Contemporary Mathematics

Front Speeds, Cut-Offs, and Desingularization: A Brief Case Study

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Abstract. The study of propagation phenomena in reaction-diffusion systems is a central topic in non-equilibrium physics. One particular aspect which has received recent attention concerns the effects of a "cut-off" of the reaction kinetics on the propagation speed of the traveling fronts often found in such systems. In this brief note, we discuss one specific example of front propagation into a metastable state in a "cut-off" Ginzburg-Landau equation. We indicate how the modified dynamics resulting from this "cut-off" can be understood from a geometric point of view. Moreover, we motivate why the leading-order correction to the unperturbed propagation speed will be a sublinear function of the "cut-off" parameter in this case.

1. Introduction

The phenomenon of front propagation in systems of reaction-diffusion equations is a fundamental topic in non-equilibrium physics. One important aspect of this topic concerns the question of the particular propagation speed that is selected by traveling fronts in such systems, as well as of the factors that influence the selection process. This subject is a vast and complex one, since one typically has to distinguish between front propagation into unstable vs. metastable states, as well as between so-called "pulled" and "pushed" fronts, among other distinctions. For a comprehensive discussion of these classifications, as well as of their physical significance, the reader is referred to the review in [**8**].

A related question that naturally arises in this context is the effect of a modification of the respective reaction kinetics on the front dynamics. In particular, the shift in propagation speed due to a "cut-off" has received much recent attention. To fix ideas, let us consider the family of scalar reaction-diffusion equations of the form

$$\phi_t = \phi_{xx} + f(\phi). \tag{1.1}$$

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Here, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $\phi(t, x) \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ denotes a (smooth) reaction function with $f(0) = 0$; in other words, we assume without loss of generality that (1.1) has a rest state at $\phi = 0$. Given (1.1), the corresponding modified, or "cut-off" equation is then defined as

$$\phi_t = \phi_{xx} + f(\phi)\Theta(\phi - \varepsilon), \tag{1.2}$$

where the "cut-off" function Θ is assumed to satisfy

$$\Theta(\phi - \varepsilon) \ll 1 \quad \text{if } \phi < \varepsilon \quad \text{and} \quad \Theta(\phi - \varepsilon) \equiv 1 \quad \text{if } \phi \geq \varepsilon, \tag{1.3}$$

with $0 < \varepsilon \ll 1$ the small "cut-off" parameter: To state it simply, Θ "annihilates" the reaction function f in (1.1) close to the zero rest state. This so-called "cut-off" is motivated by the fact that the physical situation underlying (1.1) is often discrete in nature, and, hence, that no "particles" should be present when $\phi < \varepsilon$, for ε sufficiently small.

In a pioneering study, Brunet and Derrida [**1**] investigated the effects of a cut-off on the front dynamics in the classical Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation, with $f(\phi) = \phi(1-\phi^2)$ in (1.1). One of their principal findings was that the selected front speed is reduced significantly by the cut-off; more precisely, the actual correction is proportional to the inverse square of the logarithm of ε.

More recently, the effects of a cut-off in other classes of scalar reaction-diffusion equations have been studied by a number of authors using matched asymptotics [**5**] and variational principles [**6**]. Notably, the corresponding shift in the front speed due to a cut-off has frequently been found to be not logarithmic in ε, but sublinear. Additionally, in some of these cases, the selected propagation speed is actually increased by the cut-off, in contrast to the FKPP case. For a more detailed discussion of this question of so-called "strong" vs. "weak" velocity selection, see [**5**].

In this note, we present a brief case study to elucidate how, in one of the scenarios discussed in [**5, 6**], the effects of a cut-off on the front dynamics of (1.1) can be understood from a more geometric, dynamical systems point of view. More specifically, we will consider only one exemplary scenario in the following, namely, front propagation into a metastable state in the cut-off Ginzburg-Landau equation

$$\phi_t = \phi_{xx} - [2(1 - \sigma)\phi + (\sigma - 3)\phi^2 + \phi^3]\Theta(\phi - \varepsilon), \tag{1.4}$$

where $0 < \sigma < 1$ is a (fixed) parameter. Moreover, for the sake of exposition, we will restrict ourselves to the Heaviside cut-off function in (1.3), with $\Theta \equiv 0$ for $\phi < \varepsilon$. Note that (1.4) also goes under the name (cut-off) Schlögl equation [**6**], and that we have defined ϕ so that the metastable state, which in the usual formulation is located at $\phi = -1$, is now at zero. Additional (stable) rest states of (1.4) are found at $\phi = 1 - \sigma$ and $\phi = 2$.

Following [**6**], we are interested in studying traveling front solutions of (1.4) that propagate from the stable state at $\phi = 2$ into the metastable state at $\phi = 0$. To that end, we revert to a co-moving frame by introducing the traveling wave variable $\xi = x - ct$ in (1.4), and we write $u(\xi) = \phi(t, x)$ for the corresponding front solution. A phase-plane analysis of the resulting traveling wave equation

$$u'' + cu' - [2(1 - \sigma)u + (\sigma - 3)u^2 + u^3]\Theta(u - \varepsilon) = 0$$

is best performed by rewriting this equation as a first-order system,

$$u' = v, \tag{1.5a}$$

$$v' = -cv + [2(1-\sigma)u + (\sigma - 3)u^2 + u^3]\Theta(u-\varepsilon), \tag{1.5b}$$

$$\varepsilon' = 0; \tag{1.5c}$$

here, we have appended the (trivial) ε-equation, and the prime denotes differentiation with respect to ξ. In the framework of (1.5), the desired traveling front that connects the two rest states at $\phi = 2$ and $\phi = 0$ in (1.4) now corresponds to a heteroclinic connection between $Q^- = (2, 0, \varepsilon)$ and $Q^+ = (0, 0, \varepsilon)$. (The third rest state of (1.4) at $\phi = 1 - \sigma$ is of no interest to us here.) Note that both Q^- and Q^+ are hyperbolic saddle equilibria, in (u, v), of the traveling wave equation obtained from (1.1), before the cut-off.

Our goal in this note is to indicate how the leading-order ε-dependent correction to the "unperturbed" propagation speed $c(0) = \sqrt{2}\sigma$ [**5, 6**] due to the cut-off in (1.5) can be derived using geometric techniques. More specifically, we will motivate why, for $\varepsilon > 0$ sufficiently small, the leading-order ε-asymptotics of c in (1.5) are of the form

$$c(\varepsilon) = c(0) + \Delta c(\varepsilon) = \sqrt{2}\sigma[1 + \mathcal{O}(\varepsilon^{2-\sigma})], \tag{1.6}$$

where, in particular, Δc has to be positive, in agreement with [**5, 6**]. Notably, this geometric approach was first employed in [**3**] to establish rigorously the logarithmic form of the corresponding correction in the cut-off FKPP equation and, hence, to give a mathematical proof of some of the results of Brunet and Derrida [**1**].

The motivation behind our investigation is to provide another example of why the method of geometric desingularization, or "blow-up" [**2**], is an appropriate tool for studying such phenomena. A more systematic, geometric classification of the possible propagation scenarios that can result from a cut-off in equations of the type in (1.1) will be the topic of the upcoming article [**4**], where these and similar questions will be analyzed in full rigor. The discussion here, on the other hand, is intentionally expository, and only serves to convey to the reader the flavor of the geometric approach.

The main underlying ideas of our argument are the following: First, we partition the phase space of (1.5) into three distinct regions, and we analyze the dynamics separately in each of these regions. In particular, we introduce various rescalings which allow us to define a "singular" heteroclinic connection Γ between Q^- and Q^+ for $\varepsilon = 0$ in (1.5); the corresponding analysis is outlined in Section 2. Then, in Section 3, we combine the results of Section 2 to motivate why the front speed $c(\varepsilon)$ has to be chosen as in (1.6) for a heteroclinic connection to persist close to Γ when $\varepsilon > 0$.

2. Dynamics of (1.5)

In this section, we analyze the dynamics of (1.5) separately in the three regimes where $u = \mathcal{O}(1)$, $0 < u < \varepsilon$, and $\varepsilon < u < \mathcal{O}(1)$, respectively.

2.1. "Outer" region. In the "outer" region, which is defined by $u = \mathcal{O}(1)$, the cut-off has no impact on the dynamics of (1.5), since, clearly, $\Theta \equiv 1$ then, see (1.3). Therefore, system (1.5) is precisely equivalent to the original, unmodified traveling wave equation

$$u'' + cu' - [2(1-\sigma)u + (\sigma-3)u^2 + u^3] = 0 \tag{2.1}$$

in that region. A traveling front solution of this equation is known explicitly provided $c = \sqrt{2}\sigma$ [**5**], with

$$u(\xi) = 1 - \tanh\left(\frac{\xi - \xi^{\text{in}}}{\sqrt{2}}\right) \tag{2.2}$$

for arbitrary (finite) ξ^{in}.

Observe that, given the expression for u in (2.2), v can of course easily be determined from $v = u'$. Note also that for $\xi \geq \xi^{\text{in}}$ large, u behaves asymptotically like $u^{\text{in}}\mathrm{e}^{-\sqrt{2}(\xi-\xi^{\text{in}})}$, for some ξ^{in}-dependent u^{in}, in accordance with classical "mode counting" arguments which require that (2.2) contain no linearly growing modes, cf. [**5**]. In the following, we will assume that u^{in} is not "too large," and we define the boundary of the outer region to be the hyperplane Σ^{in} given by $\{u = u^{\text{in}}\}$.

Now, the orbit which is determined by (2.2) corresponds precisely to the unstable manifold of the point Q^- for $\varepsilon = 0$ in (1.5), since it follows from (2.2) that $(u, u') \to (2, 0)$ as $\xi \to -\infty$. Hence, (2.2) describes the part of the singular heteroclinic connection Γ that lies in the outer region. (Equivalently, Γ of course corresponds to the "tail" of the traveling front solution in (2.2) that originates in the stable rest state at $\phi = 2$.) The point $(u^{\text{in}}, v^{\text{in}}, 0)$, with $v^{\text{in}} = u'(\xi^{\text{in}})$, will be denoted by P^{in}; it is the point of intersection of Γ with Σ^{in}. Finally, for $\varepsilon > 0$ small, the unstable manifold of $Q^- = (2, 0, \varepsilon)$ will be a regular perturbation of Γ, and will lie ε-close to it.

2.2. "Inner" region. To analyze the dynamics of (1.5) in the "inner" region where $u < \varepsilon$ and where, in particular, $\Theta \equiv 0$, we introduce the rescaling

$$u = \varepsilon U, \quad v = \varepsilon V, \quad \text{and} \quad \varepsilon = \varepsilon \tag{2.3}$$

in (1.5). The resulting, cut-off equations in terms of (U, V, ε) are given by

$$U' = V, \tag{2.4a}$$

$$V' = -cV, \tag{2.4b}$$

$$\varepsilon' = 0. \tag{2.4c}$$

For ε fixed, the equilibria of (2.4) are located on the U-axis; however, we are only interested in the origin, since it is the only equilibrium that can correspond to Q^+ for $\varepsilon > 0$.

In the singular limit as $\varepsilon \to 0$, the unique orbit of (2.4) for which $V(0) = 0$ is given by

$$V(U) = -\sqrt{2}\sigma U, \tag{2.5}$$

where we recall $c(0) = \sqrt{2}\sigma$. Observe that (2.5) corresponds to the strong stable manifold of the origin, and that it represents the singular heteroclinic connection Γ in the inner region. Moreover, for $\varepsilon > 0$ sufficiently small and (U, V) bounded, (2.5) will perturb, in a smooth, regular fashion, to the strong stable manifold $\{V(U) = -c(\varepsilon)U\}$ of $Q^+ = (0, 0, \varepsilon)$.

Since $u = \varepsilon$ corresponds precisely to $U = 1$, see (2.3), the boundary of the inner region is naturally given by the $\{U = 1\}$-hyperplane in (U, V, ε)-space. We will denote this hyperplane by Σ^{out}, and will write P^{out} for the point of intersection

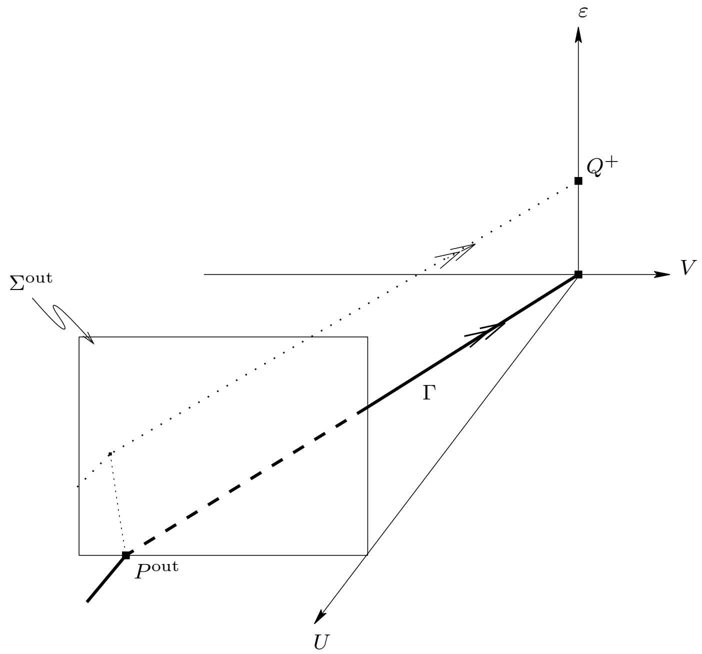

FIGURE 1. Geometry in the “inner” region.

$(U^{\mathrm{out}}, V^{\mathrm{out}}, 0) = (1, -\sqrt{2}\sigma, 0)$ of Γ with Σ^{out}. The situation in the inner region is summarized in Figure 1.

2.3. “Intermediate” region. Finally, we analyze the dynamics of (1.5) in the “intermediate” region that describes the transition between the inner and outer regions studied in the previous subsections; hence, we assume $\varepsilon < u < \mathcal{O}(1)$ now, and we observe that, consequently, $\Theta \equiv 1$, see (1.3). To study the equations in (1.5) in this regime, we introduce a new “projective” variable $W = \frac{v}{u}$; moreover, we additionally rescale the parameter ε. Hence, in sum, we define the rescaling

$$u = u, \quad v = uW, \quad \text{and} \quad \varepsilon = uE \tag{2.6}$$

for u, v, and ε in (1.5). In terms of these new variables, (1.5) becomes

$$u' = uW, \tag{2.7a}$$

$$W' = -cW - W^2 + 2(1-\sigma) + (\sigma - 3)u + u^2, \tag{2.7b}$$

$$E' = -EW. \tag{2.7c}$$

The equilibria of (2.7) are located at $P^{\mathrm{s}} = (0, -\sqrt{2}, 0)$ and $P^{\mathrm{u}} = (0, \sqrt{2}(1-\sigma), 0)$. Note that these two points correspond precisely to the stable eigendirection, respectively to the unstable eigendirection, of the linearization at Q^+ of the original, unmodified equation (2.1); therefore, the effect of (2.6) in (1.5) is to “tease apart” the possible asymptotics of trajectories close to Q^+, and, hence, to “desingularize” the dynamics there down to the singular limit of $\varepsilon = 0$.

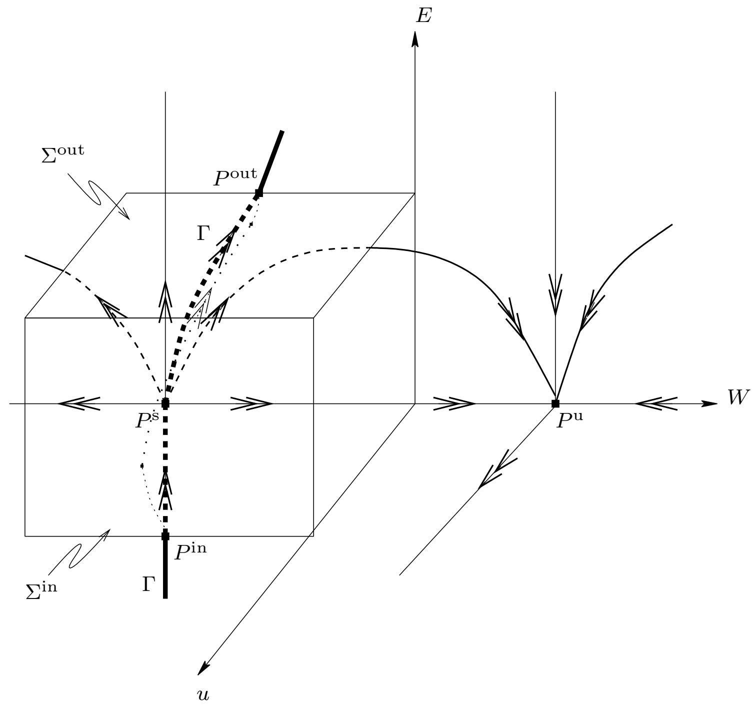

FIGURE 2. Geometry in the "intermediate" region.

Now, a simple computation shows that both P^{s} and P^{u} are hyperbolic saddle equilibria of (2.7), with eigenvalues

$$-\sqrt{2} < 0, \quad \sqrt{2}(2-\sigma) > 0, \quad \text{and} \quad \sqrt{2} > 0,$$

respectively

$$\sqrt{2}(1-\sigma) > 0, \quad -\sqrt{2}(2-\sigma) < 0, \quad \text{and} \quad -\sqrt{2}(1-\sigma) < 0,$$

where we recall $0 < \sigma < 1$. More interesting to us is the point P^{s}, since it follows from the large-ξ asymptotics of (2.2) that $W = \frac{v}{u} \to -\sqrt{2}$ must hold on Γ as $u \to 0$ and, hence, that Γ must contain P^{s}. The resulting local dynamics of (2.7) are illustrated in Figure 2.

Finally, we observe that the two hyperplanes $\{u = 0\}$ and $\{E = 0\}$ are invariant for (2.7); the corresponding limiting systems which are obtained for $u = 0$, respectively for $E = 0$, in (2.7) govern the singular dynamics of (1.5) in the intermediate region. Indeed, one can show that there exist unique orbits in $\{u = 0\}$, respectively in $\{E = 0\}$, which connect P^{in} to P^{s}, respectively P^{s} to P^{out}. Hence, the union of these orbits constitutes the part of Γ lying in between Σ^{in} and Σ^{out}, see again Figure 2.

REMARK 2.1. The eigenvalues of the linearization of (2.7) at both P^{s} and P^{u} are in resonance for $\sigma \in \mathbb{Q}$; generically, such values of σ can be expected to give rise to logarithmic (switchback) terms [**7**].

3. Motivation of (1.6)

In this section, we combine the results of Section 2 to describe the transition of orbits from $\Sigma^{\rm in}$ to $\Sigma^{\rm out}$ for $\varepsilon > 0$ in (1.5). In other words, we will derive a condition on $c(\varepsilon)$ for a heteroclinic connection between Q^- and Q^+ to persist, close to Γ, after passage through the intermediate region; this condition, in turn, will imply (1.6). The subsequent analysis is based on an adaptation of the approach developed in [**3**], and will be made fully rigorous in [**4**].

In a first step, consider again system (2.7), and note that the E-equation (2.7c) decouples. Note also that the u-terms in (2.7b) are assumed to be small, since $u \in [\varepsilon, u^{\rm in}]$, and that we can therefore neglect them to leading order. In sum, we are hence concerned with the approximate system

$$u' = uW, \tag{3.1a}$$

$$W' = -cW - W^2 + 2(1-\sigma) \tag{3.1b}$$

now. To simplify (3.1) further, we recall $c(\varepsilon) = \sqrt{2}\sigma + \Delta c(\varepsilon)$, where $\Delta c = \mathcal{O}(1)$; moreover, we introduce the new variable $Z = W + \frac{\sigma}{\sqrt{2}}$ in (3.1):

$$u' = -u\big(\tfrac{\sigma}{\sqrt{2}} - Z\big), \tag{3.2a}$$

$$Z' = \big(\tfrac{\sigma}{\sqrt{2}} - Z\big)\Delta c - Z^2 + \frac{(\sigma-2)^2}{2}. \tag{3.2b}$$

Note that for $\Delta c = 0$, the equations in (3.2) again correspond to the linearization of (2.1) about Q^+, before the various transformations.

Since we are not interested in the (time-parametrized) solutions of (3.2), but only in the corresponding orbits, we may change the time parametrization in (3.2) by dividing out the (positive) factor $\frac{\sigma}{\sqrt{2}} - Z$ from the right-hand sides:

$$\dot{u} = -u, \tag{3.3a}$$

$$\dot{Z} = \Delta c - \frac{Z^2 - \frac{(\sigma-2)^2}{2}}{\frac{\sigma}{\sqrt{2}} - Z}. \tag{3.3b}$$

Here, the overdot denotes differentiation with respect to the new "time" ζ. Now, note that (3.3a) can be solved explicitly, with $u(\zeta) = u^{\rm in}{\rm e}^{-(\zeta-\zeta^{\rm in})}$, which will allow us to estimate the transition "time" $\zeta^{\rm out}$ to $\Sigma^{\rm out}$; moreover, observe that (3.3b) is separable, and that the solution can be given in implicit form:

$$\begin{aligned}\zeta^{\rm out} - \zeta^{\rm in} &- \frac{1}{2}\ln\left|2Z^2 - (2-\sigma)^2 - (\sqrt{2}\sigma - 2Z)\Delta c\right|\Big|_{Z^{\rm in}}^{Z^{\rm out}} \\ &- \frac{\sqrt{2}\sigma + \Delta c}{\sqrt{2(2-\sigma)^2 + 2\sqrt{2}\sigma\Delta c + \Delta c^2}} \\ &\times \operatorname{arctanh}\left(\frac{2Z + \Delta c}{\sqrt{2(2-\sigma)^2 + 2\sqrt{2}\sigma\Delta c + \Delta c^2}}\right)\Big|_{Z^{\rm in}}^{Z^{\rm out}} = 0.\end{aligned} \tag{3.4}$$

To ensure the persistence of a heteroclinic connection in (1.5) for ε small, Δc in (3.4) has to be fixed such that, to leading order, the corresponding orbit remains a connection between the points $P^{\rm in}$ and $P^{\rm out}$. To that end, we note that $Z^{\rm in} =$

$W^{\rm in} + \frac{\sigma}{\sqrt{2}} \approx \frac{\sigma-2}{\sqrt{2}}$, since $W^{\rm in} = \frac{v^{\rm in}}{u^{\rm in}} \approx -\sqrt{2}$ close to $P^{\rm s}$, see (2.2). Since, moreover,

$$W^{\rm out} = \frac{v^{\rm out}}{u^{\rm out}} = \frac{V^{\rm out}}{U^{\rm out}} = -\sqrt{2}\sigma,$$

cf. (2.3), there holds $Z^{\rm out} = W^{\rm out} + \frac{\sigma}{\sqrt{2}} = -\frac{\sigma}{\sqrt{2}}$. Finally, we note that $\zeta^{\rm out} \approx -\ln\varepsilon$ due to $u^{\rm out} = \varepsilon U^{\rm out} = \varepsilon$, whereas $\zeta^{\rm in}$ is still $\mathcal{O}(1)$.

Taking into account these estimates on Z and ζ and expanding the result accordingly, we find that the leading-order contribution in (3.4) is given by

$$-\ln\varepsilon + \frac{1}{2}\ln|\Delta c| - \frac{\sigma}{2-\sigma}\operatorname{arctanh}\Big(1 - \frac{\sqrt{2}\Delta c}{(2-\sigma)^2}\Big), \tag{3.5}$$

as well as that this contribution must be $\mathcal{O}(1)$ in order for (3.4) to be satisfied. Expanding (3.5) further and rearranging terms, we see that the arctanh-term is approximately equal to $-\frac{1}{2}\ln|\Delta c|$ and, hence, that (3.5) can be simplified to

$$-2(2-\sigma)\ln\varepsilon + (2-\sigma)\ln|\Delta c| + \sigma\ln|\Delta c|.$$

In sum, we must therefore require

$$\ln\frac{|\Delta c|}{\varepsilon^{2-\sigma}} = \mathcal{O}(1)$$

for a connection close to Γ to be possible for ε small. This condition, however, is precisely equivalent to (1.6).

Finally, let us comment on the question of why $\Delta c > 0$ must hold in (1.6): For $|\Delta c|$ small, system (3.1) has an equilibrium close to $P^{\rm s}$, where the corresponding W-value is approximately given by $-\sqrt{2} - \frac{\Delta c}{2-\sigma}$. However, since $W^{\rm in} \approx -\sqrt{2}$, this equilibrium would "block" trajectories from making the connection to $P^{\rm out}$ for $\Delta c < 0$, which rules out the persistence of Γ in that case.

In conclusion, the geometric picture outlined in this note nicely motivates why the problem of front speed selection in reaction-diffusion equations of the type in (1.1) can be so delicate. Indeed, as is obvious from Figure 2, a connection from $P^{\rm in}$ to $P^{\rm out}$ will not be possible in general; rather, trajectories will typically be repelled away from $P^{\rm s}$ for "most" choices of c. In other words, the modified, cut-off traveling wave system in (1.5) will admit a heteroclinic connection between Q^- and Q^+ only if the correction Δc is tuned in precisely the "right" manner, as specified in (1.6). In that sense, the smallness of Δc (in ε) has to be such that it "balances" the strongly expansive dynamics of (2.7) close to $P^{\rm s}$.

Remark 3.1. A more rigorous geometric analysis of the dynamics of (1.5) would have to rely on the "blow-up" technique [**2**] to describe accurately the transition from $\Sigma^{\rm in}$ to $\Sigma^{\rm out}$, cf. [**3**]. In that context, the rescaled variables in (2.3) and (2.6) would correspond to the "classical" rescaling and to a "phase-directional" rescaling, respectively.

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