Spatial optical solitons in highly nonlocal media

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We theoretically investigate the propagation of bright spatial solitary waves in highly nonlocal media possessing radial symmetry in a three-dimensional cylindrical geometry. Focusing on a thermal nonlinearity, modeled by a Poisson equation, we show how the profile of the light-induced waveguide strongly depends on the extension of the nonlinear medium in the propagation direction as compared to the beamwidth. We demonstrate that self-trapped beams undergo oscillations in size, either periodically or aperiodically, depending on the input waist and power. The—usually neglected—role of the longitudinal nonlocality as well as the detrimental effect of absorptive losses are addressed.

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I. INTRODUCTION

Light-beam self-localization in nonlinear nonlocal materials, that is, in the presence of a nonlinearly induced refractive index well extending beyond the beam profile, has been widely investigated in the last few years, both experimentally and theoretically. A nonlocal response allows for the stabilization of bright $(2+1)$D spatial solitons, at variance with the collapse observed in local Kerr media (Townes soliton) [1–3]. In fact, self-localization of light and, specifically, the soliton profile and its stability, all strongly depend on the degree of nonlocality [4]. Nonlocality also entails the observation of phase-insensitive long-range interactions between solitons [5,6], stable propagation of higher-order solitons [7–11], thresholdless surface solitons [12], power-dependent steering due to interaction with boundaries [13–15], and so on. Moreover, modeling spatial solitons in the highly nonlocal limit leads to the concept of accessible solitons [16], a useful approximation in various instances [17–19]. Nonlocality in optics also plays an important role in photon condensation [20], dispersive shock waves [21–23], distributed coupling to guided waves [24,25], gradient catastrophe [26], and Anderson localization [27].

Several optical materials exhibit a highly nonlocal nonlinearity, often related to diffusive processes. Accessible solitons were first reported in nematic liquid crystals [17], with self-focusing provided by molecular reorientation [28] and nonlocality stemming from intermolecular forces [29]. High nonlocality in soliton propagation has also been exploited in thermo-optic media (heat diffusion ruled by a Poisson equation) [7,30,31], atomic vapors (molecular diffusion) [32], photorefractive crystals [33], semiconductors [34] (carrier diffusion), nanoparticle suspensions (thermophoresis) [35], silica nanowebs (optomechanics) [36], and colloidal suspensions (optical gradient forces) [37].

From a theoretical viewpoint, most highly nonlocal materials can be modeled by a diffusion-like equation describing the light-induced refractive index distribution. With a few exceptions in reorientational liquid crystals [38–40], however, longitudinal nonlocal effects are usually neglected, that is, the Laplacian operator lacks the partial derivative along the propagation direction. This is appropriate only for shape-preserving beams and/or in the absence of power dissipation; even without losses, in fact, the family of self-confined waves includes breathing solitons subject to bounded periodic oscillations of their width and peak intensity [16,17,41–43]. In such cases, the refractive index gradient along the direction of propagation affects the self-trapped wave packet. Additionally, when losses are included, a longitudinal gradient arises even when a shape-preserving soliton is launched at the input [21,44]: appreciable effects are expected when the attenuation distance is comparable with the Rayleigh length of the input beam.

In this paper, we study light propagation in a highly nonlocal, nonlinear medium in a three-dimensional (3D) cylindrical geometry with circular symmetry. The nonlinear, nonlocal refractive index well is accounted for by a Poisson equation, which applies to actual highly nonlocal materials and systems [19,45]. We first show that the nonlinear lens induced by a light beam strongly depends on the ratio between the input radius and the available propagation length. We then discuss how the light-induced refractive index well affects light propagation, with particular attention to the beamwidth. We also address the roles of the longitudinal nonlocality and of the unavoidable absorptive losses, two important aspects which are usually ignored in the literature. The paper is organized as follows. In Sec. II, we introduce the mathematical model and its range of applicability. In Sec. III, we compute the nonlinear perturbation, keeping fixed the optical excitation and using a Green’s function formalism. In Sec. IV, we employ beam propagation method (BPM) simulations to account for the effect of the nonlinear refractive index well on beam propagation. Finally, in Sec. V, we summarize the results and pinpoint further developments.

II. MODEL

Let us consider a thermo-optic medium with heat transfer dominated by conduction. We take a homogeneous cylinder...
of circular cross section, diameter $2a$ and length $l$ in $z$, with a radially symmetric input beam launched in $z = 0$ and centered in $r = 0$ ($r$ is the radial coordinate). Light propagation in the paraxial, slowly varying envelope approximation and for small nonlinear index changes is governed by

$$2ik_0n_0\frac{\partial A}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + 2n_0k_0^2\phi A + ik_0n_0\alpha A = 0,$$  

(1)

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\alpha|A|^2,$$  

(2)

where $A$ is the beam envelope (normalized so that $I = |A|^2$ is the intensity), $k_0$ is the vacuum wave number, $n_0$ is the linear refractive index, $\alpha > 0$ is the absorption coefficient, and $\phi = n(r,z,P) - n_0$ is the nonlinear (i.e., power-dependent) refractive index well, with $P(z) = 2\pi \int |A|^2 r\,dr$ the beam power at each section $z$. The constant $\gamma$ “turns on (off)” the longitudinal nonlocality when set equal to 0 (1). Equation (2) is solved together with the boundary conditions $\phi(r = a, z) = \phi(r, z = 0) = \phi(r, z = l) = 0$, valid irrespective of the power $P$. Without loss of generality, in writing Eq. (2) we assumed that the thermal conductivity and capacity are both equal to 1; moreover, we assumed that the thermo-optic coefficient is unitary as well, i.e., $\phi = \Delta T = T - T_0$, with $T$ the local temperature and $T_0$ the environment (background) temperature. In the general case, the thermal conductivity of the medium, providing the proportionality factor between heat flux and temperature gradient, would multiply the whole left-hand side of Eq. (2); in a similar way, the right-hand side of Eq. (2) should be multiplied by the thermal capacity of the material. We note that the model consisting of Eqs. (1) and (2) is quite general, as it can also describe, e.g., particle diffusion and weak all-optical reorientation in liquid crystals [39]. Finally, this set of equations can be normalized by scaling all of the spatial dimensions (including the wavelength $\lambda$ and the absorption distance $1/\alpha$) with the medium extension $l$ along $z$, providing the beam power $P$ is multiplied by $l$.

III. NONLINEAR WELL FOR A GIVEN INTENSITY PROFILE

We first discuss how the nonlinear refractive index well $\Delta n$ (corresponding to $\phi$ in our formalism) changes for a fixed excitation $I$. For the sake of simplicity, in the following we assume an infinitely extended cylinder, that is, $a \to \infty$.

A. Green’s function

Writing $\phi$ in terms of its own Hankel transform, Eq. (2) for $\gamma = 1$ becomes the ordinary differential equation

$$\frac{\partial^2 \tilde{\phi}}{\partial z^2} - k_0^2 \tilde{\phi} + \alpha \tilde{I} = 0,$$  

(3)

with $\tilde{\phi}(k_z, z) = \int_0^\infty r \phi(r,z) J_0(k_r r)\,dr$ the Hankel transform of $\phi$. A general description (magnitude, longitudinal and transverse width) of the solutions of Eq. (2), in particular their dependence on the boundary conditions across $z$, can be obtained using a Green’s function. Setting $I = \delta(r - r')\delta(z - z')/(2\pi r)$ ($[r, z']$ is the location of the impulsive ringlike excitation), the corresponding distribution of the index of refraction reads

$$\phi_{\text{ring}}(r, r', z, z') = -\alpha[G_{\text{ring}}(r, r' z, z') + G_{\text{ring}}(r', r, z, z')],$$  

(4)

having defined the auxiliary function

$$G_{\text{ring}}(r, r', z, z') = u_0(z' - z) u_0(l - z') \int_0^\infty J_0(k_r r) J_0(k_r r') e^{-k_l z'} \sinh(k_l z) \,dk_k,$$  

(5)

with $u_0$ the Heaviside step function. For $l \to \infty$, Eq. (5) gives $G_{\text{ring}} = -u_0(z' - z) \int_0^\infty J_0(k_r r) J_0(k_r r') e^{-k_l z'} \sinh(k_l z) \,dk_k$.

The overall nonlinear perturbation of the refractive index is given by the convolution of the intensity profile $I$ with the Green’s function (4); profile and size (that is, nonlocal response [39]) can be determined from the spatial behavior of $\phi_{\text{ring}}$. The function $\phi_{\text{ring}}$ is plotted in Fig. 1 for the case $r' = 0$, i.e., a ringlike excitation degenerating to a point source. At $r' = r^*$, the azimuthally averaged impulsive response $\phi_{\text{ring}}$ is singular, in agreement with the general properties of the Poisson’s equation. Clearly, $\phi_{\text{ring}}$ has a finite extent related to the nonlocal response along $z$, with size approximately equal to 0.1. Due to this finite size along $z$, the system response depends on $z'$ and gets smaller and strongly asymmetric near the boundaries [Fig. 1(a)]. The spatial size of the response along $r$ is comparable to the nonlocality along $z$: the longitudinal boundary conditions fix the transverse nonlocality as well, as $l \ll a$ in this case (see Ref. [46] for a similar case in a rectangular 2D geometry). Finally, Fig. 1(b) graphs the system response in the limit $l \to \infty$: $\phi_{\text{ring}}$ tends to the free space solution for $z' > 1/2$ due to the absence of the output interface.

B. Gaussian excitation

The influence of nonlocality on light propagation is maximum in the highly nonlinear case: solitons acquire nearly Gaussian profiles [16,19], with a varying width $w$ along $z$ due to breathing and/or power losses. Thus we write

$$I = \frac{2P_0}{\pi w^2(z)} e^{-\frac{z^2}{w^2(z)}} e^{-\alpha w} = I_0(z) e^{-\alpha w} e^{-\frac{z^2}{w^2(z)}},$$  

(6)

with $P_0$ the initial power at $z = 0$. Then we substitute the ansatz (6) into (2) in order to find the perturbation $\phi$ corresponding to a given intensity distribution $I$. Using the Green’s function $\phi_{\text{ring}}$, we find that

$$\phi(r,z) = \int_0^\infty r' dr' \int_0^l I(r',z') \phi_{\text{ring}}(r,r',z,z') \,dz'.$$  

(7)
In Eq. (7), \( \phi \) is computed [with Eq. (5)] by evaluating a triple integral. The integration along \( r' \), expressed as \( F(k_r,w) = \int_0^{\infty} r' I(r',z') J_0(k_r r') dr' \), can be performed first: \( F \) is the Hankel transform of \( I \) and contains information on the spatial extension of the beam [in fact, all terms depending on the beam size \( w(z) \) are included in it]. When \( I \) is radially Gaussian as in Eq. (6), we obtain \( F(k_r,z,w) = (w^2/4) J_0(\exp(-\alpha z) \exp(-k_r^2 w^2/8)) \).

1. Constant width

The simplest case is when the beam size \( w(z) \) is constant along \( z \). The solution of Eq. (2) is

\[
\phi(r,z) = -\alpha \int_0^{\infty} \frac{J_0(k_r r)}{\sinh(k_r l)} \frac{H(k_r,z,F(k_r,z,w)}{dk_r, (8)
\]

where

\[
H = \left[ e^{-k_r l} e^{-(k_r - \alpha)z} - e^{(k_r - \alpha)z} \right] / 2(k_r - \alpha) - e^{-k_r l} \sinh(k_r z) \sinh(k_r z - l)
\]

The correctness of Eqs. (8) and (9) was verified by a direct comparison with the numerical solutions of Eq. (2) obtained via standard relaxation algorithms. We note that for large \( k_r \), it is \( H / \sinh(k_r l) \approx -\exp(-\alpha z)/k_r \), thus ensuring the convergence of the integral (8) for finite \( w \).

Figure 2 shows a few examples of the nonlinear refractive index well \( \phi \) computed from Eq. (8). The most striking difference from the local case \( \gamma = 0 \) is the role of input and output interfaces on \( \phi \), with the formation of a transition region between the boundary and the bulk; in fact, in the approximation \( \gamma = 0 \), it is implicitly assumed that light propagates in bulk, that is, without effects from the end faces along the propagation direction \( z \). For small absorption \( \alpha \), the light-induced perturbation is almost symmetric with respect to the transformation \( z \rightarrow z - l \), whereas increased losses lead to a marked asymmetry between input and output, as is physically apparent. As expected, \( \phi \) also depends on the beamwidth \( w \).

For narrow beams, the transition between the interfaces and the bulk region is much steeper than for wide beams: in fact, for small \( w \), the derivative along \( r \) is dominant with respect to \( z \), thus the longitudinal effects can be neglected over a wider region.

The role of the transverse \( I \) profile on \( \phi \) (i.e., on the beamwidth \( w \) if we consider a Gaussian input) can be addressed by looking at Eq. (8) or at Fig. 3, where \( H / \sinh(k_r r) \) is graphed. The auxiliary function \( H_{aux} = H(k_r,z/l) / \sinh(k_r l) \) vanishes on line \( k_r = 0 \), reaching a peak for a finite \( k_r \), and then monotonically decreases (Fig. 3). In \( z \), the function \( H_{aux} \) has a sine-like behavior for small absorption, but shows a strong peak close to the input interface for large losses [Figs. 3(a)–3(c)], in full analogy with the behavior of \( \phi \) in Fig. 2. It is noteworthy that the width of \( H_{aux} \) on the \( k_r \) axis becomes larger as \( \alpha \) increases. The net effect on \( \phi \) can be understood by examining Figs. 3(d) and 3(e). For wide beams, the Hankel transform \( F \) tends to a Dirac \( \delta \), so that the integral is sampled in \( k_r = 0 \); this implies a smaller perturbation \( \phi \) (compare the two rows in Fig. 2) and an effective cutoff for high spatial frequencies \( k_r \), thus inhibiting rapid variations of \( \phi \). Conversely, for narrow beams, the Hankel transform \( F \) acts as a weight function with a width inversely proportional to \( w \). Thus, the narrower the excitation \( I \), the larger the cutoff frequency in \( k_r \); thus \( \phi \) can vary on short spatial lengths, in agreement with Fig. 2.

The advantage of an integral solution such as Eq. (8) with respect to a direct numerical solution of Eq. (2) is the possibility of computing the perturbation \( \phi \) and its derivative in a given portion of space (in our case, the line \( r = 0 \)) without the need for a complete knowledge of \( \phi \). In fact, in the limit of high nonlocality, the two quantities required to compute soliton propagation are the perturbation peak \( \phi_0(z) = \phi(r = 0, z) \) and the Taylor-series coefficient \( \phi_2(z) = \partial^2 \phi / \partial r^2 (r = 0, z) \), with the latter giving the strength of the nonlinear refractive index well [16].

Having calculated the longitudinally nonlocal case \( \gamma = 1 \), the next step is to obtain the nonlinear perturbation \( \phi \) when \( \gamma = 0 \), i.e., for a longitudinally local nonlinearity. In this case, we need to fix a finite \( a \) to ensure a finite \( \phi \); we take...
a = Nw. Integrating Eq. (2) twice yields \( \phi_0 = \frac{\alpha P_0}{2\pi} T(N)e^{-\alpha z} \), with \( T(N) = \int_0^N dt (1 - e^{-2t^2})/t \): the peak of the perturbation depends on the boundary condition via the parameter \( N \), tending to infinity for \( N \) that is arbitrarily large. Furthermore, using the Taylor expansion around \( r = 0 \), we find [18]

\[
\phi_2 = \frac{\alpha P_0}{2\pi w^2} e^{-\alpha z}.
\]

Figure 4(a) shows the numerically computed \( \phi \) along \( z = 0 \) for \( \gamma = 0 \): as anticipated, the peak of the nonlinear perturbation \( \phi_0 \) is roughly proportional to \( N \). The inset compares the full profile and the approximation with a parabolic nonlinear index well: the accuracy worsens for \( r \approx w \), in agreement with Ref. [45]. We then compare the approximate solution for \( \gamma = 0 \) with the exact profile for \( \gamma = 1 \). The effect of the longitudinal nonlocality on \( \phi \) reduces as the width \( w/l \) becomes smaller: in this regime, in fact, the transverse derivative along \( r \) is dominant [Figs. 4(b) and 4(c)]. Importantly, the differences between the local and the nonlocal cases are bigger for the amplitude perturbation \( \phi_0 \) than for \( \phi_2 \).

In agreement with the Green’s function in Fig. 1, the case \( \gamma = 0 \) cannot describe the effects of input and output facets on the overall distribution of \( \phi \), whereas it faithfully reproduces the decay in \( z \) due to a nonzero \( \alpha \). Finally, Fig. 4(d) shows the behavior of the normalized maximum nonlinear perturbation \( \phi_0 \) versus normalized absorption: the smaller \( w \), the stronger the normalized nonlinear perturbation is, unlike the original model proposed by Snyder and Mitchell which assumes a nonlinear refractive index well depending on the beam power only [16].

2. Sinusoidal breathing

Next we consider the case of a breathing soliton, that is, with width \( w \) varying along \( z \). We assume for the width a sinusoidal behavior of the form

\[
w = w_m + \Delta w \sin \left( \frac{2\pi}{\Lambda} z \right),
\]

where \( w_m \) is the mean beamwidth, and \( \Delta w \) and \( \Lambda \) are the breathing oscillation amplitude and period, respectively. The peak of the perturbation \( \phi_0 \) versus \( z \) characterizes the nonlinear refractive index well written by a breathing soliton obeying Eq. (11). In order to quantify the effects due to width modulation, we define \( \Delta \phi_0 \) as the relative change in \( \phi_0 \) with respect to the case \( \Delta w = 0 \) (lack of breathing):

\[
\Delta \phi_0 (z, w_m, \Delta w, \Lambda) = \frac{\phi_0(z, w_m, \Delta w, \Lambda)}{\max\{\phi_0(z, w_m, \Delta w = 0, \Lambda)\}},
\]

where \( \max \) indicates the maximum computed along the \( z \) axis.

In the local case \( \gamma = 0 \), both \( \phi_0 \) and \( \phi_2 \) retain the same form as in the case of a \( z \)-independent beam, as the nonlinear perturbations \( \phi \) computed in different sections are independent from each other: in other words, the nonlinear perturbation follows the beamwidth oscillation, despite how large the amplitude \( \Delta w \) or how short the period \( \Lambda \) are.

Figure 5 shows \( \phi_0 \) versus \( z \), computed from Eq. (7) after setting \( r = 0 \): generally, \( \Delta \phi_0 \) mimics the sinusoidal behavior of the beamwidth \( w \), with a nonzero mean value owing to the nonlinear relationship between \( w \) and \( I \). Clearly, the perturbation of \( \phi \) depends on how \( w \) varies with \( z \): a faster modulation of \( w \) along \( z \) yields a slightly smaller perturbation \( \Delta \phi_0 \) due to the smoothing action of \( \phi_{\text{ring}} \) (compare first and second rows in Fig. 5). Moreover, the relative modulation of \( \Delta \phi_0 \) is proportional to \( \Delta w/w_m \) with good accuracy (compare first and third rows in Fig. 5): in fact, the intensity \( I \) is proportional to \( w^{-2} \), thus \( |\Delta I|/I \propto |\Delta w|/w_m \).

The quantity \( \phi_0 \) has a minor role in beam self-confinement: it produces a power-dependent modulation of the effective propagation constant of the self-trapped beam, with the latter having relevance only when the solitary wave interferes with a reference beam [47]. The parameter ruling the beamwidth is \( \phi_2 \), quantifying the strength of the self-induced index well.

If the field intensity is given by Eq. (6), then Eqs. (5) and (7) yield

\[
\phi_2(z) = \frac{\alpha P_0}{4\pi} \int_0^\infty \frac{k_r^2 dk_r}{\sinh(k_r l)} \int_0^1 g(z, z') e^{-\alpha z} e^{-\frac{|\Delta z|^2}{4}} dz',
\]

where

\[
g(z, z') = \frac{1}{\sqrt{2\pi D(z) z'}} e^{-D(z) z'^2}.
\]
where we set
\[ g(z,z') = \sinh(kr) \sinh[kr(z' - l)]u_0(z' - z) + \sinh(kr) \sinh[kr(z - l)]u_0(z - z'). \] (14)

The results computed from Eq. (13) are presented in Fig. 6. First, due to the linearity of Eq. (2), the parameter \( \phi_2 \) is inversely proportional to \( w_m^2 \). Nonetheless, the shape of \( \phi \) along \( z \) strongly depends on the ratio \( w_m/l \). For small \( w_m/l \), the periodic modulation of the nonlinear well is retained, approaching the local limit \( \gamma = 0 \) (black curves with asterisks in Fig. 6) as \( w_m/l \) reduces; conversely, for large \( w_m/l \), the oscillations in \( z \) are smoothed out. The size of \( \phi_2 \) increases with \( \Delta w/w_m \), similar to \( \phi_2 \); moreover, the spatial filtering of the longitudinal oscillations of \( \phi_2 \) changes slightly with \( \phi_2 \). The solutions of the system given by Eqs. (10) and (14) are plotted in Fig. 7. The solutions can be normalized by introducing the scaled radial coordinate \( R = k_0 r \), the scaled propagation constant \( N_{\text{eff}} = \beta/k_0 \), and the scaled power \( \mathcal{P} = \alpha P \). The solutions of the system given by Eqs. (1) and (2) strongly depend on the chosen radius \( r_0 \) for the integration domain: here we take \( r_0 = 50/k_0 \) and limit our considerations to normalized soliton widths \( W = w_0/k_0 \) of a few units, so that the finite boundaries do not affect the results [see Fig. 4(a)]. The soliton profiles across \( r \) are nearly Gaussian, but their width is \( \sqrt{2} \) larger than that stemming from the Snyder-Mitchell model [19], with the latter providing the edge of the response function used here, whereas the Snyder-Mitchell model assumes it is continuous and differentiable everywhere in space [19,45]. In summary, the width \( w_0 \) of the Gaussian best fit to the soliton profile is expressed by

\[ w_0(P) = \frac{1}{k_0^2 \alpha P} \sqrt{\frac{16\pi}{\gamma}}. \] (17)

**IV. NONLINEAR LIGHT PROPAGATION**

Up to now, we have calculated the nonlinear refractive index well determining the intensity distribution \( I \); next we numerically solve Eq. (1) together with the constitutive equation (2). The numerical algorithm to compute the wave-packet evolution is based on a finite-difference beam propagation method. In the longitudinal case \( \gamma = 1 \), we first find the beam evolution for a given distribution of the refractive index \( \phi \), then we substitute the corresponding intensity \( I \) into Eq. (2); this procedure is iterated until we achieve convergence. For \( \gamma = 0 \), the algorithm is slightly different: the beam intensity in a plane \( z = \text{const} \) is used to find the corresponding nonlinear refractive index well through Eq. (2). The latter well is then used to calculate the new field profile in the following section.
diffraction and self-focusing. For a parabolic index well, the beamwidth obeys \[ n_0 \frac{d^4 w^2}{dz^4} - 2 \phi_2 \frac{d^2 w^2}{dz^2} - 3 \frac{d \phi_2}{dz} \frac{dw^2}{dz} - \frac{d^2 \phi_2}{dz^2} w^2 = 0, \tag{18} \]
where \( \phi_2 \) generally depends on \( z \) due to losses (\( \alpha \neq 0 \)) or breathing.

When the changes in beamwidth are negligible and \( \alpha l \ll 1 \), according to Eq. (10) the variation of \( \phi_2 \) with \( z \) is negligible: in this limit, Eq. (18) is linear in the unknown variable \( w^2 \) and its solution is \[ \frac{w^2}{w_0^2} = 1 + \frac{w^4 - w_0^4}{2w_0^4} \left[ 1 - \cos \left( \sqrt{\frac{4|\phi_2(P_0)|}{n_0}} \right) \right]. \tag{19} \]
with \( \phi_2 \) computed for the soliton case, thus depending only on the input power \( P_0 \). In deriving Eq. (19), we set \( w(z = 0) = w_0 \) and assumed \( dw^2/dz = 0 \) at \( z = 0 \) (i.e., a flat phase profile at the input). In this limit, \( \phi_2 \) is fixed by the input power and takes the form \( \phi_2(P) = -\alpha P/(2\pi w_0^2) = -k_0^2(\alpha P/F)/(32\pi^2) \) [see Eqs. (10) and (17)]. The breathing period is then \( \Lambda = 4\sqrt{2n_0}\pi/|k_0\alpha P| \).

When \( \phi_2 \) changes slowly with \( z \) due to losses, Eq. (19) predicts adiabatic variations in both breathing amplitude (determined by the ratio between \( w_0 \) and \( w_z \), thus generally nonmonotonic with \( z \)) and period \( \Lambda \) (monotonically increasing with \( z \)).

When the derivatives of \( \phi_2 \) with respect to \( z \) cannot be neglected (\( w_0 \) appreciably differing from \( w_z \)), we can substitute Eq. (10)—valid away from the interfaces for narrow beams with \( \gamma = 1 \) (see Fig. 6)—into Eq. (18), yielding \[ n_0 \frac{d^4 w^2}{dz^4} + \frac{\alpha P_0 e^{-\alpha z}}{2\pi w_0^2} \frac{d^2 w^2}{dz^2} - \frac{\alpha P_0 e^{-\alpha z}}{2\pi w_0^4} \left( \frac{dw^2}{dz} \right)^2 = 0. \tag{20} \]
Equation (20) is highly nonlinear, in general with aperiodic solutions. To this extent and for the sake of simplicity, we neglect the last term in Eq. (20), i.e., we leave out very narrow solitons. Therefore, Eq. (20) becomes harmonic (with respect to the unknown \( d^2 w^2/dz^2 \)), but with an oscillation period proportional to the width \( w \). Changes in beamwidth imply variations in periodicity, in turn leading to a rather aperiodic breathing.

2. Numerical simulations

Hereby we validate our predictions on breathing by using the BPM. To verify how width oscillations depend on the soliton radius, we assume negligibly small losses, i.e., \( \alpha l \ll 1 \). The input beam shape corresponds to a solitary wave for a given power \( P_0^* \), numerically computed from Eqs. (15) and (16) and shown in Fig. 7. In order to study the width oscillations, the input power \( P_0 \) is set different from the value \( P_0^* \) corresponding to a shape-preserving solitary wave. Unless otherwise stated, we set \( w_0 = w_z(P_0^*) \) and let \( w_z(P_0) \) vary freely.

The dynamics obtained for \( \gamma = 0 \) and for \( \gamma = 1 \) is summarized in Fig. 9. For input powers below \( P_0^* \), the beam radius oscillates quasiperiodically around a mean larger than \( w_0 \) as \( w_z \) increases. As \( P_0 \) approaches \( P_0^* \), the breathing is nearly sinusoidal, in agreement with Eq. (19). If the power is further increased, the breathing becomes aperiodic, consistent with

\[ n_0 \frac{d^4 w^2}{dz^4} - 2 \phi_2 \frac{d^2 w^2}{dz^2} - 3 \frac{d \phi_2}{dz} \frac{dw^2}{dz} - \frac{d^2 \phi_2}{dz^2} w^2 = 0, \tag{18} \]
Eq. (20). As expected, the computed breathing period $\Lambda$ decreases with $P_0^\alpha$; similarly, the oscillation amplitude $A_{\text{osc}}$ (difference between maximum and minimum width values) tends to zero when the power approaches $P_0^\alpha$, but increases above and below $P_0^\alpha$. Note that the curve is not symmetric with respect to the soliton state as, for large powers, the beam cannot shrink indefinitely due to diffraction, with the latter assuming a dominant role for wavelength or subwavelength beam sizes.

For input powers above $3P_0^\alpha$, the BPM for $\gamma = 0$ yields a strongly aperiodic breathing, in agreement with Eq. (20). It is noteworthy that the BPM solution is stable when input noise is added. In the nonlocal case ($\gamma = 1$), for large powers ($P_0 > 2.56P_0^\gamma$) the code, based on an iterative scheme (and thus introducing different numerical noise at each step), does not converge; see Fig. 10. This can be ascribed to the highly nonlinear character of Eq. (20) for small beamwidths $w$, leading to chaotic dynamics. Chaotic dynamics can occur in periodically modulated waveguides, even in the linear regime [49], but in our case the longitudinal modulation is due to nonlinear effects. This interpretation is corroborated by the appearance of dips in the intensity profiles at $r = 0$, pinpointing the simultaneous excitations of several modes of the self-induced guide.

Next we focus on a more realistic excitation, a fundamental Gaussian beam launched in $z = 0$ with a planar phase front, varying both its initial waist $w$ and power $P_0$. Figure 11 graphs the beamwidth versus $z$ for various absorption coefficients and a fixed $P_0$. For small losses, the self-trapped beam is quasiperiodic, in agreement with Refs. [16,18] and Fig. 9. When the normalized loss reaches $\alpha l = 1$, the beam widens in $z$ due to its power dropping. For $\alpha l = 10$, the beam focuses near the entrance facet due to the stronger self-focusing associated with larger absorption. The position of the focus markedly depends on the input waist, moving away from the interface as the beams get wider. At the same time, the power decays rapidly and the nonlinear effects vanish at $z/l \approx 0.2$. At the output facet, the beam is wider than in the linear limit due to the initial self-focusing. In Fig. 11, the differences between the local ($\gamma = 0$) and nonlocal ($\gamma = 1$) cases are negligible due to the smallness of $w/l$, in agreement with Sec. III.

FIG. 9. (Color online) Beamwidth vs $z/l$ for powers $P_0$ differing from the soliton power $P_0^\alpha$ for (a) the local case [$P_0/P_0^\alpha$ equals 0.56 (blue line), 0.72 (cyan line), 1.21 (red line), 1.69 (yellow line), 4 (green line), and 9 (black line)] and (d) the nonlocal case [$P_0/P_0^\alpha$ is 0.56 (blue line), 0.72 (cyan line), 1.21 (red line), 1.69 (yellow line), 2.25 (green line), and 2.56 (black line)]; beam focusing increases with $P_0$ (top to bottom). Corresponding (b) oscillation period and (c) oscillation amplitude for varying $P_0$; symbols correspond to the excitations in (a)–(d). Solid lines with □ refer to the local case and dashed lines with △ refer to the nonlocal one. At high powers, in the local case the sinusoidal behavior is lost, whereas in the nonlocal case the solutions do not converge (not shown). Here, $\alpha l = 1 \times 10^{-15}$, $P_0 = 1.23 \times 10^9$, and $l/l = 0.02$.

FIG. 10. (Color online) Beam evolution at $r = 0$ vs $z/l$ at various steps of the iteration for the nonlocal case discussed in Fig. 9. Relaxation is achieved at low input powers (see results for $P_0/P_0^\alpha = 2.25$), whereas the code does not converge for high $P_0$ (see results for $P_0/P_0^\alpha = 3.24, 9$).

FIG. 11. (Color online) Beamwidth vs $z$ for a Gaussian input; each panel corresponds to a different input waist, as marked. The curves correspond to $\alpha P_0 = 0.01$ ($\alpha l = 1 \times 10^{-15}$), $\alpha P_0 = 0.03$ ($\alpha l = 1$), and $\alpha P_0 = 0.12$ ($\alpha l = 10$), respectively (higher $\alpha P_0$ correspond to stronger focusing). Blue solid and red dash-dotted lines refer to nonlocal and local cases, respectively. Gray dashed lines represent linearly diffracting beams. Here, $\lambda/l = 1 \times 10^{-3}$.
We now turn to small losses ($\alpha_l = 1 \times 10^{-15}$) in the limit $w/l \ll 1$ where, according to Fig. 6, the solutions for $\gamma = 0$ and $\gamma = 1$ should be similar. To this extent, we simulate the evolution of a Gaussian beam of width $w/l = 0.005$ with or without longitudinal nonlocality. Figure 12 shows the computed intensity profiles and the corresponding index of refraction distributions. As predicted by the Green’s function approach, the longitudinal nonlocality does not appreciably affect the propagation in the bulk of the sample. Nonetheless, as predicted in Fig. 4, the longitudinally local case does not correctly reproduce the nonlinear refractive index well near the input and output boundaries. Thus, for instance, if long-range soliton interaction is under investigation, the local model $\gamma = 0$ is expected to overestimate the mutual attraction between solitons [6,50].

The beam evolution is illustrated in Fig. 13. Similar to the results in Fig. 9, the oscillation period gets smaller as the power increases, whereas the oscillation amplitude is proportional to the mismatch between the input width $w$ and the soliton width $w_s(P_0)$ for a given power $P_0$. We also note that the beam initially shrinks when $w > w_s$ (self-focusing overtaking diffraction), whereas the opposite occurs when $w < w_s$ (diffraction dominant on the nonlinear lens) [16,18].

V. CONCLUSIONS

We investigated the full 3D evolution of highly nonlocal solitons possessing radial symmetry based on a nonlinearity governed by a Poisson equation. We addressed in detail the role played by the longitudinally nonlocal response, usually neglected in the literature, and the dependence of the nonlinear perturbation on the cell extension in the propagation direction. We detailed the role of longitudinal (nonlocality and attenuation) effects on the beam profile and evolution, including periodic and aperiodic solitary-wave breathing and power attenuation. At variance with the simple case of accessible solitons [16] which oscillate sinusoidally in width and amplitude, nonexact solitary beams launched at the input tend to breathe aperiodically for input parameters away from the soliton existence curve [18]. Such behavior can be qualitatively explained using Ehrenfest’s theorem and a quadratic nonlinear index well.

We showed that when the nonlinearity is of thermal origin, a trade-off between the magnitude of the refractive index well and the power decay versus propagation has to be met in order to maximize beam self-trapping. Moreover, the boundary conditions break the longitudinal symmetry of the system.

FIG. 12. (Color online) Beam evolution and corresponding index of refraction distribution for Gaussian inputs of width $w/l = 0.005$ in local (first three columns) and nonlocal (last three columns) regimes, respectively, for various powers $P_0$. Here, $\alpha_l = 1 \times 10^{-15}$ and $\lambda/l = 0.02$.

FIG. 13. (Color online) (a) Width evolution with breathing and (b) index of refraction distribution at the maximum beam amplitude for a Gaussian input and $\alpha P_0 = 0.01, 0.012, 0.023,$ and $0.036$ [from top to bottom in (a), in the reversed order in (b)], respectively. Solid lines correspond to the longitudinally nonlocal case and dash-dotted lines correspond to the local one, respectively. The dashed gray line in (a) corresponds to linear diffraction. Here, $\alpha_l = 1 \times 10^{-15}$ and $\lambda/l = 1 \times 10^{-3}$. 
and prevent truly shape-preserving self-trapped waves from existing, even without losses.

Our model, accounting for losses and 3D nonlocality in any nonlinear regime, is a promising workbench for studying nonlinear dynamics and the interplay of self-localization, losses, and nonlocality. In its numerical implementation, it is an ideal tool for the assessment of experimental results when investigating spatial optical solitons in (self-focusing) highly nonlocal media (including, e.g., thermal, reorientational, and liquid crystalline materials), including higher-order solitons [51,52].

Our findings can also find application in the use of Z-scan techniques [53] applied to thermal media, in the study of thermal effects in active media [54] and in the design of light-written waveguides [29], as well as for graded-index lenses in soft matter and nonlocal dielectrics.

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