefficients in the correction due to a cut-off in a number of explicitly solvable cases, including for the Zeldovich equation discussed here; recall Remark 1. Finally, they argued that the correction  $\Delta c$  is computable in closed form whenever the function  $\hat{g}$  that maximizes (5.2) – in the absence of a cut-off – is known exactly. We conjecture that this condition is in fact equivalent to the requirement that the variational equation in (3.15) has a closed-form solution  $\frac{\partial V}{\partial c}(U, c(0))$ , as discussed in Section 3.3; moreover, it is argued there that knowledge of a front solution to Equation (1.5) (without cut-off) is a necessary, but in general not a sufficient condition for (3.15) to be solvable in closed form. Incidentally, we remark that  $\hat{g}(U) = \frac{1-U}{U}$  for  $m = 2$  in (1.1), cf. *e.g.* [8], where it is noted that  $\hat{g}$  is unique up to a multiplicative constant. Evaluating the denominator in (5.2), we find that  $-\int_0^1 \frac{\hat{g}^2(U)}{\hat{g}'(U)} dU = \frac{1}{3}$  equals  $\nu_2$ , as defined in Lemma 3.4; see also [8], where that same integral features in their closed-form expression for the leading-order coefficient in  $\Delta c$ . (Specifically, (3.21) and the corresponding formula in [8] agree up to the scaling factor  $\frac{2}{m}$ .) However, to the best of our knowledge, the precise nature of the relationship between the integral variational principle and our more geometric approach remains to be resolved. In particular, since the work of Benguria *et al.* does not seem to make explicit reference to the case of non-integer  $m$  in (1.1), such an investigation would appear particularly worthwhile in that case.

Finally, we emphasize that the degenerate family of equations studied in this article represents only one aspect of a larger and more ambitious program, the aim of which is a systematic, geometric classification of traveling front propagation in scalar reaction-diffusion systems in the presence of a cut-off. We postulate that, generally, the effects of a cut-off on the dynamics of traveling fronts can be categorized in terms of the associated normal form equations that are obtained in one of the (phase-directional) coordinate charts, after blow-up. The properties of these normal forms will determine not only the structure of the  $\varepsilon$ -dependence of  $c(\varepsilon)$ , but also the sign of the corresponding leading-order coefficient in  $\Delta c$ . (The dynamics in the rescaling chart, on the other hand, is obtained by regular perturbation off the much simplified cut-off equations and is hence largely independent of the specific choice of reaction terms in (1.1).)

One prototypical example of a reaction-diffusion system in which a variety of front propagation regimes can be realized is given by the Nagumo equation [8, 29]

$$(5.3) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a),$$

which is also known as the Schlögl equation when formulated in terms of the reaction function  $f(u) = (1-u^2)(u+\sigma)$ . (In fact, the degenerate family of equations in (1.1) has been suggested in [39] as a ‘bridge’ between the classical FKPP and Nagumo equations; see also [16].) The traveling front dynamics of (5.3) depends crucially on the value of the parameter  $a$ ; moreover, explicit front solutions are available for a wide range of  $a$ -values, cf. *e.g.* [21]. In particular, for  $a \in (0, \frac{1}{2})$ , the reaction kinetics in (5.3) are bistable, and a family of front solutions propagating between the rest states at 1 and 0 is given by  $U(\xi) = (1 + e^{\frac{\xi}{\sqrt{2}}})^{-1}$ , with propagation speed  $c = \frac{1}{\sqrt{2}} - \sqrt{2}a$  [21]. As was shown *e.g.* in [8, 29], the correction  $\Delta c$  in the associated cut-off equation is of the order  $O(\varepsilon^{1+2a})$  in that case; moreover, the leading-order coefficient in  $\Delta c$  was calculated in [8] via a variational approach. (The corresponding result for the Schlögl equation, first derived geometrically in [34] and subsequently proven rigorously in [18], reads  $\Delta c = O(\varepsilon^{2-\sigma})$ , with  $0 < \sigma < 1$ .) In the limit as  $a \rightarrow 0^+$  in (5.3), we retrieve the Zeldovich equation; correspondingly, the leading-order expansion for  $\Delta c$  obtained in [8, 18] has to reduce to  $-\frac{3}{\sqrt{2}}\varepsilon^2$  in that limit, as required by our Theorem 1.2 (after division of (1.9) by a factor of  $\sqrt{2}$ , recall Remark 1). The relationship between (5.3) and the family in (1.1) discussed here has been elucidated in detail in [18], where the propagation of ‘bistable’ fronts in the presence of a cut-off has been studied in full generality, from a geometric point of view; a comprehensive study of Equation (5.3) in the ‘pushed’ propagation regime will be provided in the upcoming article [19].

#### ACKNOWLEDGMENT

The author is grateful to Freddy Dumortier and Peter de Maesschalck for fruitful discussions during the preparation of this article, to Heidi Popović for her careful reading of the original manuscript, as well as to the anonymous referee for numerous helpful suggestions and comments.

#### APPENDIX A. GRONWALL’S LEMMA

In this appendix, we restate Gronwall’s Lemma in the formulation in which it is applied here; recall the proof of Proposition 3.3:

**Lemma A.1** (Gronwall’s Lemma). *Let  $\mathcal{U}$  be an open set in  $\mathbb{R}$ , let  $f, g : [0, T] \times \mathcal{U} \rightarrow \mathbb{R}$  be continuous, and let  $x(t)$  and  $y(t)$  be solutions of the initial value problems*

$$(A.1) \quad x'(t) = f(t, x(t)) \quad \text{with } x(0) = x_0 \quad \text{and} \quad y'(t) = g(t, y(t)) \quad \text{with } y(0) = y_0,$$

*respectively. Assume that there exists  $C \geq 0$  such that*

$$(A.2) \quad |g(t, y_2) - g(t, y_1)| \leq C|y_2 - y_1|;$$

*furthermore, let  $\varphi : [0, T] \rightarrow \mathbb{R}^+$  be a continuous function, with*

$$(A.3) \quad |f(t, x(t)) - g(t, x(t))| \leq \varphi(t).$$

*Then, there holds*

$$(A.4) \quad |x(t) - y(t)| \leq e^{Ct}|x_0 - y_0| + e^{Ct} \int_0^t e^{-C\tau} \varphi(\tau) d\tau$$

*for  $t \in [0, T]$ .*

## REFERENCES

- [1] M.A. Abramowitz and I.A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (1974). Dover Publications, Inc., New York, Ninth printing.
- [2] H. Amann, *Ordinary differential equations. An introduction to nonlinear analysis. Translated from the German by Gerhard Metzen.* de Gruyter Studies in Mathematics **13** (1990), Walter de Gruyter & Co., Berlin.
- [3] D.G. Aronson, *Density-dependent Interaction Systems.* In W.E. Stewart, W.H. Ray, and C.C. Conley (eds.), *Dynamics and Modeling of Reactive Systems* (1980), 161–176, Academic Press, Inc., New York.
- [4] D.G. Aronson and H.F. Weinberger, *Nonlinear Diffusion in Population Genetics, Combustion, and Nerve Pulse Propagation.* In *Partial Differential Equations and Related Topics, Tulane University, New Orleans, LA, 1974*, Lecture Notes in Mathematics **446** (1975), 5–49, Springer-Verlag, Berlin.
- [5] R.D. Benguria and M.C. Depassier, *Variational principle for the asymptotic speed of fronts of the density-dependent diffusion-reaction equation*, Phys. Rev. E **52**(3) (1995), 3285–3287.
- [6] R.D. Benguria and M.C. Depassier, *Speed of Fronts of the Reaction-Diffusion Equation*, Phys. Rev. Letters **77**(6) (1996), 1171–1173.
- [7] R.D. Benguria and M.C. Depassier, *Speed of pulled fronts with cutoff*, Phys. Rev. E **75**(5) (2007), 051106.
- [8] R.D. Benguria, M.C. Depassier, and V. Haikala, *Effect of a cutoff on pushed and bistable fronts of the reaction-diffusion equation*, Phys. Rev. E **76**(5) (2007), 051101.
- [9] J. Billingham and D.J. Needham, *A note on the properties of a family of travelling-wave solutions arising in cubic autocatalysis*, Dynam. Stabil. Sys. **6**(1) (1991), 33–49.
- [10] J. Billingham, *Phase plane analysis of one-dimensional reaction-diffusion waves with degenerate reaction terms*, Dyn. Stab. Syst. **15**(1) (2000), 23–33.
- [11] N.F. Britton, *Reaction-Diffusion Equations and Their Applications to Biology* (1986). Academic Press, Inc., London.
- [12] E. Brunet and B. Derrida, *Shift in the velocity of a front due to a cutoff*, Phys. Rev. E **56**(3) (1997), 2597–2604.
- [13] P. De Maesschalck, *Personal communication* (2009).
- [14] F. Dumortier, *Techniques in the Theory of Local Bifurcations: Blow-Up, Normal Forms, Nilpotent Bifurcations, Singular Perturbations.* In D. Schlomiuk (ed.), *Bifurcations and Periodic Orbits of Vector Fields*, NATO ASI Series C **408** (1993), 19–73, Kluwer Acad. Publ., Dordrecht, The Netherlands.
- [15] F. Dumortier, J. Llibre, and J.C. Artés, *Qualitative theory of planar differential systems* (2006). Universitext. Springer-Verlag, Berlin.
- [16] F. Dumortier, N. Popović, and T.J. Kaper, *The asymptotic critical wave speed in a family of scalar reaction-diffusion equations*, J. Math. Anal. Appl. **326**(2) (2007), 1007–1023.
- [17] F. Dumortier, N. Popović, and T.J. Kaper, *The critical wave speed for the FKPP equation with cut-off*, Nonlinearity **20**(4) (2007), 855–877.
- [18] F. Dumortier, N. Popović, and T.J. Kaper, *A geometric approach to bistable front propagation in scalar reaction-diffusion equations with cut-off*, Phys. D **239**(20) (2010), 1984–1999.
- [19] F. Dumortier, N. Popović, and T.J. Kaper, *Wave speeds for pushed fronts in reaction-diffusion equations with cut-offs*, in preparation (2010).
- [20] R.A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.
- [21] B.H. Gilding and R. Kersner, *Travelling waves in nonlinear diffusion-convection reaction.* Progress in Nonlinear Differential Equations and their Applications **60** (2004), Birkhäuser Verlag, Basel.
- [22] S. van Gils, M. Krupa, and P. Szmolyan, *Asymptotic expansions using blow-up*, Z. Angew. Math. Phys. **56**(3) (2005), 369–397.
- [23] M.W. Hirsch, C.C. Pugh, and M. Shub, *Invariant Manifolds.* Lecture Notes in Mathematics **583** (1977), Springer-Verlag, Berlin.
- [24] D.A. Kessler, Z. Ner, and L.M. Sander, *Front propagation: Precursors, cutoffs, and structural stability*, Phys. Rev. E **58**(1) (1998), 107–114.
- [25] A.C. King and D.J. Needham, *The effects of variable diffusivity on the development of travelling waves in a class of reaction-diffusion equations*, Philos. Trans. Roy. Soc. London Ser. A **348**(1687) (1994), 229–260.
- [26] A.N. Kolmogorov, I.G. Petrowskii, and N. Piscounov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Moscow Univ. Math. Bull. **1** (1937), 1–25.
- [27] M. Krupa and P. Szmolyan, *Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions*, SIAM J. Math. Anal. **33**(2) (2001), 286–314.
- [28] P.A. Lagerstrom, *Matched Asymptotic Expansions. Ideas and Techniques.* Applied Mathematical Sciences **76** (1988), Springer-Verlag, New York.
- [29] V. Méndez, D. Campos, and E.P. Zemskov, *Variational principles and the shift in the front speed due to a cutoff*, Phys. Rev. E **72**(5) (2005), 056113.

- [30] J.H. Merkin and D.J. Needham, *Reaction-diffusion waves in an isothermal chemical system with general orders of autocatalysis and spatial dimension*, Z. Angew. Math. Phys. **44**(4) (1993), 707–721.
- [31] D.J. Needham and A.N. Barnes, *Reaction-diffusion and phase waves occurring in a class of scalar reaction-diffusion equations*, Nonlinearity **12**(1) (1999), 41–58.
- [32] D. Panja, *Effects of fluctuations on propagating fronts*, Phys. Rep. **393** (2004), 87–174.
- [33] N. Popović, *A geometric analysis of logarithmic switchback phenomena*. In *HAMSA 2004: Proceedings of the International Workshop on Hysteresis and Multi-Scale Asymptotics, Cork 2004*, J. Phys. Conference Series **22** (2005), 164–173.
- [34] N. Popović, *Front speeds, cut-offs, and desingularization: a brief case study*. In *Fluids and waves*, Contemp. Math. **440** (2007), 187–195, Amer. Math. Soc., Providence, RI.
- [35] N. Popović and T.J. Kaper, *Rigorous asymptotic expansions for critical wave speeds in a family of scalar reaction-diffusion equations*, J. Dynam. Differential Equations **18**(1) (2006), 103–139.
- [36] N. Popović and P. Szmolyan, *Rigorous asymptotic expansions for Lagerstrom’s model equation—a geometric approach*, Nonlinear Anal. **59**(4) (2004), 531–565.
- [37] W. van Saarloos, *Front propagation into unstable states*, Phys. Rep. **386** (2003), 29–222.
- [38] F. Sánchez-Garduño and P.K. Maini, *Travelling wave phenomena in some degenerate reaction-diffusion equations*, J. Differential Equations **117**(2) (1995), 281–319.
- [39] J.A. Sherratt and B.P. Marchant, *Algebraic decay and variable speeds in wavefront solutions of a scalar reaction-diffusion equation*, IMA J. Appl. Math. **56**(3) (1996), 289–302.
- [40] T.P. Witelski, K. Ono, and T.J. Kaper, *Critical wave speeds for a family of scalar reaction-diffusion equations*, Appl. Math. Lett. **14**(1) (2001), 65–73.

UNIVERSITY OF EDINBURGH, SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, KING’S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, UNITED KINGDOM

*E-mail address:* Nikola.Popovic@ed.ac.uk