Two-color, nonlocal vector solitary waves with angular momentum in nematic liquid crystals

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The propagation and interaction of two solitary waves with angular momentum in bulk nematic liquid crystals, termed nematicons, have been studied in the nonlocal limit. These two spinning solitary waves are based on two different wavelengths of light and so are referred to as two-color nematicons. Under suitable boundary conditions, the two nematicons can form a bound state in which they spin about each other. This bound state is found to be stable to the emission of diffractive radiation as the nematicons evolve. In addition this bound state shows walk-off due to dispersion. Using an approximate method based on the use of suitable trial functions in an averaged Lagrangian of the two-color nematicon equations, modulation equations for the evolution of the individual nematicons are derived. These modulation equations are extended to include the diffractive radiation shed as the nematicons evolve. Excellent agreement is found between solutions of the modulation equations and full numerical solutions of the nematicon equations. The shed diffractive radiation is found to play a much lesser role in the nonlocal limit than in the local limit.

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I. INTRODUCTION

The study of optical nonlinear guided waves in soft matter has seen a large increase of interest in recent years, examples of such soft media being nematic liquid crystals [1–4] and colloidal suspensions [5,6]. One reason for such interest is that such media have been shown to support stable, two-dimensional solitary waves. In the context of nematic liquid crystals these solitary waves have been termed “nematicons” [3]. Soft media such as nematic liquid crystals, as well as being highly nonlinear, have the further important property of being nonlocal, in that the response of the medium to the light extends far beyond the waist of the light beam. This nonlocality is vital in that solitary wave solutions of (2+1)-dimensional nonlinear-Schrödinger- (NLS-) type equations are normally unstable, undergoing catastrophic collapse above an amplitude threshold [7]. Theoretical work by Conti et al. [2] showed that nematicons are stable due to their nonlocal interaction with the nematic. In addition the nematicon equations are the same as those governing a thermoelastic waveguide. Kuznetsov and Rubenchik [8] showed that solitary waves are stable in this additional context.

While the original work on nematicons considered the propagation of a single soliton [2–4,9–11], more recent work has considered the interaction of two nematicons of the same wavelength (color) [12] and two different wavelengths (colors) [13]. It was shown numerically [12] and experimentally [14] by Fratalocchi et al. for two interacting nematicons of the same color that if the two beams have angular momenta the attractive “force” due to the nematic can balance the repulsive centrifugal “force,” so that a bound state can form with the two self-trapped beams rotating about each other. Moreover, the spin rate of the cluster can be controlled by acting on the input power [15]. This interaction of two nematicons of the same color, both in-phase and out-of-phase, was studied theoretically by García-Reimbert et al. [16] using the averaged Lagrangian-based modulation theory developed by Minzioni et al. [17] for the evolution of a single nematicon. This modulation theory had the advantage in that it can take account of the diffractive radiation shed by a nematicon as it evolves.

In previous work Skuse and Smyth [18] considered the interaction of two-color nematicons in the local limit by using the averaged Lagrangian method of García-Reimbert et al. [19,20]. In this limit catastrophic collapse is avoided due to the nematicon equations reducing to a pair of saturating NLS equations. Skuse and Smyth [18] were interested in the case in which the two colors formed a bound state, a vector nematicon. Due to the nonsymmetric optical parameters for the two colors, momentum conservation gives that the vector nematicon experiences walk-off. Here the term walk-off refers to a diffraction effect rather than optical anisotropy, the angular deviation of the Poynting vector from the wave vector. It is then the off-axis departure of each color (beam) component as it undergoes a different amount of diffraction while the system conserves linear and angular momentum. Excellent agreement was found for this walk-off as given by solutions of their modulation equations and by full numerical solutions. It was further found that the inclusion of the effect of the shed diffractive radiation was vital for this agreement. Without the inclusion of this radiation, the modulation walk-off differed from the numerical one by up to 30%, while when the radiation was included the difference was no more than 1%.
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et al.
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much less of a role in the nematicon evolution than in the
local limit case. Neglecting this radiation results in a
decrease in agreement of around 5%, rather than the up to 30%
decrease found in the local limit.

II. APPROXIMATE EQUATIONS

In the present work the interaction of two different color
nematicons with angular momentum will be studied in the
highly nonlocal limit using the theoretical approach of Min-
zon et al. [17] for a single nematicon. As for the local limit,
the inclusion of the shed diffractive radiation will be found to
be vital in order to obtain good agreement with full numeri-
cal solutions. However, the shed radiation is found to play
much less of a role in the nematicon evolution than in the
local limit case. Neglecting this radiation results in a
decrease in agreement of around 5%, rather than the up to 30%
decrease found in the local limit.

FIG. 1. (Color online) Schematic diagram of a liquid crystal cell
with two polarized light beams of different colors.

Let us consider two polarized, coherent light beams of
two different wavelengths propagating through a cell filled
with a nematic liquid crystal, as illustrated in Fig. 1. The
light initially propagates in the z direction, with the (x,y)
plane orthogonal to this. A static electric field is applied in
the x direction so that in the absence of light the optical
director is pre-tilted at an angle \( \hat{\theta} \) to the z direction. Both
input light beams are polarized with their electric fields in
the x direction. Then let \( u \) and \( v \) be the electric field enve-
lopes of the two light beams and \( \theta \) be the perturbation of the
optical director angle from its static value due to the light
beams. In nondimensional form these equations are

\[
\frac{i}{\partial z} + \frac{1}{2} D_u \nabla^2 u + 2 A_u u \sin 2 \theta = 0, \tag{1}
\]

\[
\frac{i}{\partial z} + \frac{1}{2} D_v \nabla^2 v + 2 A_v v \sin 2 \theta = 0, \tag{2}
\]

\[
\nu \nabla^2 \theta - q \sin 2 \theta = -2 [A_u |u|^2 + A_v |v|^2] \cos 2 \theta, \tag{3}
\]

see Alberucci et al. [13]. The Laplacian \( \nabla^2 \) is in the (x,y)
plane. The coefficients \( D_u \) and \( D_v \) are the diffraction coeffi-
cients for the two colors and \( A_u \) and \( A_v \) are the coupling
coefficients between the light and the nematic for the two
colors. The parameter \( \nu \) measures the elasticity of the nematic
and \( q \) is related to the energy of the static electric field
which pretilts the nematic [20].

The usual operating regime for nematicon propagation is
the so-called nonlocal regime in which \( \nu \) is large [2,4]. In
this limit the reorientation of the nematic extends far past the
optical beam(s), as measured by the waist(s) of the beam(s).
Furthermore, in this limit, the deviation \( \theta \) of the director
from its equilibrium value is small, as can be seen from the
director Eq. (3). So in the highly nonlocal limit the nematic
Eqs. (1)–(3) can be approximated by

\[
\frac{i}{\partial z} + \frac{1}{2} D_u \nabla^2 u + 2 A_u u \theta = 0, \tag{4}
\]

\[
\frac{i}{\partial z} + \frac{1}{2} D_v \nabla^2 v + 2 A_v v \theta = 0, \tag{5}
\]

\[
\nu \nabla^2 \theta - 2q \theta = -2A_u |u|^2 - 2A_v |v|^2. \tag{6}
\]

These nonlocal two-color nematicon equations have the La-
grangian

\[
L = i(u^* u_z - uu_z) - D_u |u|^2 + 4 A_u |u|^2 + i(v^* v_z - vv_z)
- D_v |v|^2 + 4 A_v |v|^2 - \nu \nabla^2 \theta - 2q \theta^2, \tag{7}
\]

where the superscript asterisk denotes the complex conju-
gate.

As in Minzoni et al. [17] and Skuse and Smyth [18] an
approximate solution of the highly nonlocal nematicon equa-
tions will be sought using appropriate trial functions in an
averaged Lagrangian. Solutions will then be sought using
self-similar trial functions of the form

\[
u = a_u(\chi_u/w_u)e^{i\psi_u} + ig_u e^{i\phi_u},
\]

\[
u = a_v(\chi_v/w_v)e^{i\psi_v} + ig_v e^{i\phi_v},
\]

\[
\theta = a_v(f(\chi_u/\beta_u))^2 + a_v(f(\chi_v/\beta_v))^2, \tag{8}
\]

where

\[
\chi_u = \sqrt{(x - \xi_u)^2 + (y - \eta_u)^2},
\]

\[
\chi_v = \sqrt{(x - \xi_v)^2 + (y - \eta_v)^2},
\]

\[
\psi_u = \sigma_u + U_u(x - \xi_u) + V_u(y - \eta_u),
\]

\[
\psi_v = \sigma_v + U_v(x - \xi_v) + V_v(y - \eta_v). \tag{9}
\]

The electric field amplitudes \( a_u, a_v \), widths \( w_u, w_v \), nema-
ticon positions \( (\xi_u, \eta_u), (\xi_v, \eta_v) \), velocities \( (U_u, V_u), (U_v, V_v) \), phases \( \sigma_u, \sigma_v \), shelf heights \( g_u, g_v \),
and director pulse amplitudes \( a_u, a_v \), amplitudes \( \beta_u, \beta_v \) are functions of \( z \). The first
terms in the trial functions (8) for \( u \) and \( v \) are varying soli-
tary waves, while the second terms represent the diffrac-
tive radiation of low wave number which accumulates under
the evolving nematics [17,21]. This shed radiation cannot
remain flat, so it is assumed that \( g_v(\xi_u) \) is nonzero in the disk
\[
0 \leq \sqrt{(x - \xi_u)^2 + (y - \eta_u)^2} \leq R_u \quad \text{and} \quad 0 \leq \sqrt{(x - \xi_v)^2 + (y - \eta_v)^2} \leq R_v
\]
[17,21]. In the case of the (1 + 1)-dimensional NLS equa-
tion the existence of this shelf of low wave number radiation
under the pulse can be shown using inverse scattering [21].

The most widely used self-similar profiles \( f \) are a Gauss-
ian [2] and a sech [17,18]. A sech profile is the same as that

013832-2
for the soliton solution of the one-dimensional NLS equation. On the other hand, it was shown by Conti et al. [2] that the numerically determined two-dimensional nematicon solution has a Gaussian profile near its peak and its tail decays as the Bessel function $K_0$, due to circular symmetry. Hence a sech is a better approximation away from the peak of a nematicon, while a Gaussian is a better approximation around the peak. In this regard it was found by Skuse and Smyth [18] that in the limit of local interaction for $\nu$ small the particular choice of trial function made little difference to the final results. Hence both the sech and Gaussian profiles

$$f(r) = \text{sech}\ r \quad \text{and} \quad f(r) = e^{-r^2} \tag{10}$$

will be considered here.

The trial functions (8) are now substituted into the Lagrangian (7) for the nematicon equations and the averaged Lagrangian

$$\mathcal{L} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L} dxdy \tag{11}$$

is calculated, from which the variational equations for the nematicon parameters are derived. For the Gaussian trial is calculated, from which the variational equations for the sech trial function made little difference to the final results. Hence both the sech and Gaussian profiles

$$f(r) = \text{sech}\ r \quad \text{and} \quad f(r) = e^{-r^2} \tag{10}$$

will be considered here.

The trial functions (8) are now substituted into the Lagrangian (7) for the nematicon equations and the averaged Lagrangian

$$\mathcal{L} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L} dxdy \tag{11}$$

is calculated, from which the variational equations for the nematicon parameters are derived. For the Gaussian trial function all the required integrals can be evaluated. For the sech trial function all integrals, except cross integrals of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta u^2 dxdy, \tag{12}$$

can be evaluated. As in García-Reimbert et al. [19] and Minzoni et al. [17] approximate values for such integrals can be obtained using an idea of an “equivalent Gaussian,” whereby in these integrals $\text{sech}^2(r/\beta)$ is replaced by the Gaussian $\exp[-r^2/(A\beta)^2]$ and $\text{sech}^2(r/w)$ is replaced by the Gaussian $\exp[-r^2/(Bw)^2]$. The scaling parameters $A$ and $B$ are then calculated by matching the Taylor series of the integral (12) with the sech with the Taylor series with the “equivalent Gaussians” in the highly nonlocal limit $w_u/\beta_u \ll 1$. The details are as in Minzoni et al. [17], so that only the final results will be given here.

Substituting the trial functions (8) into the Lagrangian (7) then gives the averaged Lagrangian (11) as

$$\mathcal{L}_I = 2A_x v^2_a^2 w_u^2[\alpha_x \phi_0^2 \Omega_1^2 + \alpha_x \beta_0^2 \Omega_2^1 e^{-\gamma}]$$

$$+ 2A_x v^2_a^2 w_u^2[\alpha_x \beta_0^2 \Omega_1^2 + \alpha_x \beta_0^2 \Omega_2^1 e^{-\gamma}]$$

$$- 4\nu \alpha \alpha \alpha \beta \beta \beta \Omega_2^1 \Omega_2^1 (1 - \gamma_1) e^{-\gamma} - 2q \alpha \alpha \alpha \beta \beta \beta \Omega_2^1 e^{-\gamma}. \tag{14}$$

Here

$$Q_1 = A \beta_0^2 + B w_u^2, \quad Q_2 = A \beta_0^2 + B w_u^2,$$

$$Q_3 = A \beta_0^2 + B w_u^2, \quad Q_4 = A \beta_0^2 + B w_u^2,$$

$$Q_5 = \beta_0^2 + \beta_0^2, \quad \gamma_1 = \frac{\beta_0^2}{Q_2}, \quad \gamma_2 = \frac{\beta_0^2}{Q_4}, \quad \gamma_3 = \frac{\beta_0^2}{Q_5},$$

$$\rho^2 = (\xi_0 - \xi_0) + (\eta_u - \eta_u)^2.$$

The integrals $I_i, I_j$ are

$$A = -\sqrt{\frac{2}{I_{132}}} 1, \quad B = \sqrt{2}, 1, \quad I_1 = \int_0^\infty f(x) dx = 2C, 1/2,$$

$$I_2 = \int_0^\infty f(x) dx = \ln 2, 1/4,$$

$$I_22 = \int_0^\infty (f^2(x) dx = \frac{1}{3} \ln 2 + \frac{1}{6} \sqrt{2}, \quad I_{32} = \int_0^\infty f^2(x) dx$$

$$= 1.352301002 \cdots, \quad I_{42} = \int_0^\infty \left( \frac{d}{dx} f(x)^2 \right) dx = 2 \frac{1}{5} \ln 2$$

$$+ \frac{1}{60} \sqrt{8}, \tag{16}$$

where $C$ is the Catalan constant $C=0.915965594 \cdots$ [22]. The first values of $A, B$ and the integrals refer to the sech trial function, while the second values refer to the Gaussian trial function. There is no value given for $I_{132}$ for the Gaussian trial function as this integral is not needed. Finally

$$\Lambda_u = \frac{1}{2} R_u^2 \quad \text{and} \quad \Lambda_v = \frac{1}{2} R_v^2. \tag{17}$$

Taking variations of the averaged Lagrangian (13) results in the modulation equations

$$\frac{d}{dz} \left[ D_v a^2 w_u^2 + \Lambda_v a^2 \right] = 0, \tag{18}$$

$$I_1 \frac{d}{dz} (a_u w_u^2) = \Lambda_v a_u \left[ a_u' - \frac{1}{2} D_u U_a + \frac{1}{2} D_u V_a \right], \tag{19}$$

$$\frac{d}{dz} \left[ D_v a^2 w_u^2 + \Lambda_v a^2 \right] = 0, \tag{20}$$

$$I_1 \frac{d}{dz} \left( a_u w_u^2 - \Lambda_a a_u w_u^2 + \alpha \alpha \alpha \beta \beta \beta \Omega_2^1 e^{-\gamma} \right)$$

$$- \alpha \beta \beta \beta \Omega_2^1 e^{-\gamma}, \tag{21}$$

013832-3
means that the resulting radii expressions are extremely complicated. In experiments the diffraction coefficients $D_u$ and
$D_v$ and the coupling coefficients $A_u$ and $A_v$ take similar values. For instance, for the experiments of Alberucci et al.\cite{13}
the diffraction coefficients were 0.805 for red and 0.823 for
near-infrared light. To calculate the shelf radii, it is then
much simpler to take $D_u=D_v$ and $A_u=A_v$, in which case the
radius expression of Minzioni et al.\cite{17} can be used, suitably
rescaled as this work had $D=1$ and $A=1$. These expressions
are given in the Appendix.

Noether’s theorem applied to the Lagrangian (7) gives that the
nematicons satisfy the energy conservation equation

$$\frac{dH}{dz} = \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ D_u |\nabla u|^2 - 4A_u \theta |u|^2 + D_v |\nabla v|^2 - 4A_v \theta |v|^2 \right] + v |\nabla \theta|^2 + 2q \theta^2 \, dx \, dy = 0. \quad (27)$$

On substituting the trial functions (8) the final energy equation

$$\frac{dH}{dz} = \frac{d}{dz} [D_u I_{22} a_u^2 + D_u I_2 a_u w_u^2 + A_u g_u^2(U_u^2 + V_u^2) + D_v I_{22} a_v^2
+ D_v (I_2 a_v w_v^2 + A_v g_v^2(U_v^2 + V_v^2)) - 2A_u A_v^2 a_u^2 w_v^2 (\alpha_v \beta_u a_v^2 Q_1^3 + \alpha_u \beta_v a_u^2 Q_1^3 e^{-\gamma}) - 2A_u A_v^2 a_u^2 w_v^2 (\alpha_v \beta_u a_v^2 Q_1^3 + \alpha_u \beta_v a_u^2 Q_1^3 e^{-\gamma}) + 4q a_v a_u^2 \beta_v \beta_u^2 a_v (1 - A^2 \rho^2 Q_1^3) e^{-\gamma} + 2q A^2 a_v a_u^2 \beta_v \beta_u^2 a_v Q_1^3 e^{-\gamma}] = 0 \quad (28)$$

results. On making the symmetry assumption discussed
above, the fixed point of the modulation equations can then
be found from this energy equation and the relations deter-
mined by looking for the steady states of Eqs. (19) and (21)
and their $v$ color equivalents. This fixed point will be re-
quired below, particularly in the calculation of radiation loss.

Adding the momentum Eqs. (23) and (24) to their $v$ color
counterparts gives the equations for total momentum conser-
vation in the $x$ and $y$ directions as

$$\frac{d}{dz} \left[ (I_2 a_u^2 w_u^2 + A_u g_u^2) U_u + (I_2 a_u^2 w_u^2 + A_v g_v^2) U_v \right] = 0 \quad (29)$$

and

$$\frac{d}{dz} \left[ (I_2 a_u^2 w_u^2 + A_u g_u^2) V_u + (I_2 a_u^2 w_u^2 + A_v g_v^2) V_v \right] = 0, \quad (30)$$

respectively. Let us consider boundary conditions for which
the two nematicons form a bound state and denote these
boundary values by a subscript 0. Let us further assume that
the nematicons evolve to have the same position as $z \to \infty$
and denote final steady state values by a hat superscript.
Then on noting mass conservation (18) and total momentum
conservation in the $x$ and $y$ directions and that $g_{x0} = g_{y0} = 0$
and $\hat{g}_x = \hat{g}_y = 0$, we have that the walk-off of the bound state
nematicon is given by

013832-4
and
\[ \ddot{\eta} = \ddot{\eta}_u = \ddot{\eta}_v = \frac{D_u D_v P_{u0}}{I_2(D_u \partial^2_w U_{u0} + D_u \partial^2_v U_{v0})}, \]  
\[ \ddot{\xi} = \ddot{\xi}_u = \ddot{\xi}_v = \frac{D_u D_v P_{v0}}{I_2(D_u \partial^2_w U_{u0} + D_u \partial^2_v U_{v0})}, \]  
\[ P_{u0} = I_2 \partial^2_w U_{u0} + I_2 \partial^2_v U_{v0} \]  
\[ P_{v0} = I_2 \partial^2_w V_{u0} + I_2 \partial^2_v V_{v0} \]
which are the initial x and y momenta. Combining the two momentum conservation expressions (31) and (32) gives the radial walk-off of the vector nematicon as
\[ \rho_\gamma = \sqrt{\xi^2 + \eta^2}. \]  

III. KEPLER COORDINATES

The modulation Eqs. (18)–(26) for the \( u \) color and their symmetric \( v \) counterparts consist of two basically independent oscillations, one oscillating being an amplitude and width oscillation in \( a_{u(\gamma)} \), \( w_{u(\gamma)} \), and \( \theta_{u(\gamma)} \), and the other a position and velocity oscillation in \( \xi_{u(\gamma)}, \eta_{u(\gamma)}, U_{u(\gamma)}, \) and \( V_{u(\gamma)} \). The position and velocity oscillation possesses a close connection with the two body gravitational problem. This can be more clearly seen by transforming the position and velocity Eqs. (20), (23), and (24) and their symmetric \( v \) equivalents to polar coordinates centered on the “center of mass” of the two-color nematics. This has the added advantage of making the position and velocity equations much cleaner.

We define the “masses” \( M_u \) of the \( u \) color and \( M_v \) of the \( v \) color as
\[ M_u = I_2 \partial^2_w U_{u0} + \Lambda_u g_u^2 \quad \text{and} \quad M_v = I_2 \partial^2_v V_{u0} + \Lambda_v g_v^2. \]  
The positions of the two nematics can be written in vector form as \( \tilde{\xi}_u = (\xi_u, \eta_u) \) for the \( u \) color and \( \tilde{\xi}_v = (\xi_v, \eta_v) \) for the \( v \) color. The polar coordinate system is then based on the “center of mass” of the two-color nematics
\[ \tilde{R} = \frac{D_u M_u \tilde{\xi}_u + D_v M_v \tilde{\xi}_v}{D_u M_u + D_v M_v}, \]  
with the relative displacement of the nematics
\[ \tilde{\rho} = \tilde{\xi}_u - \tilde{\xi}_v. \]

In this center-of-mass system, the distance from the center-of-mass is \( \rho \) and the polar angle is \( \phi \). The equations for the position of the two-color nematics are then the conservation of angular momentum equation
\[ \rho^2 \frac{d \phi}{dz} = L, \]  
where \( L \) is a constant and the radial equations
\[ \frac{d \rho}{dz} = 0 \]  
\[ \frac{d^2 \rho}{dz^2} - L^2 \rho^{-3} = -\frac{D_u M_u + D_v M_v}{M_u M_v} \frac{\partial \Phi}{\partial \rho}. \]  
The potential \( \Phi \) is
\[ \Phi = -2A_u A^2 B^2 \alpha_u \beta_u^2 Q_1^{2} e^{-\gamma} \]  
\[ -2A_u A^2 B^2 \alpha_u \beta_u^2 Q_2^{2} e^{-\gamma} \]  
\[ + 4\nu\alpha_u \alpha_v \beta_u^2 \beta_v^2 Q_2^{2} (1 - A_u^2 \rho^2 Q_2^{4}) e^{-\gamma} \]  
\[ + 2A_u A^2 \alpha_u \beta_u^2 \beta_v^2 Q_1^{2} e^{-\gamma}. \]  

IV. DIFFRACTIVE RADIATION LOSS

The modulation equations of the previous two sections are not complete yet as the effect of the diffractive radiation shed by the nematics as they evolve has not yet been included. As the amplitude of the shed radiation is much less than the amplitude of the nematics, this radiation is governed by the linearized electric field equations
\[ i \frac{\partial u}{\partial z} + \frac{1}{2r} D_u \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0, \]  
\[ i \frac{\partial v}{\partial z} + \frac{1}{2r} D_v \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) = 0, \]  
which are just the Schrödinger equation. This radiation problem has already been studied by García-Reimbert et al. [17], so that the details will not be repeated here. The final result is that the mass conservation Eq. (18) and Eq. (21) for \( g_u \) are replaced by
\[ \frac{d}{dz} \left( I_1 a^2 w^2 + \Lambda_u g_u^2 \right) = -2D_u \delta_u \tilde{\xi}_u \kappa_u^2 \]  
and
\[ \frac{d g_u}{dz} = \frac{1}{2} D_u I_2 a_0 w^2 u^2 - A_u A^2 B^2 \alpha_u \beta_u^2 Q_1^{2} + \alpha_u \beta_v^2 Q_2^{2} e^{-\gamma} \]  
\[ - \alpha_u \beta_v^2 (A_u^2 \rho^2 Q_2^{4}) e^{-\gamma} \]  
\[ - 2D_u \delta_u g_u, \]  
respectively, where the loss coefficient \( \delta_u \) is
\[ \delta_a = -\frac{\sqrt{2\pi I_1}}{2e\kappa_u \bar{\Lambda}_u} \int_0^z \pi \kappa_u(z') \ln\left(\frac{z-z'}{\bar{\Lambda}_u}\right) \times \left\{ \left( \frac{1}{2} \ln\left(\frac{z-z'}{\bar{\Lambda}_u}\right) + \frac{3\pi^2}{4} \right)^2 + \pi^2 \left(\ln\left(\frac{z-z'}{\bar{\Lambda}_u}\right)\right)^2 \right\}^{-1} \frac{dz'}{(z-z')} \] (46)

and

\[ \kappa_u^2 = \frac{1}{\bar{\Lambda}_u} \left[ I_2 a_u^2 \omega_u^2 - I_2 a_u^2 \omega_u^2 + \bar\Lambda_u \delta_u^2 \right]. \] (47)

The variable \( \kappa_u \) measures the difference between the mass of the \( u \) color at \( z \) and its mass at the fixed point. The mass equation for the \( v \) color and the equation for \( \delta_v \) are obtained by the obvious symmetric substitutions. Finally

\[ \bar{\Lambda}_u = \frac{1}{2} (7 \beta_{u(1/2)}^2), \quad \beta_{u(1/2)} = \beta_u \operatorname{sech}^{-1}(1/\sqrt{2}). \] (48)

V. RESULTS

In this section solutions of the modulation Eqs. (19), (20), (22)–(26), (44), and (45) and their symmetric \( v \) color counterparts will be compared with full numerical solutions of the two-color nematicon Eqs. (1)–(3). The modulation equations were solved using the standard fourth order Runge-Kutta method. The full nematicon Eqs. (1)–(3) were solved using the method described in Skuse and Smyth [18], so this method will not be described again here.

In deriving the modulation equations it was assumed that the director angle \( \theta \) was small, so that \( \sin 2\theta \) and \( \cos 2\theta \) could be approximated by the first terms in their Taylor series. It was found from numerical solutions that for this approximation to be valid, the nonlocality parameter \( \nu \) was required to be \( O(100) \), in which case \( \theta \approx -0.1 \). In contrast, in Minzoni et al. [17] for a single nematicon the small \( \theta \) approximation was valid for \( \nu \approx 10 \), while for two-color nematics \( \theta \approx -0.3-0.4 \) for \( \nu = 10 \). The difference for two interacting nematics is that the nematics are in fairly close proximity and overlap to some extent, so that as the director is forced by the sum of the powers of the two colors, as can be seen from the director Eq. (3), the forcing is increased over a single nematicon. Now as \( \nu \) increases, \( \theta \) decreases. Furthermore as \( \nu \) increases, the further apart the two colors can be and still form a bound state due to nonlocality. So for the examples of this section, the value \( \nu = 500 \) was chosen, which is at the upper end of the usual experimental range from \( O(10) - O(100) \) [2].

Let us first consider comparisons for the sech initial condition. Figure 2 shows the walk-off comparison for the initial values and parameters indicated in this figure

\[ \rho_u = \left[ \xi_u^2 + \eta_u^2 \right]^{1/2} \quad \text{and} \quad \rho_v = \left[ \xi_v^2 + \eta_v^2 \right]^{1/2}, \] (49)

so that \( \rho_u \) and \( \rho_v \) measure the distances of the nematicons from the axis, which is a measure of their walk-off [11]. It can be seen that there is good agreement for the walk-off as given by the full numerical and modulation solutions, particularly up to \( z = 100 \). After this point, there is some period difference between the solutions. Furthermore the modulation solution settles to a more harmonic oscillation faster than the numerical solution. Also shown in this figure is the walk-off of the bound state given by Eq. (35). This line shows that the walk-off of the orbiting nematics is given nearly exactly by momentum conservation, so that there is little momentum being shed into diffractive radiation, in agreement with experimental reports [15]. This is in contrast to the local limit for which momentum loss to radiation is significant [18]. In Skuse and Smyth [18] a comparison was made between the walk-off angle as given by the full numerical and modulation solutions. In this local limit the nematicons did not have angular momentum and their trajectories rapidly settled to a straight line, making such a comparison possible. However in the present nonlocal limit the nematicons remain in an orbit about each other and do not settle to a straight walk-off. Furthermore the numerical walk-off path is not harmonic, even after \( z = 400 \). Therefore it is not possible to make a similar comparison here.

The comparison of Fig. 2 was for small angular velocities. Figure 3 shows comparisons for much larger angular velocities. The walk-off comparison shown in Fig. 3(a) shows similar good agreement between the numerical and modulation solutions, again particularly up to \( z = 100 \), with some period difference for larger \( z \). However for this example, there is about a 5% difference between the momentum conservation trajectory (35) and the mean of the modulation trajectories, showing that radiation loss is of some importance for higher angular velocities. The main difference between the results for small and relatively larger angular velocities is shown in Fig. 3(b), where an amplitude comparison is shown. It can be seen that while there is agreement in the means of the amplitude oscillation, the numerical amplitude oscillation picks up a second frequency. This sec-
functions. However, it was found that for the Gaussian trial function, so no results will be reported here.

VI. CONCLUSIONS

The interaction of two nematicons of different wavelengths (colors) has been considered in the highly nonlocal limit. This interaction was analyzed using suitable trial functions in an averaged Lagrangian formulation, resulting in modulation equations for the nematicon parameters. These modulation equations were shown to have a close connection to the equations for the two body gravitational problem, but with a potential that was Gaussian (and attractive). Therefore a vector nematicon solution consisting of the two nematicons orbiting about each other exists, which was found to be stable with respect to mutual diffraction but in contrast to the opposite local limit, in which the diffractive radiation was found to have a dominant role [18]. This decreasing effect of radiation as nonlocality increases is in agreement with the work of Minzoni et al. [17] for the evolution of single nematicons. The close connection between the equations governing multicolored nematicons and gravitation should allow further multicolored nematicon interactions to be investigated. This is the subject of current investigations.

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APPENDIX: SHELF RADIUS

The shelf radii $R_u$ and $R_v$ are determined by linearizing the modulation Eqs. (18)–(26) and their symmetric $v$ counterparts about their fixed point. After some algebra this results in the simple harmonic oscillator equation

$$\frac{d^2 g_u}{dc^2} - \frac{\Theta \Lambda \Sigma_u}{\hat{p}_1^2 (\hat{w}_u^2 + 2\hat{a}_u \hat{w}_u \varphi)} g_u = 0 \quad (A1)$$

and a similar equation for $g_v$, where

$$\Sigma_u = \hat{a}_u - \frac{1}{2} D_u \hat{U}_u - \frac{1}{2} D_u \hat{V}_u \quad (A2)$$

$$\varphi = \frac{D_u \Sigma_I^2 A^2 B^2 \hat{w}_u \hat{w}_u \hat{a}_u \hat{a}_u}{A_u A^2 B^2 \hat{a}_u \hat{a}_u \hat{b}_u \hat{b}_u \Phi} \quad (A3)$$

$$\Phi = 2A^2 \hat{a}_u \hat{b}_u \hat{w}_u \hat{w}_u \hat{a}_u \hat{a}_u W_2 \Gamma + 2B^2 \hat{w}_u \hat{a}_u W_4, \quad (A4)$$

$$W_0 = \frac{2q (I_4 + A^2 /4) B^2 \hat{w}_u \hat{w}_u \hat{a}_u \hat{a}_u + \nu (8I_4 + 1) B^2 \hat{w}_u \hat{a}_u}{2q \hat{a}_u (I_4 + A^2 /4) (2A^2 \hat{b}_u \hat{b}_u - A^2 \hat{b}_u \hat{b}_u)} \quad (A5)$$

FIG. 3. Comparisons for the initial values $a_u=1.8$, $a_v=1.8$, $w_u=3.0$, $w_v=3.0$, $\xi_u=1.0$, $\xi_v=-1.0$, $\eta_u=\eta_v=0.0$, $U_u=0.1$, $U_v=-0.1$, $V_u=0.1$, $V_v=-0.15$ with $\nu=500$, $q=2$, $A_u=1.0$, $A_v=0.95$, $D_u=1.0$, $D_v=0.98$. Full numerical solution, $u$ color ($\cdots$); $v$ color ($\cdots$); solution of modulation equations for $u$ color ($\cdots$); $v$ color ($\cdots$); (a) Radial positions, momentum conservation result (35) ($\cdots$ $\cdots$ $\cdots$); (b) amplitudes.
\[ W_1 = \frac{4 A_u A^2 B^4 \hat{a}_u \hat{w}_u^2}{q(I_2 + A^2/4) \Gamma^2}, \quad \Gamma = A^2 \hat{\beta}_u^2 + B^2 \hat{w}_u^2, \quad (A6) \]

\[ W_2 = \frac{8 A_u A^4 B^2 \hat{a}_u \hat{w}_u^2 \hat{\beta}_u^2}{q(I_2 + A^2/4) \Gamma^3} (\beta_u - \hat{w}_u W_0), \quad (A7) \]

\[ \Theta = \frac{D_{u2}}{2 \hat{w}_u} \left[ Q_0 - \frac{Q_1 + Q_2}{\hat{a}_u \hat{\beta}_u} \right], \quad Q_0 = 1 - \frac{2 \hat{\alpha}_u \varphi}{\hat{w}_u}, \quad (A8) \]

\[ Q_1 = \hat{w}_u \hat{\beta}_u [\hat{a}_u + \hat{\alpha}_u (W_1 + \varphi W_2)], \quad (A9) \]

\[ Q_2 = 2 \hat{\alpha}_u \hat{\alpha}_u \varphi (A^2 \hat{\beta}_u^2 - B^2 \hat{w}_u^2) (\hat{\beta}_u - W_0) \Gamma^{-1}. \quad (A10) \]

As in Kath and Smyth [21] and Minzoni et al. [17], the frequency of the simple harmonic oscillator (A1) is then matched to the nematic oscillation frequency \( \Sigma'_u \), given by Eq. (22) as

\[ \Sigma'_u = 2 A_u A^2 B^2 \hat{a}_u \hat{w}_u \hat{\beta}_u (A^2 \hat{\beta}_u^2 - B^2 \hat{w}_u^2) (I_1 \Gamma^{-2})^{-1}. \quad (A11) \]

This results in

\[ \Lambda_u = - \frac{\Sigma'_u \hat{w}_u (\hat{w}_u^2 + 2 \hat{\alpha}_u \varphi)}{\Theta}. \quad (A12) \]

The shelf radius \( R_u \) and \( \Lambda_u \) for the \( u \) color is found by symmetry.