THE ASYMPTOTIC CRITICAL WAVE SPEED IN A FAMILY OF SCALAR REACTION-DIFFUSION EQUATIONS

FREDDY DUMORTIER, NIKOLA POPOVIĆ, AND TASSO J. KAPER

Abstract. We study traveling wave solutions for the class of scalar reaction-diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u),$$

where the family of potential functions $\{f_m\}$ is given by $f_m(u) = 2u^m(1-u)$. For each $m \geq 1$ real, there is a critical wave speed $c_{\rm crit}(m)$ that separates waves of exponential structure from those which decay only algebraically. We derive a rigorous asymptotic expansion for $c_{\rm crit}(m)$ in the limit as $m \to \infty$. This expansion also seems to provide a useful approximation to $c_{\rm crit}(m)$ over a wide range of m-values. Moreover, we prove that $c_{\rm crit}(m)$ is $\mathcal{C}^\infty$-smooth as a function of m^{-1}. Our analysis relies on geometric singular perturbation theory, as well as on the blow-up technique, and confirms the results obtained by means of asymptotic methods in [D.J. Needham and A.N. Barnes, Nonlinearity, 12(1):41-58, 1999] and in [T.P. Witelski, K. Ono, and T.J. Kaper, Appl. Math. Lett., 14(1):65-73, 2001].

1. Introduction

We consider traveling wave solutions for the class of scalar bistable reaction-diffusion equations given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u), \tag{1}$$

where the family of potential functions $\{f_m\}$ is defined via $f_m(u) = 2u^m(1-u)$, with $m \geq 1$ real. The restriction to $m \geq 1$ is necessary, since it has been shown in [17, 27] that no traveling waves for (1) can exist when $m < 1$, see also [21].

The class of problems in (1) includes the classical Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with quadratic nonlinearity ($m = 1$) [10, 12], as well as a bistable equation with degenerate cubic nonlinearity ($m = 2$) [25]. In particular, it has been studied in [25] as a bridge between the classical FKPP equation and the family of nondegenerate bistable cubic equations with potential $f(u) = u(u-a)(1-u)$, $a \in (0, \frac{1}{2})$. In the former, $u = 0$ is an unstable state (in the PDE sense), whereas in the latter, it is a stable state of the PDE. The motivation for studying (1) in [25] was that it is a family of equations for which the state $u = 0$ is neutrally stable and, hence, that it lies "in between" the two classical cases. Interesting mathematical phenomena concerning the stability of wave fronts were reported in [25], see also [18, 15]. We hope that the existence analysis presented here will be useful for further investigating the stability of these solutions.

Let the traveling wave solutions to (1) be denoted by $u(x,t) = U(\xi)$, with $\xi = x - ct$ the traveling wave variable and c the wave speed. Moreover, let

$$\lim_{\xi \to \infty} U(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to -\infty} U(\xi) = 1.$$

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It is well-known that for each $m \geq 1$, there is a critical wave speed $c_{\mathrm{crit}}(m) > 0$ such that traveling wave solutions exist for $c \geq c_{\mathrm{crit}}(m)$ in (1) [2, 1]. The speed $c_{\mathrm{crit}}(m)$ is critical in the sense that waves decay exponentially ahead of the wave front (i.e., as $\xi \to \infty$) when $c = c_{\mathrm{crit}}(m)$, whereas the decay is merely algebraic in ξ for $c > c_{\mathrm{crit}}(m)$.

The family of equations in (1) has been studied in the regimes where m is near 1 or 2. Perturbation analyses off these classical cases have been carried out for $m = 1+\varepsilon$ using matched asymptotic expansions [17] and geometric singular perturbation theory [21], showing that the limit as $\varepsilon \to 0$ is non-uniform, with the critical wave speed given by

$$c_{\mathrm{crit}}(1+\varepsilon) = 2\sqrt{2} - \sqrt{2}\Omega_0 \varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon) \qquad \text{for } \varepsilon \in (0, \varepsilon_0).$$

Here, $\varepsilon_0 > 0$ is small, and Ω_0 is the first real zero of the Airy function. The corresponding result for $m \approx 2$ is

$$c_{\mathrm{crit}}(2+\varepsilon) = 1 - \frac{13}{24}\varepsilon + \mathcal{O}(\varepsilon^2) \qquad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

see also [27].

In the following, we study (3) in the limit of $m \to \infty$. This problem was considered in [27] via the method of matched asymptotic expansions; independently, it was analyzed in [18] using a slightly different approach. In particular, it has been shown that $c_{\mathrm{crit}}(m) \sim \frac{2}{m}$ to leading order for the critical wave speed c_{crit} that separates solutions in (1) which decay exponentially from those for which the decay is merely algebraic.

Here, the aim is to derive a rigorous asymptotic expansion for $c_{\mathrm{crit}}(m)$ in the large-m limit, and thereby to justify the matched asymptotic analysis of [27] and [18] within a geometric framework. At the same time, we will also obtain an alternative proof for the existence of the corresponding traveling wave solutions in (1). Two additional factors motivated the analysis of the large-m limit. First, both the asymptotic analysis and the numerical results in [27, 18] suggest that $c_{\mathrm{crit}}(m)$ decreases monotonically to zero as $m \to \infty$, which is confirmed in Theorem 1.1 below. Second, the expansion for $c_{\mathrm{crit}}(m)$ as $m \to \infty$ agrees well with the numerics over a wide range of m-values, even down to $m = 2$, see [27, Figure 3(a)]. Hence, the results obtained in the large-m regime seem to provide a useful approximation to $c_{\mathrm{crit}}(m)$ also for finite values of m.

The following is the principal result of this work:

Theorem 1.1. *There exists a function $c_{\mathrm{crit}}(m)$ and an $m_0 \in \mathbb{R}$ sufficiently large such that for $m \geq m_0$, $c = c_{\mathrm{crit}}(m)$ is the critical wave speed for* (1)*. Moreover, $c_{\mathrm{crit}}(m)$ is $\mathcal{C}^\infty$-smooth in m^{-1}, and there holds*

$$c_{\mathrm{crit}}(m) = \frac{2}{m} + \frac{\sigma}{m^2} + \mathcal{O}(m^{-3}), \tag{2}$$

where σ is defined as

$$\sigma = \lim_{\omega_0 \to \infty} \int_0^{\omega_0} \left[\frac{\omega^2 \mathrm{e}^{-\omega}}{\sqrt{1-(1+\omega)\mathrm{e}^{-\omega}}} - \frac{\omega^3}{2}\mathrm{e}^{-\omega} \right] d\omega \approx -0.3119.$$

The main technique on which our proof of Theorem 1.1 is based is the global blow-up technique, also known as geometric desingularization of families of vector fields. To the best of our knowledge, this method was first used in studying the limit cycles near a cuspidal loop in [7]. The blow-up technique has since been successfully applied in the study of numerous bifurcation problems. It has for instance been introduced in [5] as an extension of the more classical geometric singular perturbation theory [9, 11] to problems in which normal hyperbolicity is lost. For further examples, we refer the reader to [3, 6, 4, 13, 14, 22].

This article is organized as follows. In Section 2, we define the geometric framework for the analysis of (3). In Section 3, we introduce the blow-up transformation required for the desingularization of the corresponding "inner problem." In Section 4, we combine the results of the previous sections into the proof of Theorem 1.1.

2. A Geometric Analysis of (3)

We will prove Theorem 1.1 by studying the corresponding global bifurcation problem in the traveling wave ODE associated to (1). Recall that $\xi = x - ct$ denotes the traveling wave variable and that $U(\xi) = u(x,t)$. Then, traveling waves of velocity c are given by heteroclinic trajectories for the nonlinear second-order equation

$$U'' + cU' + 2U^m(1-U) = 0 \tag{3}$$

that connect the two rest states at $U = 1$ and $U = 0$; here, the prime denotes differentiation with respect to ξ.

For a geometric analysis of (3), it is convenient to first recast the equation in Liénard form, i.e., to consider the autonomous first-order system

$$\begin{aligned} U' &= V - cU, \\ V' &= -2U^m(1-U). \end{aligned} \tag{4}$$

The equilibria of (4) are located at $Q^+ : (U,V) = (0,0)$ and $Q^- : (U,V) = (1,c)$. Traveling wave solutions of (1) correspond to heteroclinic connections between these two points in (4), with

$$\lim_{\xi \to \pm\infty} (U,V)(\xi) = Q^{\pm}.$$

We only consider $m > 1$ and $c \geq 0$; then, a simple calculation shows

Lemma 2.1. *The point Q^- is a hyperbolic saddle for any $c \geq 0$, with eigenvalues and the corresponding eigendirections given by*

$$-\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2+8} \quad \text{and} \quad \Big(-\frac{c}{4} \pm \frac{1}{4}\sqrt{c^2+8}, 1\Big)^T,$$

respectively. The point Q^+ is a saddle-node for $c > 0$, with eigenvalues $-c$ and 0 and eigendirections $(1,0)^T$ and $(1,c)^T$. For $c = 0$, zero is a double eigenvalue, with one eigendirection $(1,0)^T$ (and the generalized eigendirection $(0,1)^T$).

We will be interested in the unstable manifold $\mathcal{W}^u(Q^-)$ of Q^- and in those values of c for which it connects to the strong stable manifold $\mathcal{W}^s(Q^+)$ of Q^+. Geometrically, the dependence of solutions to (4) on c can be understood as follows. Whenever $c > c_{\text{crit}}(m)$, $\mathcal{W}^u(Q^-)$ approaches Q^+ on a center manifold, which is locally tangent to the span of $(1,0)^T$. Hence, solutions decay algebraically as $\xi \to \infty$. Precisely for $c = c_{\text{crit}}(m)$, $\mathcal{W}^u(Q^-)$ coincides with $\mathcal{W}^s(Q^+)$; thus, solutions approach Q^+ tangent to $(1,c)^T$ and decay exponentially as $\xi \to \infty$. For $c < c_{\text{crit}}(m)$, no heteroclinic solutions to (4) exist, as $\mathcal{W}^u(Q^-)$ does not enter the basin of attraction of Q^+. Therefore, for $m > 1$, a global bifurcation occurs at $c = c_{\text{crit}}(m)$ due to the switchover from one type of connection to another in (4).

***Remark* 1.** For $m = 1$, c_{crit} is determined by a local transition condition, with Q^+ changing from being a stable node via a degenerate node to a stable spiral.

2.1. A preliminary rescaling for (4). We define the new parameter $\varepsilon = m^{-1}$ and hence consider the limit as $\varepsilon \to 0$ in the following. Given that the function $f_m(U)$ assumes its maximum at $U = \frac{m}{m+1}$ and that

$$f_m(\tfrac{m}{m+1}) = 2\Big(\frac{m}{m+1}\Big)^m \frac{1}{m+1} \sim \frac{2}{\mathrm{e}}\varepsilon$$

for m sufficiently large, we rescale V via $V = \varepsilon\widetilde{V}$. Also, we know formally and numerically that $c_{\mathrm{crit}} = \mathcal{O}(1)$ as $\varepsilon \to 0$ [18, 27]; therefore, we write $c = \varepsilon\tilde{c}$.

Under these rescalings, the equations in (4) become

$$\dot{U} = \widetilde{V} - \tilde{c}U, \tag{5a}$$

$$\dot{\widetilde{V}} = -\frac{2}{\varepsilon^2}U^{\frac{1}{\varepsilon}}(1-U); \tag{5b}$$

here, the overdot denotes differentiation with respect to the rescaled traveling wave coordinate

$$\tilde{\xi} = \varepsilon\xi. \tag{6}$$

We investigate (5) for $\varepsilon \to 0$. More precisely, we will decompose the analysis of (5) into two separate problems, the "outer problem" and the "inner problem," which are defined for $0 \le U < 1$ and for $U \approx 1$, respectively. This decomposition is naturally suggested when one introduces $U^{\frac{1}{\varepsilon}} = \mathrm{e}^{\frac{1}{\varepsilon}\ln U}$ in (5b), since this term is exponentially small if $U < 1$. The desired expansion for $c_{\mathrm{crit}}(\varepsilon)$ will then be obtained by constructing a solution which is uniformly valid on the entire domain $[0, 1]$.

2.2. The "outer problem". For $U \in [0, 1)$, the potential $f_m(U)$ is essentially zero for m large. More specifically, for $U \in [0, U_0]$ with $U_0 < 1$ constant, the right-hand side in (5b) is exponentially small in ε. Therefore, we find that on this "outer domain" the dynamics are governed to leading order by the system

$$\dot{U} = \widetilde{V} - \tilde{c}U, \tag{7a}$$

$$\dot{\widetilde{V}} = 0, \tag{7b}$$

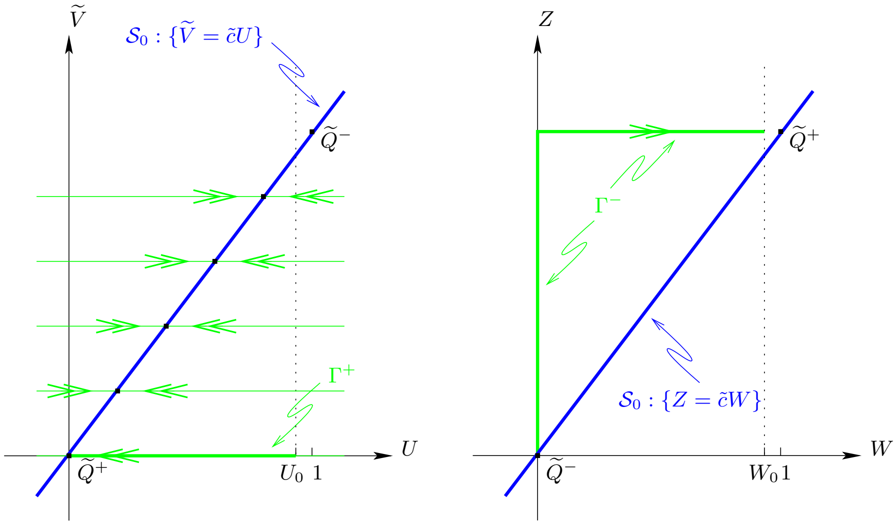
which is labeled the "outer problem" or the reduced slow system. For system (7), the invariant manifold defined by $\mathcal{S}_0 := \big\{(U, \widetilde{V}) \,\big|\, \widetilde{V} = \tilde{c}U, U \in [0, U_0]\big\}$ is normally hyperbolic; in fact, this manifold is normally attracting, since $\tilde{c} > 0$ by assumption. The corresponding fast foliation $\mathcal{F}_0$ consists of axis-parallel fibers $\{\widetilde{V} = \widetilde{V}_0\}$. The situation is illustrated in Figure 1(a).

By standard persistence theory [8, 9], it follows that for $\varepsilon > 0$ sufficiently small, both $\mathcal{S}_0$ and $\mathcal{F}_0$ will persist; we will denote the corresponding slow manifold and its associated foliation by $\mathcal{S}_\varepsilon$ and $\mathcal{F}_\varepsilon$, respectively. Since the only ε-dependence in (5a) is encoded in $\tilde{c}$, the slow manifold $\mathcal{S}_\varepsilon$ is to all orders given by the straight line of slope $\tilde{c}$ in $(U, \widetilde{V})$-space,

$$\mathcal{S}_\varepsilon = \big\{(U, \widetilde{V}) \,\big|\, \widetilde{V} = \tilde{c}U, U \in [0, U_0]\big\},$$

where $\tilde{c} = \tilde{c}(\varepsilon)$ is ε-dependent now. Similarly, given (5b), we see that the fibers of $\mathcal{F}_\varepsilon$ will be exponentially close (in ε) to the lines $\{\widetilde{V} = \widetilde{V}_0\}$, with $\widetilde{V}_0$ constant.

The fiber $\Gamma^+ : \{\widetilde{V} = 0\}$, i.e., the U-axis, will be of particular interest. It gives, to leading order, the strong stable manifold $\mathcal{W}^s(\widetilde{Q}^+)$ of $\widetilde{Q}^+$, where $\widetilde{Q}^+$ denotes the origin which lies on $\mathcal{S}_\varepsilon$ for any value of ε.



(a) The "outer problem" (7).

(b) The "inner problem" (10).

FIGURE 1. The geometry for $\varepsilon = 0$.

2.3. The "inner problem". For $U \approx 1$, the potential $f_m(U)$ gives a finite contribution even as $m \to \infty$. Moreover, in a neighborhood of $U = 1$ (referred to as an "inner region") f_m varies rapidly, which signals the existence of a boundary layer there. More precisely, close to the point $\widetilde{Q}^- : (U, \widetilde{V}) = (1, \tilde{c})$, the right-hand side in (5b) is significant in the limit as $\varepsilon \to 0$, and there is a rapid transition for ε positive, but small.

To analyze the dynamics of (5) in the boundary layer near $U = 1$, we first introduce the new variables $W = 1 - U$ and $Z = -(\widetilde{V} - \tilde{c})$ in (5):

$$\dot{W} = Z - \tilde{c}W, \tag{8a}$$

$$\dot{Z} = \frac{2}{\varepsilon^2}(1 - W)^{\frac{1}{\varepsilon}} W. \tag{8b}$$

Hence, the point $\widetilde{Q}^-$ has been moved to the origin in the new (W, Z)-coordinates, while the critical manifold $\mathcal{S}_0$ is now given by $\{Z = \tilde{c}W\}$, and is still a line of slope $\tilde{c}$.

Next, we write $(1 - W)^{\frac{1}{\varepsilon}} = \mathrm{e}^{\frac{1}{\varepsilon}\ln(1-W)}$ and expand the logarithm as

$$\ln(1 - W) = -\sum_{j=1}^{\infty} \frac{W^j}{j}, \tag{9}$$

since we are interested in W small. In sum, we have obtained the system

$$\dot{W} = Z - \tilde{c}W, \tag{10a}$$

$$\dot{Z} = \frac{2}{\varepsilon^2} W \mathrm{e}^{-\frac{W}{\varepsilon}(1+\mathcal{O}(W))}. \tag{10b}$$

Even though the right-hand side in (10b) is not defined at $\varepsilon = 0$, we will show in Section 3 that the corresponding limiting dynamics (the "inner problem" for (4)) can be obtained by geometric desingularization (blow-up) [3]. In particular, the inner limit of (10b) as $(W, \varepsilon) \to (0, 0)$ is non-uniform. Heuristically, the limiting dynamics for $\varepsilon \to 0$ should be described by the singular orbit

$$\Gamma^- := \{(0, Z) \mid Z \in [0, \tilde{c}]\} \cup \{(W, \tilde{c}) \mid W \in [0, W_0]\}, \tag{11}$$

where $W_0 = 1 - U_0$ (with U_0 defined as above). The orbit Γ^- consists of that portion of the Z-axis which to lowest order describes the boundary layer at $W = 0$, as well as of a segment of $\{Z = \tilde{c}\}$ which corresponds to the fiber $\{\widetilde{V} = 0\}$ in the "outer" coordinates, see Figure 1(b). This intuition will be made rigorous using geometric desingularization to analyze the dynamics of (10) in a neighborhood of the Z-axis.

3. The blow-up transformation for (10)

To desingularize the dynamics of (10) close to the Z-axis, we define the cylindrical blow-up transformation

$$W = \bar{r}\bar{w}, \quad Z = \bar{z}, \quad \varepsilon = \bar{r}\bar{\varepsilon}, \tag{12}$$

where $(\bar{w}, \bar{\varepsilon}) \in \mathbb{S}^1_+ = \{(\bar{w}, \bar{\varepsilon}) \,|\, \bar{w}^2 + \bar{\varepsilon}^2 = 1, \bar{w}, \bar{\varepsilon} \geq 0\}$, $\bar{z} \in [0, z_0]$, and $\bar{r} \in [0, r_0]$.

***Remark* 2.** The central idea underlying the blow-up technique is to rescale both phase variables and parameters in a manner that transforms a non-hyperbolic situation into a hyperbolic one, with fixed points (respectively lines of non-isolated fixed points) typically being blown-up into spheres (respectively cylinders). Mathematically, an n-dimensional equation depending on p parameters is transformed into an $(n+1)$-dimensional equation which depends on $p-1$ parameters. In general, if there is a lack of normal hyperbolicity along a q-dimensional submanifold $\mathcal{W}$ with $q < n$, then $\mathcal{W}$ can be represented in local coordinates as $\mathbb{R}^q \times \{\mathbf{0}\} \subset \mathbb{R}^q \times \mathbb{R}^{n-q}$, and we can identify the parameter space with $\mathbb{R}^p$. During the blow-up procedure, one first writes the parameter λ as $(\lambda_1, \ldots, \lambda_p) = (\varepsilon^{i_1}\bar{\lambda}_1, \ldots, \varepsilon^{i_p}\bar{\lambda}_p)$ with $(\bar{\lambda}_1, \ldots, \bar{\lambda}_p) \in \mathbb{S}^{p-1}$, for "well-chosen" powers $i_1, \ldots, i_p \in \mathbb{N}$. (Here, $\mathbb{S}^{p-1}$ denotes the $(p-1)$-sphere in $\mathbb{R}^p$.) Then, one adds ε as an additional variable to $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and one replaces $\mathbb{R}^q \times \{\mathbf{0}\} \subset \mathbb{R}^q \times \mathbb{R}^{n-q+1}$ by $\mathbb{R}^q \times \mathbb{S}^{n-q}$. For example, $\{\mathbf{0}\} \subset \mathbb{R}^{n+1}$ would be replaced by a sphere $\mathbb{S}^n$, while $\mathbb{R} \times \{\mathbf{0}\} \subset \mathbb{R} \times \mathbb{R}^n$ would be transformed into $\mathbb{R} \times \mathbb{S}^{n-1}$. In our case, we have $n = 2$ and $q = 1$. We refer the reader to the references cited above for more information.

The dynamics of the blown-up vector field are best analyzed by introducing charts. We employ two charts here, the "rescaling" chart K_2 defined by $\bar{\varepsilon} = 1$ and a "phase-directional" chart K_1 with $\bar{w} = 1$. The following lemma describes the transition between these charts on their overlap domain:

Lemma 3.1. *The coordinate change* $\kappa_{21} : K_2 \to K_1$ *is given by*

$$r_1 = r_2 w_2, \quad z_1 = z_2, \quad \text{and} \quad \varepsilon_1 = w_2^{-1}.$$

***Remark* 3.** Given any object $\square$, we will denote the corresponding blown-up object by $\overline{\square}$; in chart K_j $(j = 1, 2)$, the same object will appear as $\square_j$.

***Remark* 4.** In [27], the modified potential $\tilde{f}_m(U) = 2U(1-U)\mathrm{e}^{-(m-1)(1-U)}$ is introduced to analyze (10) via a comparison principle. Incidentally, the modified dynamics resulting from replacing f_m by $\tilde{f}_m$ in (10) will correspond precisely to the leading-order behavior obtained after blow-up.

3.1. Dynamics in chart K_2.

In chart K_2, (12) is given by

$$W = r_2 w_2, \quad Z = z_2, \quad \varepsilon = r_2.$$

Substituting this transformation into (10), we obtain

$$\begin{aligned} \dot{w}_2 &= \frac{1}{r_2}(z_2 - r_2\tilde{c}w_2), \\ \dot{z}_2 &= \frac{2}{r_2}w_2\mathrm{e}^{-w_2(1+\mathcal{O}(r_2 w_2))}, \\ \dot{r}_2 &= 0. \end{aligned} \tag{13}$$

To desingularize the flow on $\{r_2 = 0\}$, we multiply through the right-hand sides in (13) by a factor of r_2; this desingularization corresponds to a reparametrization of "time," leaving the phase portrait unchanged,

$$
\begin{aligned}
w_2' &= z_2 - r_2\tilde{c}w_2, \\
z_2' &= 2w_2 \mathrm{e}^{-w_2(1+\mathcal{O}(r_2 w_2))}, \\
r_2' &= 0.
\end{aligned} \tag{14}
$$

Here, the prime denotes differentiation with respect to the new variable $\tilde{\xi} r_2^{-1}$, which, in chart K_2, is precisely the original ξ, recall (6).

***Remark* 5.** The fact that (13) is desingularized by multiplying the equations by a positive power of $\bar{r}$ (instead of by dividing out some positive power of $\bar{r}$) reflects the nature of the singular limit in (10). More precisely, the vector field is unbounded as $\varepsilon \to 0$, which contrasts with the more standard non-hyperbolic case, where desingularization is achieved by dividing out the appropriate power of $\bar{r}$.

The only finite equilibrium of (14) is the origin. This equilibrium, which we call $\widetilde{Q}_2^-$, is a hyperbolic saddle point for $\tilde{c} > 0$ and $r_2 \in [0, r_0]$ sufficiently small:

Lemma 3.2. *For $r_2 \in [0, r_0]$ fixed, the eigenvalues of* (14) *at $\widetilde{Q}_2^-$ are given by*

$$
-\frac{r_2\tilde{c}}{2} \pm \frac{1}{2}\sqrt{r_2^2\tilde{c}^2 + 8} \quad \textit{and} \quad 0,
$$

with corresponding eigendirections

$$
\Big(-\frac{r_2\tilde{c}}{4} \pm \frac{1}{4}\sqrt{r_2^2\tilde{c}^2 + 8}, 1, 0\Big)^T \quad \textit{and} \quad (0,0,1)^T,
$$

respectively.

Note that $\widetilde{Q}_2^-$ corresponds to the origin in (Z, W)-coordinates before blow-up and, hence, to the original saddle point located at $\widetilde{Q}^- : (U, \widetilde{V}) = (1, \tilde{c})$.

For $r_2 = 0$ in (14), we obtain the integrable system

$$
\begin{aligned}
w_2' &= z_2, \\
z_2' &= 2w_2 \mathrm{e}^{-w_2}.
\end{aligned} \tag{15}
$$

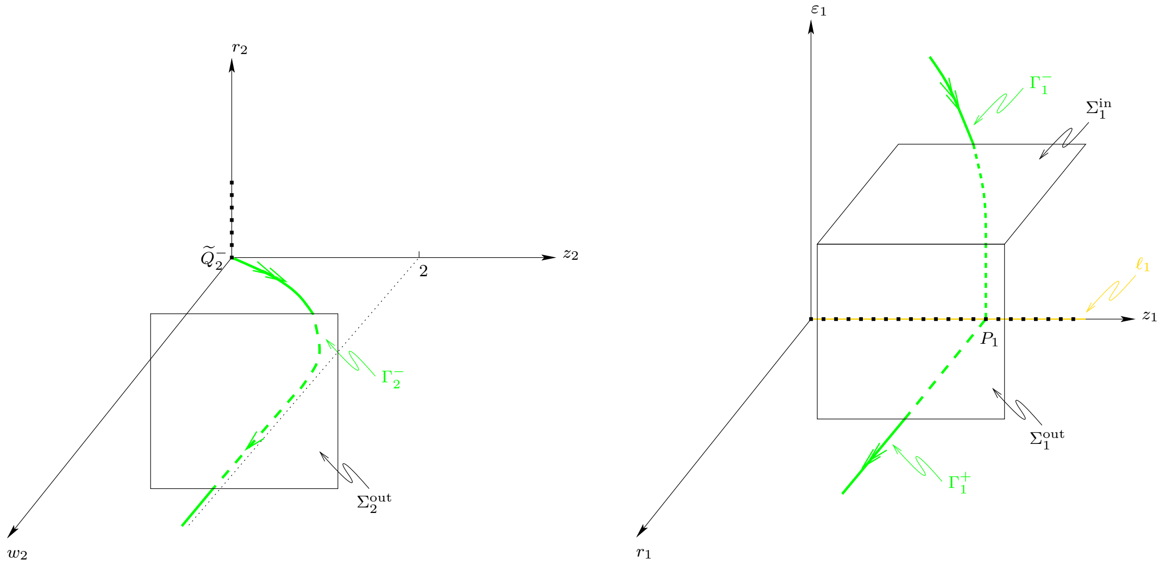
Equivalently, we can rewrite (15) as $z_2 \frac{dz_2}{dw_2} = 2w_2 \mathrm{e}^{-w_2}$, which can be solved explicitly for $z_2 = z_2(w_2)$. The only two solutions with $z_2(0) = 0$ are given by $z_2(w_2) = \pm 2\sqrt{1 - (1 + w_2)\mathrm{e}^{-w_2}}$. The corresponding orbits are associated to the two eigendirections $(\pm\frac{\sqrt{2}}{2}, 1, 0)^T$ with eigenvalues $\pm\sqrt{2}$, respectively. To lowest order, they give the stable and unstable manifolds $\mathcal{W}_2^s(\widetilde{Q}_2^-)$ and $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of $\widetilde{Q}_2^-$. Note that for $w_2 \to \infty$, $z_2 \to \pm 2$.

We will be concerned with

$$
\Gamma_2^- : z_2(w_2) = 2\sqrt{1 - (1 + w_2)\mathrm{e}^{-w_2}} \tag{16}
$$

here, since it corresponds to the singular orbit Γ^- before blow-up. See Figure 2(a) for a summary of the geometry in chart K_2.

***Remark* 6.** Equations (15) correspond precisely to the leading-order "inner system" obtained in [27] by means of asymptotic analysis.



(a) The "rescaling" chart K_2. (b) The "phase-directional" chart K_1.

FIGURE 2. The dynamics in the two charts.

3.2. Dynamics in chart K_1. In chart K_1, we have

$$W = r_1, \quad Z = z_1, \quad \varepsilon = r_1\varepsilon_1$$

for the blow-up transformation in (12), which implies

$$\begin{aligned} r_1' &= r_1(z_1 - r_1\tilde{c}), \\ z_1' &= \frac{2}{\varepsilon_1^2}\mathrm{e}^{-\frac{1}{\varepsilon_1}(1+\mathcal{O}(r_1))}, \\ \varepsilon_1' &= -\varepsilon_1(z_1 - r_1\tilde{c}) \end{aligned} \tag{17}$$

for the equations in (10) after desingularization, i.e., after multiplication by r_1.

Since we assume that r_1 is small, the equilibria of (17) are located on the line $\ell_1 = \{(0, z_1, 0) \,|\, z_1 \in [0, z_0]\}$. Note that although the vector field in (17) is, at first sight, not defined for $\varepsilon_1 = 0$, it extends for $\varepsilon_1 \to 0$ to a $\mathcal{C}^\infty$ vector field, since $\mathcal{O}(r_1)$ stands for an analytic function which is strictly positive; in fact, all of the coefficients in $\mathcal{O}(r_1)$ are positive, see (9). Therefore, given the above analysis of the dynamics in K_2, it follows with $z_1 = z_2$ that we can restrict ourselves to $|z_1 - 2| \le \alpha$ here, with $\alpha > 0$ small. We will denote the point $(0, 2, 0) \in \ell_1$ by P_1 in the following.

Lemma 3.3. *The eigenvalues of* (17) *at* $P_1 \in \ell_1$ *are given by* -2, 0, *and* 2, *with corresponding eigendirections* $(0, 0, 1)^T$, $(0, 1, 0)^T$, *and* $(1, 0, 0)^T$, *respectively.*

Given (16) and Lemma 3.1, we obtain an explicit expression for the singular orbit Γ_1^- on the blown-up locus $\{r_1 = 0\}$ in chart K_1 via

$$\Gamma_1^- : z_1(\varepsilon_1) = 2\sqrt{1 - (1 + \tfrac{1}{\varepsilon_1})\mathrm{e}^{-\frac{1}{\varepsilon_1}}};$$

in particular, $z_1 \to 2$ as $\varepsilon_1 \to 0$, where $z_1(\varepsilon_1)$ is an infinitely flat function at $\varepsilon_1 = 0$ (i.e., at P_1).

The geometry in chart K_1 is summarized in Figure 2(b), while the global, blown-up situation is illustrated in Figure 3.

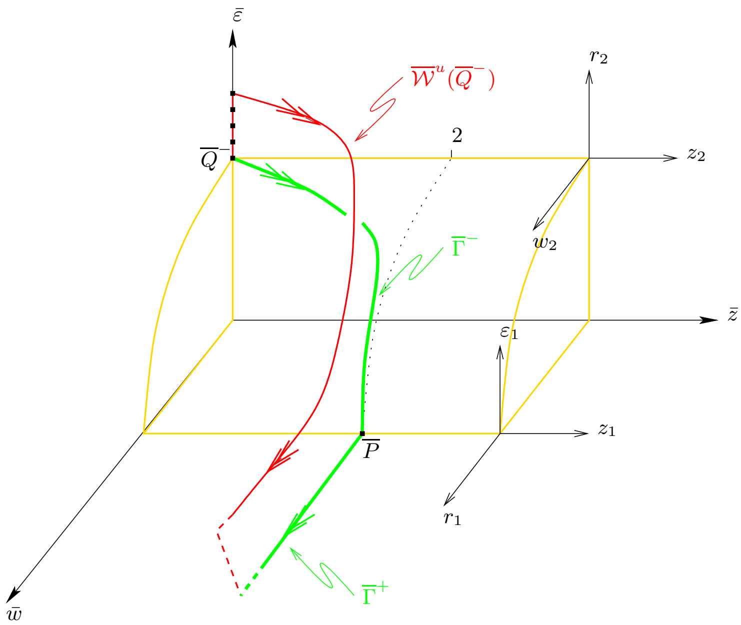


FIGURE 3. The situation in blown-up coordinates. (Here, the coordinate frames for charts K_1 and K_2 only serve to recall the relevant variables, and not to set the respective origins.)

3.3. Regularity of the transition in K_1. For the proof of Theorem 1.1, we will require a smoothness result on the transition past ℓ_1 under the flow of (17). For convenience, we introduce two sections $\Sigma_1^{\rm in}$ and $\Sigma_1^{\rm out}$, with $\varepsilon_1 = \delta$ in $\Sigma_1^{\rm in}$ and $r_1 = \rho$ in $\Sigma_1^{\rm out}$ for δ, ρ sufficiently small and positive. Note that both δ and ρ are constant, i.e., independent of ε. More precisely, we define

$$\Sigma_1^{\rm in} = \big\{(r_1^{\rm in}, z_1^{\rm in}, \delta) \,\big|\, r_1^{\rm in} \in [0,\rho],\, |z_1^{\rm in} - 2| \le \alpha\big\}, \tag{18a}$$

$$\Sigma_1^{\rm out} = \big\{(\rho, z_1^{\rm out}, \varepsilon_1^{\rm out}) \,\big|\, |z_1^{\rm out} - 2| \le \alpha,\, \varepsilon_1^{\rm out} \in [0,\delta]\big\}, \tag{18b}$$

with $\alpha > 0$ a small constant, as before, and write $\Pi_1 : \Sigma_1^{\rm in} \to \Sigma_1^{\rm out}$ for the corresponding transition map, see again Figure 2(b).

Proposition 3.4. *The map*

$$\Pi_1 : \begin{cases} \Sigma_1^{\rm in} \to \Sigma_1^{\rm out}, \\ (\varepsilon\delta^{-1}, z_1^{\rm in}, \delta) \mapsto (\rho, z_1^{\rm out}, \varepsilon\rho^{-1}) \end{cases}$$

is $\mathcal{C}^\infty$-smooth in $z_1^{\rm in}$, as well as in the parameters ε and $\tilde{c}$.

Proof. For convenience, we simplify the equations in (17) by dividing out a factor of $(z_1 - r_1\tilde{c})$ from the right-hand sides,

$$r_1' = r_1, \tag{19a}$$

$$z_1' = \frac{2}{\varepsilon_1^2(z_1 - r_1\tilde{c})} \mathrm{e}^{-\frac{1}{\varepsilon_1}(1+\mathcal{O}(r_1))}, \tag{19b}$$

$$\varepsilon_1' = -\varepsilon_1. \tag{19c}$$

Here, the prime now denotes differentiation with respect to a rescaled variable ξ_1. The equations for r_1 and ε_1 are readily solved, since it follows from (19a) and (19c) as well as from $r_1^{\rm in} = \varepsilon\delta^{-1}$

and $\varepsilon_1^{\text{in}} = \delta$ that

$$r_1 = \frac{\varepsilon}{\delta}e^{\xi_1} \quad \text{and} \quad \varepsilon_1 = \delta e^{-\xi_1}. \tag{20}$$

In particular, the transition "time" from Σ_1^{in} to Σ_1^{out} under Π_1 can be obtained explicitly as $\Xi_1 = -\ln\frac{\varepsilon}{\delta\rho}$, since $\varepsilon_1^{\text{out}} = \varepsilon\rho^{-1}$.

It only remains to investigate the regularity of $z_1^{\text{out}} = z_1^{\text{out}}(z_1^{\text{in}}, \varepsilon, \tilde{c})$. To that end, we introduce the new variable $\tilde{z}_1$ via $z_1 = 2 + \tilde{z}_1$ and then expand $(2 + \tilde{z}_1 - r_1\tilde{c})^{-1} = \frac{1}{2}(1 + \mathcal{O}(\tilde{z}_1, r_1\tilde{c}))$ in (19b) to obtain

$$\tilde{z}_1' = \frac{1}{\varepsilon_1^2}e^{-\frac{1}{\varepsilon_1}(1+\mathcal{O}(r_1))}\big(1 + \mathcal{O}(\tilde{z}_1, r_1\tilde{c})\big).$$

We now define $x_1 = \delta^{-1}e^{\xi_1}$ and $\widetilde{Z}_1(x_1) = \tilde{z}_1(\xi_1)$. Note that $x_1 \in [\delta^{-1}, \rho\varepsilon^{-1}]$ and, hence, that $\varepsilon x_1 \in [\varepsilon\delta^{-1}, \rho] \subset [0, \rho]$; in particular, it follows that εx_1 is bounded. We obtain

$$\frac{d\widetilde{Z}_1}{dx_1} = x_1 e^{-x_1(1+\mathcal{O}(\varepsilon x_1))}\big(1 + \mathcal{O}(\widetilde{Z}_1, \varepsilon x_1\tilde{c})\big),$$

or, equivalently,

$$\frac{d\widetilde{Z}_1}{d\tilde{\xi}_1} = 1 + \mathcal{O}(\widetilde{Z}_1, \varepsilon x_1\tilde{c}), \tag{21a}$$

$$\frac{dx_1}{d\tilde{\xi}_1} = \frac{1}{x_1}e^{x_1(1+\mathcal{O}(\varepsilon x_1))} \tag{21b}$$

for some $\tilde{\xi}_1$. Now, it is important to note that

$$x_1 e^{-x_1(1+\mathcal{O}(\varepsilon x_1))} \in \left[\tfrac{\rho}{\varepsilon}e^{-\frac{\rho}{\varepsilon}(1+\mathcal{O}(\rho))}, \tfrac{1}{\delta}e^{-\frac{1}{\delta}(1+\mathcal{O}(\frac{\varepsilon}{\delta}))}\right] \subset \left[0, \tfrac{1}{\delta}e^{-\frac{1}{\delta}}\right];$$

here, we have used the fact that $\mathcal{O}(\varepsilon x_1)$ in (21b) stands for an analytic function which is strictly positive, see (9). We can solve (21b) by separation of variables,

$$d\tilde{\xi}_1 = x_1 e^{-x_1(1+\mathcal{O}(\varepsilon x_1))}dx_1 = d\Psi(x_1, \varepsilon x_1),$$

which gives

$$\tilde{\xi}_1(x_1) = \Psi(x_1, \varepsilon x_1) - \Psi(\delta^{-1}, \varepsilon\delta^{-1})$$

if we impose $\tilde{\xi}_1(\delta^{-1}) = 0$. Here, Ψ is $\mathcal{C}^\infty$-smooth due to the analyticity of the vector field in (21) for $x_1 > 0$. Moreover, Ψ is bounded, since

$$0 < \frac{d\tilde{\xi}_1}{dx_1} < x_1 e^{-x_1}.$$

Therefore, we conclude that we can solve for $x_1 = x_1(\tilde{\xi}_1)$ in a unique manner, with x_1 $\mathcal{C}^\infty$-smooth. In turn, since εx_1 is bounded, there exists a unique solution $\widetilde{Z}_1 = \widetilde{Z}_1(\widetilde{Z}_1^{\text{in}}, \tilde{\xi}_1(x_1), \tilde{c})$ to (21a) which is $\mathcal{C}^\infty$-smooth in all its arguments as long as we restrict ourselves to $\tilde{\xi}_1 \in [0, \tilde{\xi}_1^{\text{out}}]$, where $\tilde{\xi}_1^{\text{out}} = \tilde{\xi}_1(\rho\varepsilon^{-1}) = \Psi(\rho\varepsilon^{-1}, \rho) - \Psi(\delta^{-1}, \varepsilon\delta^{-1})$. Reverting to the original variables z_1 and ξ, we find that $z_1^{\text{out}} = z_1^{\text{out}}(z_1^{\text{in}}, \varepsilon, \tilde{c})$ is $\mathcal{C}^\infty$-smooth in z_1^{in}, as well as in ε and $\tilde{c}$. This completes the proof. ■

Remark 7. We conjecture that Π_1 is "infinitely close" to the identity, since the right-hand side in (19b), as well as all its derivatives, go to zero as $\varepsilon \to 0$. A proof would, however, be outside the scope of this work.

***Remark* 8.** Lemma 3.3 shows that the equilibrium at P_1 is resonant, in the sense that the eigenvalues of the corresponding linearization are in resonance. This implies that resonant terms of the form $r_1^k z_1^\ell \varepsilon_1^k$ $(k, \ell \in \mathbb{N})$ will potentially occur in the normal form for (19b), which, in turn, might induce logarithmic (switchback) terms [13, 23, 26, 20] in the expansion of Π_1. However, Proposition 3.4 implies that no such terms will arise in our case, as Π_1 is regular in ε.

4. Proof of Theorem 1.1

The proof of our main result, Theorem 1.1, will be split up into the proofs of several subresults; indeed, Lemma 4.1, Proposition 4.2, and Lemma 4.3 below together immediately yield Theorem 1.1. First, we derive the leading-order behavior of $\tilde{c}$:

Lemma 4.1. *There holds* $\tilde{c} = 2 + \mathcal{O}(1)$.

Proof. Recall that the analysis in chart K_2 implies $z_2 \to 2$ as $w_2 \to \infty$ to lowest order on $\mathcal{W}^u(\widetilde{Q}_2^-)$, see the expression for Γ_2^- in (16). Since $w_2 \to \infty$ is equivalent to $\varepsilon_1 \to 0$, cf. Lemma 3.1, and since $z_2 = z_1$, it follows that $(r_1, z_1, \varepsilon_1) \to (0, 2, 0) = P_1 \in \ell_1$. Recalling the definition of $Z = -(\widetilde{V} - \tilde{c})$, as well as that $Z = z_1$, we have $\widetilde{V} - \tilde{c} \to -2$. Since $\mathcal{W}^s(\widetilde{Q}^+)$ is to leading order given by $\Gamma^+ : \{\widetilde{V} = 0\}$, we have $\tilde{c} \sim 2$, which is the desired result. ∎

The argument in Lemma 4.1 reflects the criticality of the wave speed $c_{\text{crit}}(m) \sim \frac{2}{m}$ corresponding to $\tilde{c} \sim 2$. On $\mathcal{W}^u(\widetilde{Q}^-)$, there holds $Z \to 2$ in the limit as $\varepsilon \to 0$, which implies $\widetilde{V} \to -2 + \tilde{c}$. Hence, for $\tilde{c} \lesssim 2$ in (5), $\mathcal{W}^u(\widetilde{Q}^-)$ is to leading order asymptotic to $\{\widetilde{V} = \widetilde{V}_0\}$ for some $\widetilde{V}_0 < 0$; therefore, solutions on $\mathcal{W}^u(\widetilde{Q}^-)$ leave the domain on which $U \geq 0$, and we do not study them further. Conversely, for $c \gtrsim 2$, $\mathcal{W}^u(\widetilde{Q}^-)$ asymptotes to a fiber with $\widetilde{V}_0 > 0$, and is exponentially attracted to $\mathcal{S}_\varepsilon$. On $\mathcal{S}_\varepsilon$, the slow flow is given by

$$\dot{U} = -\frac{1}{\tilde{c}}\frac{2}{\varepsilon^2}U^{\frac{1}{\varepsilon}}(1 - U) < 0,$$

see (5), i.e., it is exponentially slow in ε and is directed towards $\widetilde{Q}^+$. Therefore, there exists a connection between $\widetilde{Q}^-$ and $\widetilde{Q}^+$, and the decay rate of the corresponding traveling wave to zero will be algebraic, since the approach is along a center manifold. Both situations are illustrated in Figure 4.

Next, we show that c_{crit} depends on ε in a $\mathcal{C}^\infty$-manner:

Proposition 4.2. *For $\varepsilon \geq 0$ but sufficiently small, there exists a function $\tilde{c} = \tilde{c}(\varepsilon)$ which is $\mathcal{C}^\infty$-smooth in ε such that $c_{\text{crit}}(\varepsilon) = \varepsilon\tilde{c}(\varepsilon)$ for the critical wave speed c_{crit} in* (4).

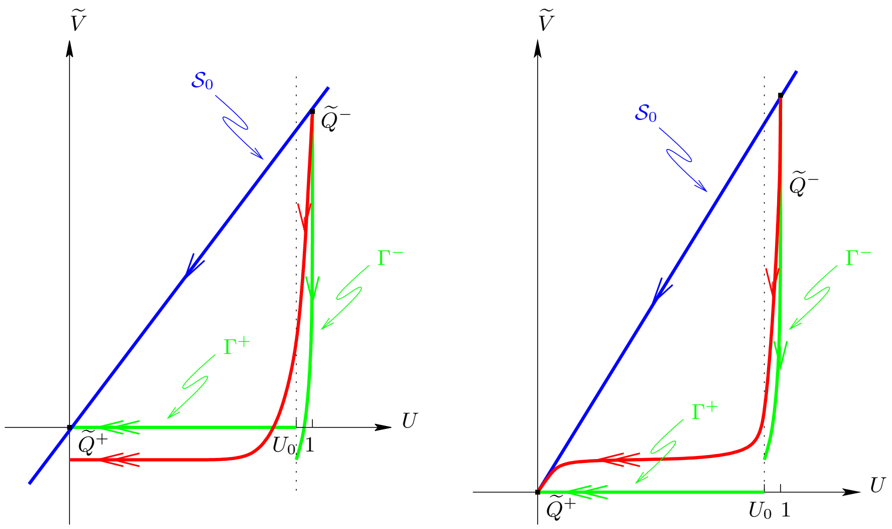
Proof. We define the section $\Sigma_2^{\text{out}} = \{(\delta^{-1}, z_2^{\text{out}}, r_2^{\text{out}}) \,|\, |z_2^{\text{out}} - 2| \leq \alpha,\ r_2^{\text{out}} \in [0, \rho]\}$, and note that $\kappa_{21}(\Sigma_2^{\text{out}}) = \Sigma_1^{\text{in}}$, see (18). Since the unstable manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of $\widetilde{Q}_2^-$ is analytic in $(w_2, z_2, \tilde{c}, r_2)$, its intersection with Σ_2^{out} can be written as the graph of an analytic function,

$$z_2^{\text{out}} = z_2^{\text{out}}(\delta^{-1}, \tilde{c}, r_2) = \varphi_2^{\text{out}}(\tilde{c}, r_2); \tag{22}$$

recall that $\varepsilon = r_2$ in K_2, cf. Lemma 3.1. Therefore, and since $z_2 = z_1$, it follows that in chart K_1, we can represent (22) by $z_1^{\text{in}} = \varphi_1^{\text{in}}(\tilde{c}, \varepsilon)$ in Σ_1^{in}, with $\varphi_1^{\text{in}} \equiv \varphi_2^{\text{out}}$. The graph of φ_1^{in}, in turn, is mapped, under the $\mathcal{C}^\infty$ mapping Π_1, to the graph of a $\mathcal{C}^\infty$-smooth function in Σ_1^{out},

$$z_1^{\text{out}} = z_1^{\text{out}}(\rho, \tilde{c}, \varepsilon\rho^{-1}) = \varphi_1^{\text{out}}(\tilde{c}, \varepsilon), \tag{23}$$

see Proposition 3.4. Hence, in sum, (23) represents the intersection of $\kappa_{21}(\mathcal{W}_2^u(\widetilde{Q}_2^-))$ with Σ_1^{out}. Moreover, since (14) does not depend on $\tilde{c}$ when $r_2 (= \varepsilon) = 0$, it follows that $\frac{\partial}{\partial \tilde{c}}\varphi_1^{\text{out}}(2, 0) = 0$.



(a) The geometry for $\tilde{c} \lesssim 2$. (b) The geometry for $\tilde{c} \gtrsim 2$.

FIGURE 4. The criticality of $\tilde{c} \sim 2$.

Next, in Σ_1^{out}, we can also represent the intersection of $\mathcal{W}^s(\widetilde{Q}^+)$ as the graph of a $\mathcal{C}^\infty$-smooth function,

$$z_1^{\text{out}} = \psi_1^{\text{out}}(\tilde{c}, \varepsilon).$$

Furthermore, it follows from (5) that $\frac{\partial}{\partial \tilde{c}} \psi_1^{\text{out}}(2, 0) = 1$.

Finally, combining the two results from above, we see that the function $\tilde{c}(\varepsilon)$ is determined by the implicit equation

$$\mathcal{D}(\tilde{c}, \varepsilon) := \varphi_1^{\text{out}}(\tilde{c}, \varepsilon) - \psi_1^{\text{out}}(\tilde{c}, \varepsilon) = 0.$$

In addition, the above analysis shows that

$$\mathcal{D}(2, 0) = 0 \quad \text{and} \quad \frac{\partial \mathcal{D}}{\partial \tilde{c}}(2, 0) = -1 \neq 0.$$

Therefore, the result follows locally near $(\tilde{c}, \varepsilon) = (2, 0)$ by the Implicit Function Theorem. ∎

Finally, we compute the second-order coefficient in the expansion for c_{crit}.

Lemma 4.3. *There holds* $\tilde{c}(\varepsilon) = 2 + \sigma\varepsilon + \mathcal{O}(\varepsilon^2)$, *where*

$$\sigma = \lim_{\omega_0 \to \infty} \int_0^{\omega_0} \left[\frac{\omega^2 \mathrm{e}^{-\omega}}{\sqrt{1 - (1 + \omega)\mathrm{e}^{-\omega}}} - \frac{\omega^3}{2} \mathrm{e}^{-\omega} \right] d\omega \approx -0.3119.$$

Proof. The unstable manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of $\widetilde{Q}_2^-$ is analytic in w_2, z_2, $\tilde{c}$, and r_2. Hence, it follows from regular perturbation theory that, on any bounded domain, we can make the ansatz

$$z_2(w_2, \tilde{c}, r_2) = \sum_{j=0}^{\infty} Z_{2_j}(w_2, \tilde{c}) r_2^j \quad \text{and} \quad \tilde{c}(r_2) = \sum_{j=0}^{\infty} C_j r_2^j, \tag{24}$$

with $Z_{2_j}(0, \tilde{c}) = 0$ for $j \geq 0$, in K_2. We will consider $w_2 \in [0, \delta^{-1}]$ in the following; recall the definition of Σ_2^{out}. Substituting (24) into (14), making use of the Chain Rule, expanding

$\exp\big[-w_2\big(\frac{r_2 w_2}{2}+\frac{r_2^2 w_2^2}{3}+\dots\big)\big]$, and collecting like powers of r_2, we obtain a recursive sequence of differential equations for Z_{2_j} which depend on C_j $(j\geq 0)$:

$$\mathcal{O}(1):\ \frac{dZ_{2_0}}{dw_2}Z_{2_0}=2w_2\mathrm{e}^{-w_2}, \tag{25}$$

$$\mathcal{O}(r_2):\ \frac{d}{dw_2}(Z_{2_0}Z_{2_1})=C_0 w_2\frac{dZ_{2_0}}{dw_2}-w_2^3\mathrm{e}^{-w_2}. \tag{26}$$

Equation (25) is equivalent to (15); hence, Z_{2_0} equals z_2 as defined in (16). Next, we can solve (26) using integration by parts,

$$\begin{aligned} Z_{2_1}(w_2,\tilde{c}) &= \frac{1}{\sqrt{1-(1+w_2)\mathrm{e}^{-w_2}}}\int_0^{w_2}\left[\frac{\omega^2\mathrm{e}^{-\omega}}{\sqrt{1-(1+\omega)\mathrm{e}^{-\omega}}}-\frac{\omega^3}{2}\mathrm{e}^{-\omega}\right]d\omega \\ &= 2w_2-\frac{1}{\sqrt{1-(1+w_2)\mathrm{e}^{-w_2}}}\int_0^{w_2}\left[2\sqrt{1-(1+\omega)\mathrm{e}^{-\omega}}+\omega^3\mathrm{e}^{-\omega}\right]d\omega, \end{aligned} \tag{27}$$

where the constant of integration is chosen such that $Z_{2_1}(0,\tilde{c})=0$ and we have used $C_0=2$. In particular, in Σ_2^{out}, the expansion for $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ is given by

$$z_2^{\mathrm{out}}=z_2(\delta^{-1})\sim Z_{2_0}(\delta^{-1})+\varepsilon Z_{2_1}(\delta^{-1},C_0). \tag{28}$$

We now need to investigate the asymptotics of $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ as $w_2\to\infty$. This is readily done in K_1, i.e., we will study the transition from $\Sigma_1^{\mathrm{in}}=\kappa_{21}(\Sigma_2^{\mathrm{out}})$ to Σ_1^{out}.

Let Π_1 be defined as in Proposition 3.4, and assume that a curve of initial conditions for Π_1 is given by $(\varepsilon\delta^{-1},z_1^{\mathrm{in}},\delta)\in\Sigma_1^{\mathrm{in}}$, with $z_1^{\mathrm{in}}=z_2^{\mathrm{out}}$ as in (28). Since Π_1 is $\mathcal{C}^\infty$-smooth in ε, see Proposition 3.4, we may expand z_1 as

$$z_1(\varepsilon_1,\tilde{c},\varepsilon)\sim Z_{1_0}(\varepsilon_1,\tilde{c})+\varepsilon Z_{1_1}(\varepsilon_1,\tilde{c}).$$

Substituting this expansion, as well as the expansion for $\tilde{c}$ from (24), into the equations in (17) and comparing powers of ε, we obtain the equations

$$\mathcal{O}(1):\ \frac{dZ_{1_0}}{d\varepsilon_1}Z_{1_0}=-\frac{2}{\varepsilon_1^3}\mathrm{e}^{-\frac{1}{\varepsilon_1}}, \tag{29}$$

$$\mathcal{O}(\varepsilon):\ \frac{d}{d\varepsilon_1}(Z_{1_0}Z_{1_1})=\frac{2}{\varepsilon_1}\frac{dZ_{1_0}}{d\varepsilon_1}+\frac{\mathrm{e}^{-\frac{1}{\varepsilon_1}}}{\varepsilon_1^5}, \tag{30}$$

which correspond precisely to (25) and (26) after transformation to K_1. One can check that the corresponding solutions Z_{1_0} and Z_{1_1} are given by $\kappa_{21}(Z_{2_0})$ and $\kappa_{21}(Z_{2_1})$, respectively. In particular, given (28) as well as $\varepsilon_1^{\mathrm{out}}=\varepsilon\rho^{-1}$, we find that $z_1^{\mathrm{out}}=\Pi_1(z_1^{\mathrm{in}})$ is obtained as

$$z_1^{\mathrm{out}}\sim 2\sqrt{1-(1+\tfrac{\rho}{\varepsilon})\mathrm{e}^{-\frac{\rho}{\varepsilon}}}+\frac{\varepsilon}{\sqrt{1-(1+\frac{\rho}{\varepsilon})\mathrm{e}^{-\frac{\rho}{\varepsilon}}}}\underbrace{\int_{\frac{\varepsilon}{\rho}}^{\infty}\left[\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta^4\sqrt{1-(1+\frac{1}{\eta})\mathrm{e}^{-\frac{1}{\eta}}}}-\frac{1}{2}\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta^5}\right]d\eta}_{=:\mathcal{I}(\frac{\varepsilon}{\rho})}. \tag{31}$$

To determine C_1, we have to match $\mathcal{W}^u(\widetilde{Q}^-)$ to $\mathcal{W}^s(\widetilde{Q}^+)$ on the overlap domain between the inner and outer regions. Without loss of generality, the matching will be done in Σ_1^{out}. Recalling that $\widetilde{V}=0$ to all orders in ε on $\mathcal{W}^s(\widetilde{Q}^+)$, we conclude that $Z=\tilde{c}$ and, hence, that $z_1^{\mathrm{out}}\sim 2+\varepsilon C_1$ in Σ_1^{out} for the contribution from the outer problem. To leading order, we retrieve $\tilde{c}=2$ (up to exponentially small terms in ε). To match the $\mathcal{O}(\varepsilon)$-terms in (31) to εC_1, we note that

$$\mathcal{I}(\tfrac{\varepsilon}{\rho})=\mathcal{I}(0)+\mathcal{O}(\mathrm{e}^{-\frac{\kappa}{\varepsilon}})\qquad\text{for some }\kappa>0,$$

since the corresponding integrand is exponentially small on $[0, \frac{\varepsilon}{\rho}]$ and since $\rho > 0$. Evaluating $\mathcal{I}(0)$ numerically, we find $C_1 \sim \mathcal{I}(0) \approx -0.3119$. This completes the proof. ∎

The numerical value of σ coincides with the result obtained in [19] by means of asymptotic matching. In fact, the above analysis is closely related to the approach one would take to determine an expansion for $c_{\rm crit}$ via the method of matched asymptotics: The "inner expansion" coming from chart K_2 is "matched" to the "outer expansion" derived in chart K_1 on the overlap domain between the two charts. Finally, we note that this domain corresponds to the classical "intermediate region" where one would typically match by defining an "intermediate variable".

***Remark* 9.** Given the regularity of Π_1, it is not surprising that the analysis in K_1 is analogous to that in K_2, and that the resulting expansions are equal up to the coordinate change κ_{21}. Although one could probably restrict oneself to K_2 for $\varepsilon > 0$, it seems more natural to analyze the asymptotics for $w_2 \to \infty$ in K_1.

***Remark* 10.** Numerical evidence [27] suggests that the one-term truncation of the asymptotic expansion for $c_{\rm crit}$ in (2), $c_{\rm crit}(m) \sim \frac{2}{m}$, is optimal for $m \in [2, m_1)$, where $m_1 \approx 4$. Similarly, it appears that the two-term truncation is optimal on some finite m-interval (m_1, m_2), with m_1 defined as before. This would indicate that the formal expansion for $c_{\rm crit}(m)$ might well have Gevrey properties, cf. e.g. [24]. A rigorous analysis of this question, including the calculation of the corresponding optimal truncation points, seems to be an interesting problem for further study. The geometric desingularization presented in this article might well be useful for such an analysis. See e.g. [16] for an example of how the blow-up technique can be employed to study Gevrey asymptotics.

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E-mail address: freddy.dumortier@uhasselt.be, popovic@math.bu.edu, tasso@math.bu.edu

Universiteit Hasselt, Campus Diepenbeek, Agoralaan Gebouw D, B-3590 Diepenbeek, Belgium

Boston University, Center for BioDynamics and Department of Mathematics and Statistics, 111 Cummington Street, Boston, MA 02215, U.S.A.