While curved waveguides are fundamental elements in photonics, those induced all optically in nonlinear uniform dielectrics tend to be straight. In uniaxial soft matter with a reorientational response, such as nematic liquid crystals (NLCs), light beams in the extraordinary polarization undergo self-focusing via an increase in refractive index and eventually form spatial solitons, i.e., self-induced waveguides. Hereby we investigate the bending of such waveguides by analyzing the trajectory of solitons in NLCs—nematicons—in the presence of a linearly varying transverse orientation of the optic axis. To this extent, we use and compare two approaches: i) a slowly varying (adiabatic) approximation based on momentum conservation of the nematicon in a Hamiltonian sense; and ii) the Frank–Oseen elastic theory coupled with a fully vectorial and nonlinear beam propagation method. The models provide comparable results in such a non-homogeneously oriented uniaxial medium and predict bent soliton paths with either monotonic or non-monotonic curvatures, enabling the design of curved channel waveguides induced by light beams.

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1. INTRODUCTION

All-optical or light-induced waveguides through the self-focusing response of optical materials are commonly referred to as spatial optical solitons and have been studied in several media [1], including liquid crystals [2]. Nematic liquid crystals (NLCs) are anisotropic, typically uniaxial, soft matter consisting of thread-like molecules that exhibit orientational but no spatial order [3]. The anisotropic molecules are in a fluid state, linked by elastic forces, and exhibit two refractive index eigenvalues, ordinary and extraordinary, for light polarized perpendicular or parallel to the optic axis, termed the molecular director and usually denoted by the unit vector \( \mathbf{n} \). The refractive index of extraordinary polarized light has a nonlinear optical dependence through the reorientational response: the electric field of the light beam induces dipoles in the NLC molecules, so that they tend to rotate toward the field vector to minimize the system energy until the elastic response balances this electromechanical torque [3]. The resulting change in molecular orientation then changes the extraordinary refractive index toward the largest eigenvalue, so that the beam undergoes self-focusing. When the latter compensates linear diffraction, a \((2 + 1)D\) solitary wave can form, often termed a nematicon [2,4]. Nematicons are non-diffracting solitary beams in NLCs, confined by their own graded-index waveguides. They have been extensively investigated over a number of years in many different scenarios, including planar cells [5–10], capillaries [11,12], one-dimensional waveguides [13], coupled waveguides [4], and bulk [14]. When the wave vector of the light beam and the molecular director are neither perpendicular nor parallel, the Poynting vector of the nematicon walks off the wave vector at a finite angle owing to the tensorial nature of the dielectric susceptibility [15]. Such an angular deviation of the energy flux depends on the refractive index eigenvalues, \( n_\parallel \) and \( n_\perp \) for electric fields parallel and perpendicular to the director, respectively, and the angle \( \psi \) between the director and wave vector. Nematicon walk-off can be exploited in optical devices, e.g., signal demultiplexers or routers [2,16–19].

Similar to other nonlinear optical media [1], in uniform NLCs nematicons propagate in rectilinear trajectories along their Poynting vector. The graded index waveguides associated
with these spatial solitons are therefore straight. Curved nematics have been investigated by introducing extra elements, such as graded interfaces [17,18,20], localized refractive index perturbations [2,16,21], and interactions with boundaries [8,22] as well as other solitons [2,5,9,23,24]. A detailed approach to designing and modeling beam-induced waveguides undergoing bending versus propagation in a medium without external perturbations, however, has never been developed before.

In this paper, we introduce and study—for the first time, to the best of our knowledge—curved solitons as they propagate in NLCs subject to a linearly varying orientation of the optic axis across one of the transverse coordinates, in the principal plane defined by optic axis and wave vector. We consider nematicons excited in a planar cell of fixed (uniform) thickness, with upper and lower interfaces treated to ensure planar anchoring of the NLC molecules. This geometry is radically different from those entailing spin-orbit interactions of light with matter [25,26], as the optic axis and the wave vector are not mutually orthogonal because the light beam is an extraordinary wave. As the molecular alignment varies across the sample, both the extraordinary refractive index and the birefringent walk-off vary as well. These two variations determine the resulting trajectory of extraordinarily polarized beams in the cell, including the path of self-confined nematicons. To investigate nonrectilinear nematicon paths in a transversely modulated uniaxial, we use two different approaches in the weakly nonlinear regime (i.e., power independent walk-off): (i) numerical solutions of the full governing Maxwell’s equations employing a fully vectorial beam propagation method for the beam and the Frank–Oseen elastic theory for the NLC response [2]; and (ii) an adiabatic (slowly varying) approximation to yield simplified forms of these equations, invoking momentum conservation (MC) [2,27,28]. The adiabatic approximation is based on the high nonlocality of the NLCs, which implies that the nonlinear response extends far beyond the transverse size of the optical wave packet [2,29] and decouples the amplitude/width evolution of the beam from its trajectory [28]. In this study, the background director angle is slowly varying, typically 0.002 rad/µm in a cell of width 200 µm, so that the nematicon trajectory can be determined by MC, in the sense of invariances of the Lagrangian for the NLC equations. The latter approach yields simple equations that have an exact solution and provides excellent agreement with the full numerical solutions, proving more than adequate to model beam evolution in non-uniform birefringent media.

2. GEOMETRY AND GOVERNING EQUATIONS

We consider the propagation of a linearly polarized, coherent light beam in a planar cell with undoped, positive uniaxial NLCs. The extraordinary polarized beam is taken to initially propagate forward in the z direction, with electric field E oscillating in the y transverse direction and x completing the coordinate triad. To eliminate the Freédericksz threshold [3] and maximize the nonlinear optical response [2], the cell interfaces perpendicular to x are rubbed so that the molecular director makes an angle $\theta_0$ with z in the (y,z) plane everywhere in the bulk owing to elastic interactions, as sketched in Fig. 1. An additional y-dependent rotation $\theta(y)$ is given to the nematic director to modulate the uniaxial medium, as illustrated in Fig. 2(b). Due to the nonlinearity, the light beam can rotate the optic axis by an extra angle $\theta$, so that the director forms a total angle $\Psi(y) = \theta_0 + \theta(y) + \theta(y)$ to the z axis in the (y,z) plane [2].

A. Beam Propagation Method and Elastic Theory

One of the approaches used to study the nonlinear evolution of a light beam in NLCs is the fully vectorial beam propagation method (FV BPM) [30] in conjunction with elastic theory based on the Frank–Oseen model for the NLC response [3,31,32]. The FV BPM can be derived directly from Maxwell’s equations [33,34], considering harmonically oscillating electric and magnetic fields in an anisotropic dielectric:

$$\frac{\partial H_x}{\partial y} = i\frac{\omega}{\mu_0 \epsilon} (\frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial z})$$

$$\frac{\partial H_y}{\partial z} = i\frac{\omega}{\mu_0 \epsilon} (\frac{\partial E_x}{\partial y} - \frac{\partial E_z}{\partial x})$$

$$\frac{\partial H_z}{\partial x} = i\frac{\omega}{\mu_0 \epsilon} (\frac{\partial E_y}{\partial z} - \frac{\partial E_x}{\partial y})$$

$$H_x = -\frac{1}{\mu_0 \omega} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial z} \right)$$

$$H_y = -\frac{1}{\mu_0 \omega} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_z}{\partial x} \right)$$

$$H_z = -\frac{1}{\mu_0 \omega} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_x}{\partial y} \right).$$

(1)
Here the complex amplitudes \( \tilde{E} \) and \( \tilde{H} \) are the electric and magnetic fields, respectively, \( \varepsilon \) is the electric permittivity tensor, \( \omega \) is the angular frequency, and \( \mu_0 \) is the vacuum permeability. These coupled partial differential equations can be solved numerically, as the \( x \) and \( y \) derivatives can be approximated using standard central differences and the solution can be propagated forward along \( z \) using a standard fourth-order Runge–Kutta scheme. In this work the step size is chosen to be \( \Delta z = 10 \) nm. At the cell boundaries, reflective Dirichlet boundary conditions are imposed, so that \( \mathbf{E} = 0 \) and \( \mathbf{H} = 0 \) at the NLC/glass interfaces. The electric tensor in Eq. (1) is

\[
\varepsilon = \begin{bmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_{1\parallel} + \Delta \varepsilon \sin^2 \psi & \Delta \varepsilon \sin \psi \cos \psi \\
0 & \Delta \varepsilon \sin \psi \cos \psi & \varepsilon_{1\perp} + \Delta \varepsilon \cos^2 \psi
\end{bmatrix},
\]

with \( \Delta \varepsilon = n_0^2 - n_0^2 \) the optical anisotropy. These electromagnetic equations are coupled to the NLC response, given by the Frank–Oseen expression for the energy density in the non-chiral case [3,31,32]:

\[
f = \frac{1}{2} K_{11}(\nabla \tilde{n})^2 + \frac{1}{2} K_{22}(\tilde{n} \cdot (\nabla \times \tilde{n}))^2
\]

\[
+ \frac{1}{2} K_{33}(\tilde{n} \times (\nabla \times \tilde{n}))^2 - \frac{1}{2} \varepsilon_0 \Delta \varepsilon (\tilde{n} \cdot \tilde{E})^2.
\]

(3)

Here, \( K_{11}, K_{22}, K_{33} \) are the Frank elastic constants for splay, twist, and bend deformations of the molecular director \( \tilde{n} \), respectively [2]. The equation for the NLC elastic response is obtained by taking variations of this free energy. However, doing so results in a large system of equations [35]. To overcome this complexity, we note that in the examined configuration the molecular director and the electric field of the beam lie in the same (principal) plane \( (y, z) \); hence, as nonlinear reorientation occurs in this same plane and the azimuthal components can be neglected, the director \( \tilde{n} \) can be expressed in polar coordinates \( \tilde{n} = [0, \sin \psi, \cos \psi] \). Because the changes in molecular orientation along \( z \) are slow as compared with the wavelength of light, the derivatives with respect to \( z \) can also be neglected. In this approximation, the variations of the free energy [Eq. (3)] yield the Euler–Lagrange equation, where the subscripts on \( \psi \) denote derivatives,

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial \psi_x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial \psi_y} \right) - \frac{\partial f}{\partial \psi} = 0,
\]

leading to the director rotation in the form

\[
K_{22} \frac{\partial^2 \psi}{\partial x^2} + (K_{11} \cos^2 \psi + K_{33} \sin^2 \psi) \frac{\partial^2 \psi}{\partial y^2}
\]

\[
- \frac{1}{2} \sin 2\psi (K_{11} - K_{33}) \left( \frac{\partial \psi}{\partial y} \right)^2
\]

\[
+ \frac{\varepsilon_0 \Delta \varepsilon}{2} [2E_x E_y \cos 2\psi + \sin 2\psi (E_x^2 - E_y^2)] = 0.
\]

(5)

Numerical solutions of this elliptic equation [Eq. (5)] are found using successive overrelaxations (SOR) with relaxation parameter \( \Omega = 1.8 \) [36]. When combined with the numerical solution of the electromagnetic model [Eq. (1)], solutions for beam propagation in NLCs with varying orientation can be obtained. The director reorientation is recalculated after each 100 nm of propagation; after the first step in \( z \), the solution for \( \psi \) is the initial guess for the SOR iterations, ensuring rapid convergence. The accuracy of the method described above can be estimated from the ratio of total input and output powers, which should be unity because absorption is neglected and the boundary conditions are purely reflective. Defining the relative error as \( \eta = (P_{\text{out}} - P_{\text{in}})/P_{\text{in}} \), we aim to achieve \( |\eta| < 0.5\% \) for all the cases considered here. In this work, the typical cell dimensions (thickness \( \times \) width \( \times \) length) are 30 \( \mu \)m \( \times \) 200 \( \mu \)m \( \times \) 500 \( \mu \)m, and two simple anchoring conditions are analyzed, uniform and linearly varying; see Fig. 2. For the sake of a realistic analysis, we choose the material parameters corresponding to the common NLC 6CHT, with Frank elastic constants \( K_{11} = 8.57 \) pN, \( K_{22} = 3.7 \) pN, and \( K_{33} = 9.51 \) pN and indices \( n_0 = 1.6335 \) and \( n_1 = 1.4967 \) at temperature \( T = 20^\circ\text{C} \) and wavelength \( \lambda = 2\pi/k_0 = 1.064 \mu \text{m} \) [15,37]. The input beam is Gaussian and \( y \) polarized, with a full width at half-maximum of 7 \( \mu \)m and power 1 mW. Typical computer runs to obtain the results presented hereby in a single Intel Core i7 at 3.60 GHz took between 100 and 120 minutes for propagation lengths of 500 \( \mu \)m.

B. Momentum Conservation

The full system [Eqs. (1) and (5)] governing the propagation of a light beam in a non-uniform NLC cell is extensive and amenable to numerical solutions only. However, these equations can be simplified to yield a reduced system for which an adiabatic approximation applies based on the slow variation of the director orientation. This adiabatic approximation shows that the beam trajectory is determined by an overall MC equation. This is not a physical momentum but momentum in the sense of the invariances of the Lagrangian in the reduced system. Such a reduction of the full system and the resulting MC equation will now be derived.

The first approximation is that the imposed linear modulation \( \theta_0 \) in the director orientation is much smaller than the constant background \( \theta_0, |\theta_0| \ll \theta_0 \). For the examples considered here, typical values are \( \theta_0 = 45^\circ \) and maximum \( |\theta_0| \) ranging from 5° to 20°. While the largest \( |\theta_0| \) is not strictly much smaller than \( \theta_0 \), nevertheless, the asymptotic results are found to be in good agreement with the numerical ones even at this upper limit. As discussed in the previous section, we denote the additional nonlinear reorientation by \( \theta \), so that the total pointwise orientation is \( \psi = \theta_0 + \theta_b + \theta \). In the paraxial, slowly varying envelope approximation, the equations [Eqs. (1) and (5)] governing the propagation of the light beam through the NLC can be reduced to [2]

\[
\frac{\partial E_y}{\partial z} + 2ik_0n_0\Delta(\psi) \frac{\partial E_y}{\partial y} + \nabla^2 E_y
\]

\[
+ k_0^2 (n_0^2 \cos^2 \psi + n_0^2 \sin^2 \psi)
\]

\[
- n_0^2 \sin^2 \theta_0 - n_0^2 \sin^2 \theta_0 E_y = 0,
\]

(6)

\[
K \nabla^2 \psi + \frac{1}{4} \varepsilon_0 \Delta \varepsilon |E_y|^2 \sin 2\psi = 0.
\]

(7)

As for the full equations of Section 2.A, \( E_y \) is the complex valued envelope of the electric field of the beam, because in the paraxial approximation the components \( E_x \) and \( E_z \) are
neglected. The Laplacian $\nabla^2$ is in the transverse $(x,y)$ plane. In the single constant approximation, the parameter $K$ is a scalar on the assumption that bend, splay, and twist in the full director equation [Eq. (5)] have comparable strengths. The wavenumber $k_0$ of the input light beam is intended as in a vacuum, and $n_e$ is the background extraordinary refractive index of the NLC. [2],

$$n_e^2(\psi) = \frac{n_0^2 n_\parallel^2}{n_0^2 \cos^2 \psi + n_\parallel^2 \sin^2 \psi},$$  \hspace{1cm} (8)

in the linear limit $\theta = 0$. The coefficient $\Delta$ is related to the birefringent walk-off angle $\delta$ of the extraordinary wave beam, with $\tan \delta = \Delta$ in the $(y,z)$ plane, and is given by

$$\Delta(\psi) = \frac{\Delta e \sin 2\psi}{\Delta e + 2n_\parallel^2 + \Delta e \cos 2\psi}.$$ \hspace{1cm} (9)

Throughout this work, despite the nonlinear dependence of $\Delta$ on the beam power through the reorientation $\theta$ [38], we assume $\Delta = \Delta(\theta_0 + \theta)$ in the low power limit. In the single elastic constant approximation, the director equations (5) and (7) differ by a factor of $1/2$ in the dipole term involving $e_0 \Delta e$, owing to definitions of the electric field based on either the maximum amplitude or the root mean square value. In this context, this difference is equivalent to a rescaling of $K$, with the latter constant $K$ canceling out in the adiabatic MC approximation.

The reduced Eqs. (6) and (7) can be set in non-dimensional form via the variable and coordinate transformations:

$$x = Wx, \quad y = Wy, \quad z = Bz, \quad E_2 = Au,$$  \hspace{1cm} (10)

where $W = \frac{\lambda}{\pi \sqrt{\Delta e} \sin 2\theta_0}$, $B = \frac{2n_\parallel \lambda}{\pi \Delta e \sin 2\theta_0}$, $A^2 = \frac{2\dot{p}_0}{\pi W^2}$, $\Gamma = \frac{1}{2}e_0cn_e$  \hspace{1cm} (11)

for a Gaussian input beam power of $\dot{p}_0$ and wavelength $\lambda$ [27]. With these non-dimensional variables, Eqs. (6) and (7) become

$$i \frac{\partial u}{\partial Z} + i\gamma(\theta_0 + \theta_b) \frac{\partial u}{\partial Y} + \frac{1}{2} \nabla^2 u + 2(\theta_0 + \theta_b + \theta)u = 0,$$

$$\nu \nabla^2 \theta = -2|u|^2.$$ \hspace{1cm} (12)

In deriving these equations, we assumed that the NLC director rotation from $\theta_0$ is small, i.e., $|\theta_b| \ll \theta_0$, as discussed above. We further assumed that the nonlinear response is small, with $|$|$ | |\theta| | | \ll \theta_0$. The trigonometric functions in the dimensional Eqs. (6) and (7) have been expanded in Taylor series. The scaled parameters in these non-dimensional equations are

$$\gamma = \frac{2n_\parallel}{\sqrt{\Delta e} \sin 2\theta_0} \quad \text{and} \quad \nu = \frac{8K}{e_0 \Delta e A^2 W^2 \sin 2\theta_0}.$$ \hspace{1cm} (14)

Equations (12) and (13) have the Lagrangian formulation

$L = i(u^* u_Z - uu_Z^*) + i\gamma(\theta_0 + \theta_b)(uu_{Y} - uu_{Y}^*)$

$$- |\nabla u|^2 + 4(\theta_0 + \theta_b + \theta)|u|^2 - \nu|\nabla \theta|^2,$$ \hspace{1cm} (15)

where the * superscript denotes the complex conjugate, and the subscripts $Z$ and $Y$ denote derivatives. Equations (12) and (13) have no general exact solitary wave, or nematicon, solution; the only known exact solutions are for specific, related values of the parameters [39]. For this reason, variational and conservation law methods have proved to be useful to study nematic evolution [39,40], as they give solutions in good agreement with numerical and experimental results [27,39,40]. In particular, they provide accurate results for the refraction of nematics due to variations in the dielectric constant [21,41–43]. Conservation laws based on the Lagrangian (15) are used below to determine the nematicon trajectory in a cell with an imposed linear modulation of the orientation angle $\theta_0 + \theta_b$.

The easiest way to obtain the approximate MC equations for Eqs. (12) and (13) is from the Lagrangian [Eq. (15)] [28]. We assume the general functional forms

$$u = ag(\rho)e^{\sigma + i\nu Y} \quad \text{and} \quad \theta = \sigma^2(\mu),$$ \hspace{1cm} (16)

where

$$\rho = \sqrt{X^2 + (Y - \xi)^2}, \quad \mu = \sqrt{X^2 + (Y - \xi)^2}. \hspace{1cm} (17)$$

for the nematicon and the director responses, respectively [28]. The actual beam profile $g$ is not specified, as the trajectory is found to be independent of this functional form. In response to the change in the NLC refractive index, the extraordinary wave beam undergoes refraction as well as amplitude and width oscillations. If the length scale of the refractive index change is larger than the beam width, the beam refraction decouples from the amplitude/width oscillations [21,28,42]. Consistent with this decoupling, the electric field amplitude $a$ and the width $w$ of the beam, the amplitude $a$ and width $\beta$ of the director response can be taken as constant if just the beam trajectory is required. Only the beam center position $\xi$ and (transverse) velocity $V$ are then taken to depend on $Z$, as well as the phase $\sigma$. This approximation is equivalent to MC for the Lagrangian Eq. (15) [44].

Substituting the profile forms [Eq. (16)] into the Lagrangian Eq. (15) and averaging by integrating in $X$ and $Y$ from $-\infty$ to $\infty$ [45] gives the averaged Lagrangian [40]:

$L_m = -2S_2(\sigma' - V \xi' a^2 w^2 - S_{22} a \xi')$

$$- S_2(V^2 + 2VF - 4F)a^2 w^2 + \frac{2A^2 B^2 q^2 a^2 w^2}{A^2 \beta^2 + B^2 w^2}$

$$- 4\nu S_{42} a^2 - 2q S_{42} a^2 \beta^2,$$ \hspace{1cm} (18)

where primes denote differentiation with respect to $Z$. Here $F$ and $F_1$, which determine the beam trajectory, are expressed by

$$F(\xi) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\theta_0 + \theta_b) g^2 dX dY}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2 dX dY},$$  \hspace{1cm} (19)

$$F_1(\xi) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla(\theta_0 + \theta_b) g^2 dX dY}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2 dX dY}. \hspace{1cm} (20)$$

The integrals $S_2, S_4$, and $S_{22}$ and $S_{42}$ appearing in this averaged Lagrangian are
Taking variations of this averaged Lagrangian with respect to \( \xi \) and \( V \) yields the modulation equations

\[
\frac{dV}{dZ} = 2 \frac{dF}{d\xi} - V \frac{dF_1}{d\xi}, \tag{22}
\]

\[
\frac{d\xi}{dZ} = V + F_1, \tag{23}
\]

which determine the beam trajectory. Equation (22) is the momentum equation.

A simple reduction of the trajectory equations (22) and (23) can be carried out when the beam width is much less than the length scale for the variation of the refractive index, i.e., the length scale of the variation of \( \theta_b \) \cite{28}. For the examples in this work, \( \theta'_b \sim 0.002 \, \text{rad/\mu m} \). Hence, a length scale for the variation of \( \theta_b \) is 500 \( \mu \text{m} \), while the typical beam width is 7 \( \mu \text{m} \).

The linear variation of the angle \( \theta_b \) from the background angle \( \theta_0 \) starts at \( \theta_b = 0 \) at \( Y = 0 \). Because the beam is launched at the midsection of the cell \( Y = L/2 \), where the total angle in the absence of light is \( \theta_0 + \theta_b(L/2) = \theta_m \), it is more accurate to expand the walk-off \( \Delta \) in a Taylor series about \( \theta_m \) rather than \( \theta_0 \). If we set \( \theta_b = \theta_b - \theta_b(L/2) \), the integrals \( F \) [Eq. (19)] and \( F_1 \) [Eq. (20)] can be approximated by

\[
F(\xi) \sim \theta_0 + \theta_b(\xi),
\]

\[
F_1(\xi) \sim \gamma \Delta(\theta_m + \theta_b(\xi)) = \gamma \Delta(\theta_m) + \gamma \Delta'(\theta_m) \theta_b(\xi) + ... \tag{24}
\]

We note that \( F_1 \) has been further approximated by expanding \( \Delta \) in a Taylor series about \( \theta_m \) on taking \( |\theta_b| \ll \theta_0 \), discussed above. With this simplification, the trajectory equations (22) and (23) become

\[
\frac{dV}{dZ} = (2 - V \gamma \Delta'(\theta_m) \theta'_{b}(\xi)), \tag{25}
\]

\[
\frac{d\xi}{dZ} = V + \gamma \Delta(\theta_m) + \gamma \Delta'(\theta_m) \theta_b(\xi). \tag{26}
\]

The simplicity of the beam trajectory equations (25) and (26) enables exact solutions for simple angle modulations \( \theta_b \). The simplest is the linear case:

\[
\theta_b(Y) = \frac{\theta}{L} Y, \tag{27}
\]

sketched in Fig. 2(b). For this linear case, \( \theta_b \) goes from 0 at \( Y = 0 \) to \( \theta_b \) at \( Y = L \). This variation of \( \theta_b \) enables the momentum equations (25) and (26) to be solved exactly and gives the position of the beam center \( \xi \) as

\[
\xi = \left[ \xi_0 + \frac{1 + \gamma^2 \Delta'(\theta_m) \Delta(\theta_m)}{\gamma^2 \Delta^2(\theta_m) \theta'_b} \right] \xi^2 \Delta(\theta_m) \theta'_b Z \]

\[
+ \frac{2 + \gamma^2 \Delta'(\theta_m) \Delta(\theta_m)}{\gamma^2 \Delta^2(\theta_m) \theta'_b} \frac{1}{\gamma^2 \Delta^2(\theta_m) \theta'_b} e^{\gamma \Delta(\theta_m) \theta'_b Z}, \tag{28}
\]

as \( \theta_b = \theta'_b \), is a constant. We assumed that the beam is launched at \( \xi = \xi_0 \) with \( V = 0 \) at \( Z = 0 \).

Because \( \theta_b \) is slowly varying, the trajectory solution given by Eq. (28) can be expanded in a Taylor series to yield

\[
\xi \sim \left[ \xi_0 + \gamma \Delta(\theta_m) Z \right] \]

\[
+ \left[ \xi_0 \left( \gamma \Delta'(\theta_m) \theta'_b Z + \frac{1}{2} \gamma^2 \Delta^2(\theta_m) \theta'_b^2 Z^2 \right) \right] + ... \tag{29}
\]

The first term in square brackets is the trajectory in a uniform NLC, and the terms in the second set of square brackets are the correction due to a changing orientation. For the examples hereby, \( \theta_b \sim 0.002 \, \text{rad/\mu m} \) and \( \Delta' \sim 0.05/\mu m \). So, to first order in small quantities

\[
\xi \sim \xi_0 + \gamma \Delta(\theta_m) Z + \theta'_b Z^2 \tag{30}
\]

as \( \Delta'(\theta_m) \) is small. Hence, the trajectory is described by the term for a uniform medium and a quadratic correction; the walk-off change due to the varying background director orientation dominates the change in the nematicon trajectory.

To convert the non-dimensional solution [Eq. (28)] back to dimensional variables, the scalings [Eq. (11)] are used. In particular, for the \( x \) scaling factor \( B \), the angle for the extraordinary form NLC, and the terms in the second set of square brackets need to be calculated. The obvious choice is to use the total director angle \( \theta_0 \) and \( \theta_b \) in the absence of light. The imposed component \( \theta_b \) is not constant but a slowly varying (linear) function of \( Y \), as discussed above, so its local value can be used to transform back to dimensional variables, consistent with a multiple scales analysis \cite{46}. This local variation in the scaling factor for \( x \) gives a metric change in this coordinate, with a small, slowly varying alteration of the trajectory. Nevertheless, the overall effect of this small local change is significant over propagation distances of 500 \( \mu \text{m} \) and larger.

### 3. RESULTS AND DISCUSSION

Figure 3 shows a comparison of nematicon trajectories in the modulated NLC as given by the adiabatic momentum approximation [Eq. (28)] and by the FVBPM solution of the full system [Eqs. (1) and (5)]. The considered cell has a range of linear variations in the background director angle \( \theta_b \) of the form Eq. (27). Each individual case, \( \theta_0 + \theta_b \), is indicated in the figure. A Gaussian beam is launched at the center of the cell, with its trajectory becoming curved due to the non-uniform director alignment. In a uniform medium the (straight) nematicon trajectory is determined solely by the walk-off, which leads to a rectilinear path in the \((y,z)\) plane. For the modulated uniaxial medium, not only the walk-off changes due to the varying anchoring but the phasefront of the wave packet is also distorted as the dielectric properties are modified and the NLC behaves like a lens with an index distribution \( n \), given by Eq. (8). Clearly, the MC approximation gives trajectories in close agreement with the numerical results. This validates the approximations made to arrive at the MC equations (25) and (26), in particular
the assumption that the beam trajectory is not influenced by its amplitude–width oscillations. Furthermore, it shows how powerful such adiabatic approximations can be. Nonetheless, the momentum result is a kinematic approximation and so does not give all the information for the evolving beam, whereas the full system (1) and (5) can also provide the amplitude–width evolution. A final point regarding Fig. 3 is that if the background angle for the extraordinary refractive index [Eq. (8)] in the z scaling [Eq. (11)] was chosen as $\theta_0$ rather than $\theta_0 + \theta_1$, there would have been a noticeable difference between the MC and numerical results. The local variation of the propagation metric $z$ due to the modulated director angle in the absence of light, in fact, has a significant effect on beam propagation.

These results are further analyzed in Fig. 4(a). The data is plotted to a logarithmic scale with an exponential regression fitted through the numerical trajectories. As $z$ increases, the trajectories are well approximated by an exponential evolution, in agreement with the MC solution [Eq. (28)], as for large $z$ the decaying exponential is negligible and the growing exponential dominates. Furthermore, when the rectilinear nematicon path in a uniform NLC is subtracted from the trajectory in the modulated case, the resulting beam position has a quadratic evolution in $z$, as shown in Fig. 4(b). These exponential and quadratic fittings of the trajectories are consistent with $\theta_0$ and $\Delta \psi$ being small, as demonstrated by reducing the full trajectory Eq. (28) to the quadratic approximation Eq. (30) via Eq. (29).

For a positive change of the anchoring conditions, i.e., $\theta_2 > \theta_0$, walk-off and phase distortion both increase the beam deviation. In the opposite case for which $\theta_1 < \theta_0$ these two phenomena counteract. The influence of walk-off and phase change on the nematicon path was analyzed for the case of the director orientation changing by $30^\circ$/200 $\mu$m, as shown in Fig. 5. When $\theta_2 > \theta_0$ the beam bends strongly due to both the walk-off and phase distortions acting in the same direction, as illustrated in Fig. 5(a). The phase change is strongest at the launch position, as the molecules are oriented at approximately $45^\circ$ there, so walk-off [given by Eq. (9) with $\psi = \theta_0 + \theta_1$] is close to its maximum. All the trajectories are monotonic, and the beam transverse deviation increases with propagation distance. As for the comparisons in Fig. 3, the agreement between the MC and numerical trajectories is near perfect, except for the lowest angle variation from $5^\circ$ to $35^\circ$, for which the agreement is still satisfactory. In the latter case, the initial director angle at the input is far from the walk-off maximum at $45^\circ$, so the trajectory bending is weak. Small errors in the momentum approximation then become relevant.

In the opposite case $\theta_0 > \theta_2$, the walk-off and the phase change along the cell counteract, resulting in the solitary beam reversing its transverse velocity, as illustrated in the comparison of Fig. 5(b). The agreement between the MC and numerical trajectories is nearly perfect, except for two noticeable cases. The first is for the modulation from $35^\circ$ to $5^\circ$, which is the same interval but opposite sign to that examined in the previous paragraph. The reason for the disagreement is again the weak bending of the beam and the enhanced role of small errors in the momentum approximation. The other case is the $75^\circ$ to $45^\circ$ modulation. It can be seen from Fig. 5(b) that as the range of $\theta_2$ varies, the beam reaches a maximum deviation in $y$. The $75^\circ$ to $45^\circ$ variation is just after this turning point. As for...
the boundary conditions in a pointwise manner. Experimental results have been recently reported in Ref. [47] using the former approach. Based on comparisons with numerical solutions obtained by FVBPM and elastic theory for self-localized light beam propagation in non-uniform NLCs, we found that MC is an excellent approximation for modeling soliton paths in highly nonlocal media. It provides simple results for these trajectories and a highly intuitive explanation for their evolution, at variance with the highly coupled form of the full governing equations. While full numerical solutions can well describe nematicon evolution under generic conditions, the simplicity of the MC theory and its analytical solution speak in its favor for specific limits within the adiabatic category. Due to the slow variation of the anchoring conditions, both models show that the nematicon trajectory can be described as propagation in a uniform medium with a quadratic correction. Additionally, the power needed to excite reorientational solitons in either uniform or linearly non-uniform NLCs is comparable. Ongoing studies will address the role of longitudinal director modulations as well as combinations of transverse and longitudinal changes, unveiling scenarios for the design of arbitrary soliton paths and corresponding all-optical waveguides. The latter results pave the way to novel generations of light-induced and light-controlled guided-wave circuits with two- and three-dimensional architectures.

**4. CONCLUSIONS**

We have studied the propagation of reorientational optical spatial solitons in NLCs encompassing a transversely modulated orientation of the optic axis (director). Even in the simplest limit of a linear change in anchoring angle, as considered here, non-uniform walk-off and wavefront distortion determine the bending of the resulting trajectory from the usual straight line, leading to curved paths and correspondingly curved optical waveguides induced by light beams through reorientation. Such modulations of the molecular director anchoring could be realized through electron-beam photolithography or photo-alignment techniques with light-sensitive layers to define the 35° to 5° case, small errors in the MC approximation can then result in large trajectory deviations, in particular errors in the θ₀ changes required for the maximum displacement in y.

Finally, we note that comparable beam powers are needed to obtain nematicons in uniform and linearly modulated NLCs, as a 1 mW input beam is sufficient to excite them in both cases, i.e., the rate of change in anchoring does not significantly modify the threshold power for reorientational solitons.

**Fig. 5.** Comparison of nematicon trajectories for (a) θ₁ > θ₀ and (b) θ₁ < θ₀. Numerical solutions of the nematic equations using FVBPM with elastic theory (dashed lines) and the MC Eq. (28) (solid lines). (a) Red lines (labeled 1) θ₀ = 5° to θ₁ = 35°; green lines (labeled 2) θ₀ = 15° to θ₁ = 45°; blue lines (labeled 3) θ₀ = 30° to θ₁ = 60°; yellow lines (labeled 4) θ₀ = 45° to θ₁ = 75°. (b) Red lines (labeled 1) θ₀ = 35° to θ₁ = 5°; green lines (labeled 2) θ₀ = 45° to θ₁ = 15°; blue lines (labeled 3) θ₀ = 60° to θ₁ = 30°; yellow lines (labeled 4) θ₀ = 75° to θ₁ = 45°. In all cases, the rate of change is 30°/200 μm.

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