Active TM mode envelope soliton propagation
in a nonlinear nematic waveguide

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Abstract

In this note the propagation of an envelope optical soliton in a nematic waveguide is examined. The waveguide is subject
to an elastic distortion and this distortion propagates as a front, which in turn influences the propagation of the soliton. It is
found that the re-orientation of the optical director can produce either defocusing or focusing of the soliton, thus influencing
its motion. Both approximate equations and numerical solutions show that a modulation in the defocusing optical regime can
catch up with the re-orientation front and then form an envelope soliton in the focusing region ahead of the front. Furthermore,
numerical solutions show that slow modulations in the defocusing region disintegrate into radiation. A limiting solution of a
coherent optical soliton locked onto the front also exists, but it is meta-stable and eventually disintegrates.
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1. Introduction

The theoretical possibility of the production of envelope solitons in nematic waveguides has been considered by
[1], and more recently by [2]. It was shown that such solitons are governed by the focusing nonlinear Schrödinger
(NLS) equation, with the cubic nonlinear term in this equation arising from the interaction between the optical
director and the wave field and the dispersive term arising from the linear modal dispersion introduced by the
waveguide thickness. In this work, the dynamics of the orientational field was neglected. The purpose of the present
work is to extend this previous work by deriving equations for the dynamics of the soliton which take into account
the elastic dynamics of the crystal and to derive and study relevant solutions of these equations.

The derivation of the equations governing the propagation of a soliton in a nematic will closely follow that of
[3]. In this derivation, it is assumed that the externally produced modulations are on the same time scale as the
re-orientation time. Under this assumption, the governing equations are the NLS equation coupled to the standard

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nonlinear diffusion equation for the re-orientation of the optical axis [1,3]. Clearly, the previous equations of [4] are obtained as a special case when the focusing steady solution for the orientational field is chosen.

In the present work, the situation of a waveguide with directors at its left end in the defocusing regime for the NLS equation is studied. At the right end, it is assumed that the directors are in the focusing regime for the NLS equation. This gradient in the directors’ direction produces a front of re-orientation which joins the defocusing to the focusing region. A finite amplitude modulation is then added to the left end of the waveguide. The evolution of this modulation is the main subject of the present work.

Approximate equations for the evolution of the input modulation will be derived using the method of [5] for the NLS equation. These approximate equations are then coupled to those for the dynamics of the front. The approximate equations give a simple qualitative explanation for the motion of the modulation and the front. In addition, the approximate equations show that a rapidly travelling modulation in the defocusing region overtakes the front and focuses to form a soliton. These predictions of the approximate equations are confirmed by full numerical solutions of the governing equations. The approximate and numerical solutions are found to be in good agreement for the pulse position, but only good qualitative agreement for the pulse amplitude evolution. The reasons for the disagreement between the approximate and numerical solutions for the pulse amplitude are discussed. Numerical solutions of the governing equations show the formation of two solitons in the focusing region and meta-stable solutions which show the concentration of the electric field at the distortion front. The physical parameter ranges over which such solutions can occur are shown to be accessible to experimental verification.

The rest of the paper is organised as follows. Section 2 deals with the formulation and simplifications leading to the governing equations. In Section 3, the approximate evolution equations are derived and their qualitative behaviour studied. Section 4 is devoted to the numerical study of the governing equations and the comparison of the numerical and approximate solutions. In Section 5, conclusions are presented and the advantages and limitations of the results are discussed.

2. Formulation

Let us consider a nematic waveguide as in [3]. The axis of the cell is orientated in the $x$ direction with the thickness of the cell transverse to this in the $z$ direction (see Fig. 1). The thickness of the cell is $2\ell$ and its length is $L$. The length of the cell is taken to be much larger than its width, so that $L \gg \ell$. It is further assumed that the optical axis re-orientates in the $(x, z)$ plane only and that the orientational angle $\theta$ is measured from the $z$-axis. The permittivity

![Fig. 1. Schematics of a homeotropically aligned liquid crystal film in the presence of a TM mode.](image)
tensor is given by [3]

\[
\begin{align*}
\epsilon_{xx} &= \epsilon_0 \epsilon_{\perp} \left( 1 + \frac{\epsilon_a}{\epsilon_{\perp}} \sin 2\theta \right), \\
\epsilon_{xz} &= \epsilon_0 \epsilon_a \sin \theta \cos \theta, \\
\epsilon_{zz} &= \epsilon_0 \epsilon_{\perp} \left( 1 + \frac{\epsilon_a}{\epsilon_{\perp}} \cos^2 \theta \right), \\
\epsilon_{yy} &= \epsilon_0 \epsilon_{\perp}, \\
\epsilon_{yz} &= \epsilon_{zy} = \epsilon_{yx} = \epsilon_{xy} = 0.
\end{align*}
\]  

(1)

The re-orientational dynamics of the liquid crystal are then described by the diffusion equation

\[
\frac{\partial \theta}{\partial t_0} = \nu \frac{\partial^2 \theta}{\partial x^2} + \nu \frac{\partial^2 \theta}{\partial z^2} + q_e \frac{\epsilon_a}{\epsilon_{\perp}} \left[ (|E_z|^2 - |E_x|^2) \sin 2\theta + (E_x E_z^* + E_y E_z) \cos 2\theta \right] + q_s \sin 2(\theta + \Psi).
\]  

(2)

Here \( t_0 \) is the non-dimensional time, which is related to the physical time by

\[
t_0 = \frac{\nu \kappa}{\gamma \epsilon_{\perp}^2},
\]  

(3)

where \( \nu \kappa \) is the elastic constant of the crystal and \( \gamma \) is the mobility. Here \( \epsilon_a = \epsilon_{||} - \epsilon_{\perp} \). The fields in the diffusion equation (2) are non-dimensional and

\[
q_e = \frac{\epsilon_a \epsilon_{\perp} \epsilon_{||} \ell^2 E_0^2}{\kappa}
\]  

(4)

is the ratio of the electric energy density to the elastic field energy. The boundary condition for the diffusion equation (2) is the strong anchoring homeotropic boundary condition

\[
\theta(x, \pm \ell, t) = 0.
\]  

(5)

The \( q_s \) term in the diffusion equation (2) is due to the static applied field, which provides the bi-stability [3].

The equations for the optical field are in terms of the \( y \) component \( H_y \) of the magnetic field since it is assumed that there is a TM mode in the waveguide. The ratio of the electric time scale

\[
t_f = \frac{\ell}{\sqrt{\epsilon_{\perp} \epsilon_{||}} c}
\]  

(6)

to the elastic time scale

\[
t_e = \frac{\gamma \ell^2}{\kappa}
\]  

(7)

is small, so that

\[
\alpha = \frac{t_f}{t_e} \ll 1.
\]  

(8)

Here \( c \) is the speed of light in the waveguide. It was then shown in [3] that Maxwell’s equations simplify to

\[
\epsilon_{\perp} \epsilon_{||} \frac{\ell^2}{c^2} \frac{\partial^2 H_y}{\partial t^2} - \left\{ \frac{\partial}{\partial x} \left( \epsilon_{xx} \frac{\partial H_y}{\partial x} + \epsilon_{xz} \frac{\partial H_y}{\partial z} \right) + \frac{\partial}{\partial z} \left( \epsilon_{zz} \frac{\partial H_y}{\partial z} + \epsilon_{xz} \frac{\partial H_y}{\partial x} \right) \right\} = 0.
\]  

(9)

The boundary conditions for \( H_y \) are

\[
\frac{1}{\epsilon_{||}} \frac{\partial H_y}{\partial z} = 0
\]  

(10)

at the ends of the waveguide.

One final reduction can be achieved by expanding in the \( z \) modes. The \( z \) modes for the magnetic field \( H_y \) are \( \cos n \pi z \), while the modes for the optical director are \( \cos (n - 1/2) \pi z \). It was argued in [3] that due to the fast
relaxation time of the high modes, only the first mode is needed in the modal expansion for the optical angle $\theta$. It was further shown by [3] that the coupling between the optical modes is weak. Therefore the modal expansions can be approximated by

$$H_y = \cos \pi z H_1^y(x,t), \quad \theta = \cos \frac{\pi z}{2} \theta_1(x,t_0). \quad (11)$$

The modal approximations (11) are now substituted into the diffusion equation (2). Averaging in the $z$ direction we obtain

$$\frac{\partial \theta}{\partial t_0} = \nu \frac{\partial^2 \theta}{\partial x^2} - \frac{\pi^2}{4} \theta + \frac{\pi^3}{4} q_{e\theta} |H_1^y|^2 \sin \theta + \frac{\pi^3}{4} q_s \sin 2(\theta + \Psi)$$

$$+ \frac{\pi^2}{4} \left( 1 + \frac{\epsilon_a}{\epsilon_{\perp}} \cos^2 \theta \right) \theta_1(x,t_0), \quad (12)$$

$$\frac{\ell^2 \epsilon_{\perp}}{c^2} \frac{\partial^2 H_1^y}{\partial t^2} = \frac{\partial}{\partial x} \left( \frac{\epsilon_{\perp}}{4} + \frac{\epsilon_a}{4} \sin^2 \theta \right) \frac{\partial H_1^y}{\partial x} - \epsilon_0 \frac{\pi^2}{4} \left( 1 + \frac{\epsilon_a}{\epsilon_{\perp}} \cos^2 \theta \right) H_1^y$$

on replacing $\theta_1$ by $\theta$ for simplicity.

The approximate modal equations (12) and (13) recover, in the case of a monochromatic wave, the corresponding equations obtained in [3] for the problem of induced transparency. It should be noted that equations (12) and (13) for the elastic and magnetic properties contain very different time scales. Because of this difference, it is clearly not possible to find a solitary wave solution in which the director plays a role. However, when a slow modulation, on the $t_e$ time scale, is produced, it will be shown that it evolves on the $t_e$ time scale according to an NLS equation coupled to the equation for the optical director.

In the present work, we are interested in crystal configurations for which $\sin^2 \theta \ll \cos^2 \theta$. In this limit the magnetic equation (13) reduces to the simpler equation

$$\frac{\ell^2 \epsilon_{\perp}}{c^2} \frac{\partial^2 H_1^y}{\partial t^2} + \epsilon_0 \frac{\pi^2}{4} \left( 1 - \frac{\epsilon_a}{2\epsilon_{\perp}} \cos^2 \theta \right) H_1^y - \frac{\partial}{\partial x} \epsilon_{\perp} \frac{\partial H_1^y}{\partial x} = 0. \quad (14)$$

A modulation solution for $H_1^y$ is now sought in the form

$$H_1^y = A \left( \alpha^{1/2} \left( x - \frac{c_g t}{t_0} \right), \frac{\alpha t}{t_0} \right) e^{i(kx - \omega(k)t/t_0)}, \quad (15)$$

where $\alpha$ is defined in (8). The envelope $A$ is thus assumed to be slowly varying. Substituting this modulation solution into Eq. (14) for the magnetic field, we find that the carrier wave satisfies the dispersion relation

$$\omega^2 = k^2 + n^2 \epsilon_{\perp} \left( 1 - \frac{\epsilon_a}{2\epsilon_{\perp}} \right). \quad (16)$$

The slowly varying envelope $A$ takes into account the modulations induced by the slow term $\epsilon_a \cos \theta/(2\epsilon_{\perp})$ in the magnetic field equation (14).

Using the modulation form (15), a standard multiple scales analysis on the magnetic field equation (14) gives the NLS-type equation

$$i \frac{\partial A}{\partial t_0} + \frac{\alpha}{2} \omega'(k) \frac{\partial^2 A}{\partial x^2} + i \alpha^{1/2} c_g \frac{\partial A}{\partial x} + \frac{\pi^2}{2} \epsilon_a \cos 2\theta A = 0. \quad (17)$$

The group velocity $c_g$ is $c_g = \omega'(k)$. We therefore have that the diffusion equation (12) and the NLS-type equation (17) describe the evolution of the envelope soliton in the waveguide. We note that (17) is a nonlinear equation as $A$ depends on $\theta$, the solution of (12). The diffusion equation (12) for the director can be written in the form

$$\frac{\partial \theta}{\partial t_0} = \nu \frac{\partial^2 \theta}{\partial x^2} - \frac{\pi^2}{4} \theta + \frac{\pi^3}{4} q_s \sin 2(\theta + \Psi) + \frac{\pi^3}{4} q_{e\theta} |A|^2 \sin 2\theta.$$  

(18)
One last simplification will now be made. This simplification is obtained on assuming that \( \theta + \Psi \) is small, so that only small deformations from the orientation of the static field are considered. Under this assumption the terms

\[
F(\Psi) = \frac{\pi^2}{4} \theta - \frac{\pi^3}{4} q_s \sin(2(\theta + \Psi))
\]

in the diffusion equation (18) can be approximated by expanding \( \sin(2(\theta + \Psi)) \) up to cubic terms [1]. Once this is done, it is possible to adjust the parameters \( q_s \) and \( \Psi \) to obtain two stable steady states \( \theta_3 = \frac{\pi}{6} \) and \( \theta_2 = -\frac{\pi}{6} \) for the diffusion equation (18). We therefore have that the diffusion equation (18) reduces to

\[
\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} - \mu (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3) + \frac{\pi^3}{4} q_s \epsilon_a |A|^2 \sin 2\theta,
\]

where

\[
\mu = \frac{\pi^3}{3} q_s.
\]

Finally, on taking \( q_s |A|^2 \) small, the director equation reduces to

\[
\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} - \mu \left[ \theta - \tilde{\theta}_1(|A|^2) \right] \left[ \theta - \tilde{\theta}_2(|A|^2) \right] \left[ \theta - \tilde{\theta}_3(|A|^2) \right],
\]

where the roots \( \tilde{\theta}_1, \tilde{\theta}_2 \) and \( \tilde{\theta}_3 \) are the first-order corrections in \( |A|^2 \) to \( \theta_1, \theta_2 \) and \( \theta_3 \). These roots are given by

\[
\tilde{\theta}_1 = -\frac{\pi}{12} + 5\beta |A|^2, \quad \tilde{\theta}_2 = -\frac{\pi}{6} + 3.6\beta |A|^2, \quad \tilde{\theta}_3 = \frac{\pi}{6} + 1.2\beta |A|^2.
\]

The coefficient \( \beta = q_s \epsilon_a \) is free. It will be taken to be small, \( \beta \approx 0.1 \).

In previous work [2,4], the standard NLS equation was obtained from the NLS-type equation (17) by taking \( \theta \) to be one of the steady states \( \theta_i(|A|^2) \) of the director equation (12). Depending on the value of \( \cos(2\theta_i(|A|^2)) \), either the focusing or defocusing NLS equation is obtained. In the present work, the solution for \( \theta \) is not steady.

To derive the NLS equation in this unsteady case, it is convenient to take

\[
\theta = \varphi + \theta_1(|A|^2).
\]

Since \( |A| \) is small, in the NLS-type equation (17) we have

\[
\cos 2\theta A = \cos 2(\varphi + \theta_1|A|^2)A = \cos \left( 2\varphi + \frac{\pi}{6} \right) A - 10\beta |A|^2 \sin \left( 2\varphi + \frac{\pi}{6} \right).
\]

With this expansion, the NLS-type equation becomes

\[
\frac{1}{i\alpha} \frac{\partial A}{\partial t} + i c_s \alpha^{1/2} \frac{\partial A}{\partial x} + \frac{1}{2} \frac{\partial^2 A}{\partial x^2} + \rho \cos \left( 2\varphi + \frac{\pi}{6} \right) A - \lambda \sin \left( 2\varphi + \frac{\pi}{6} \right) |A|^2 A = 0,
\]

where \( \lambda = 10\beta \) and

\[
\rho = \frac{\pi^2}{2} \epsilon_a.
\]

The equation for the optical axis takes the form

\[
\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} - \mu \varphi(\varphi - (\tilde{\theta}_2 - \tilde{\theta}_1))(\varphi - (\tilde{\theta}_3 - \tilde{\theta}_1)).
\]
The final set of equations to be studied in the present work consists of Eq. (26) for the magnetic field and Eq. (28) for the optical axis. The axis equation possesses front solutions which are layer-type solutions obtained by joining the stable steady states \( \tilde{\theta}_3 - \tilde{\theta}_1 \) and \( \tilde{\theta}_2 - \tilde{\theta}_1 \). From [1], the asymptotic solution of the axis equation (28) is then a moving layer of the form

\[
\varphi = \frac{1}{2} (\tilde{\theta}_3 - \tilde{\theta}_2) \left[ 1 - \frac{\tanh a(x - \xi(t))}{2} \right] + (\tilde{\theta}_2 - \tilde{\theta}_1),
\]

where

\[
a = \sqrt{2} \left( \tilde{\theta}_3 - \tilde{\theta}_2 \right) \sqrt{2}.
\]

The front moves according to the ordinary differential equation

\[
\dot{\xi} = \frac{1}{\sqrt{2}} \left( \tilde{\theta}_3 + \tilde{\theta}_2 - 2\tilde{\theta}_1 \right),
\]

which is evaluated at the front position \( x = \xi(t) \). This approximate equation for the motion of the front is then given in terms of \(|A|\) evaluated at the front position.

3. Approximate equations for soliton

To obtain approximate equations for the evolution of the optical modulation, the method of Kath and Smyth [5] for the NLS equation is used. As in this work, an approximate solution for the NLS equation (26) is taken in the form

\[
A = \eta \sech \frac{x - y}{w} e^{i(\sigma + V(x - y))} + \frac{g(\sigma + V(x - y))}{w}.
\]

(32)

Here the parameters \( \eta, w, y, \sigma, V \) and \( g \) are functions of time \( t \). These parameters need to be determined so that (32) is a good approximation to the solution of the NLS equation (26). In this regard, we note that in the focusing region, the approximate solution (32) is the soliton solution of the NLS equation provided that \( g = 0 \) and \( \eta \) and \( w \) satisfy the soliton amplitude–width relation

\[
\eta w \sqrt{\lambda \sin \left( \frac{2\varphi + \frac{\pi}{6}}{2} \right)} = 1.
\]

(33)

On the other hand, in the defocusing region (32) is not a solution of the NLS equation (26). Indeed, in the defocusing region the NLS equation has no coherent structure as a solution. Since we are interested in the study of a front for which \( \theta \to \theta_3 \) as \( x \to -\infty \) (the defocusing region) and \( \theta \to \theta_2 \) as \( x \to \infty \) (the focusing region), we take the approximate solution (32) in all regions, where the tildes on the \( \theta_i \) have been dropped from now on for simplicity. It is expected that when \( V \gg 1 \) the pulse will overtake the front and focus. In this case, it is expected that (32) will be a good approximation to the solution.

The approximate solution (32) consists of two parts. As discussed above, the first term is a varying soliton-like pulse. As the pulse evolves, it sheds dispersive radiation. The second term in (32) represents the long wavelength radiation under the pulse [5]. This shelf forms as the dispersion relation for the linear NLS equation for (26) shows that low wavenumber waves have low velocity relative to the pulse. This shelf of radiation under the evolving pulse can be clearly seen in Fig. 2(b). The form of the radiation shed from the vicinity of the pulse will be considered later. The shelf of radiation under the pulse cannot remain independent of \( x \) as otherwise the mass would be infinite. Hence the shelf term \( g \) in the approximate solution (32) is assumed to be non-zero for \( y - \epsilon/2 \leq x \leq \)
Fig. 2. Case for which one soliton forms in focusing region. Initial conditions are $\eta = 1.0$, $w = 1.0$, $\zeta = 0$, $y = -5.0$, $c_g = 1.0$ and $V = 10.0$. The parameter values are $\nu = 0.1$, $\alpha = 0.1$, $\rho = 0.1$, $\lambda = 1.0$ and $\beta = 0.1$. (a) (—) Initial pulse; (---) initial $\theta$. (b) Numerical solution at $t = 20$. (—) Pulse; (---) $\theta$. (c) Pulse amplitude $\eta$. (—) Numerical solution; (---) approximate solution. (d) Pulse position $y$. (—) Numerical solution; (---) approximate solution.

$y + \ell/2$. The value of $\ell$ will be determined later. The radiation is of small amplitude relative to the pulse, so that $|g| \ll \eta$.

Equations for the parameters for the approximate solution (32) are obtained via the Lagrangian for the NLS equation (32). This Lagrangian is

$$L = i(A^* A_t - AA_t^*) + ic_g \alpha^{1/2}(A^* A_x - AA_x^*) - 2\rho \cos 2\theta |A|^2 - |A_x|^2 - \lambda \sin 2\theta |A|^4,$$

(34)
where $\theta = \varphi + \pi/3$. Substituting the approximate solution (32) into the Lagrangian (34) and integrating over $x$ from $-\infty$ to $\infty$, we obtain the averaged Lagrangian

$$
\mathcal{L} = -2(\sigma' - V:\eta' (2\eta^2 w + \ell g^2) - 2\pi \eta w g' + 2\pi g w \eta' + 2\pi g \eta w' - 2c_g \alpha^{1/2} V (2\eta^2 w + \ell g^2)$$

$$- \frac{2\eta^2}{3w} - (2\eta^2 w + \ell g^2) V^2 - 2\rho \eta^2 w \left[ \cos 2\theta_3 \left( 1 + \tanh \frac{\zeta - y}{w} \right) + \cos 2\theta_2 \left( 1 - \tanh \frac{\zeta - y}{w} \right) \right]$$

$$- \lambda \eta^4 w \left[ \sin 2\theta_3 \left( \frac{2}{3} + \tanh \frac{\zeta - y}{w} - \frac{1}{3} \tanh \frac{3(\zeta - y)}{w} \right) + \sin 2\theta_2 \left( \frac{2}{3} - \tanh \frac{\zeta - y}{w} + \frac{1}{3} \tanh \frac{3(\zeta - y)}{w} \right) \right],$$

(35)
to \(O(g^2)\). Equations for the pulse parameters are now obtained from variations of this averaged Lagrangian. These equations are

\[
(\eta w)' = \frac{\ell g}{\pi} \left( \sigma' - V y' + \frac{1}{2} V^2 + c_g \alpha^{1/2} V \right),
\]

\[
g' = \frac{2\eta}{3\pi w} + \frac{\rho \eta}{\pi} \left[ \cos 2\theta_3 - \cos 2\theta_2 \right] \frac{\xi - y}{w} \text{sech}^2 \left( \frac{\xi - y}{w} \right) + \frac{\lambda \eta^3}{2\pi} \left[ \sin 2\theta_3 \left( \frac{2}{3} + T - \frac{1}{3} T^3 \right) \right.
\]

\[
+ \sin 2\theta_2 \left( \frac{2}{3} - T + \frac{1}{3} T^3 \right) \left( \sin 2\theta_3 - \sin 2\theta_2 \right) \frac{\xi - y}{w} \text{sech}^4 \left( \frac{\xi - y}{w} \right) \right],
\]

\[
(36)
\]

\[
\sigma' - V y' = -\frac{1}{2} V^2 - c_g \alpha^{1/2} V - \frac{1}{2 w^2} \rho \left[ \cos 2\theta_3 (1 + T) + \cos 2\theta_2 (1 - T) + \cos 2\theta_3 - \cos 2\theta_2 \right] \frac{\xi - y}{w} \text{sech}^2 \left( \frac{\xi - y}{w} \right)
\]

\[
- \frac{\lambda \eta^3}{4} \left[ 3 \sin 2\theta_3 \left( \frac{2}{3} + T - \frac{1}{3} T^3 \right) + 3 \sin 2\theta_2 \left( \frac{2}{3} - T + \frac{1}{3} T^3 \right) \right.
\]

\[
+ \left. (\sin 2\theta_3 - \sin 2\theta_2) \frac{\xi - y}{w} \text{sech}^4 \left( \frac{\xi - y}{w} \right) \right],
\]

\[
(37)
\]

\[
y' = c_g \alpha^{1/2} + V
\]

\[
(38)
\]

plus the mass and momentum equations

\[
\frac{d}{dt}(2\eta^2 w + \ell g^2) = 0,
\]

\[
(40)
\]

\[
\frac{d}{dt}[2(2\eta^2 w + \ell g^2)V] = 2\rho \eta^2 \text{sech}^2 \left( \frac{\xi - y}{w} \right) (\cos 2\theta_3 - \cos 2\theta_2) + \lambda \eta^4 \text{sech}^4 \left( \frac{\xi - y}{w} \right) (\sin 2\theta_3 - \sin 2\theta_2),
\]

\[
(41)
\]

respectively, where

\[
T = \tanh \left( \frac{\xi - y}{w} \right).
\]

\[
(42)
\]

These variational equations are supplemented by the front equation (31) with

\[
|A|^2 = \eta^2 \text{sech}^2 \left( \frac{\xi - y}{w} \right).
\]

\[
(43)
\]

The final quantity to be determined is the length \(\ell\) of the shelf under the pulse. This length was calculated in [5] from the requirement that the period of oscillation of the variational equations (36)–(41) near their soliton fixed point (in the focusing region)

\[
\hat{\eta}^2 \hat{w}^2 \lambda |\sin 2\theta_2| = 1
\]

matches the soliton oscillation frequency

\[
\frac{1}{2} \lambda \hat{\eta}^2 |\sin 2\theta_2| - \rho \cos 2\theta_2.
\]

\[
(44)
\]

It can be found that this requirement gives

\[
\ell = \frac{3\pi^2 \lambda |\sin 2\theta_2| \hat{\eta}^2 - 2\rho \cos 2\theta_2}{8 \frac{\lambda^{3/2} |\sin 2\theta_2|^{3/2} \hat{\eta}^3}}.
\]

\[
(46)
\]
This expression agrees with that of [5] in the special case for which the $\cos 2\theta_2$ term is missing and $\lambda = 1/|\sin 2\theta_2|$. The calculation of $\dot{\eta}$ will be considered after the following discussion of the method for incorporating loss from the pulse to dispersive radiation.

The variational equations (36)–(41), while including the shelf of low wavenumber radiation under the pulse, do not include the dispersive radiation shed by the pulse as it evolves. Since the shed dispersive radiation is of small amplitude relative to the pulse, it is governed by the linearised form of the NLS equation (26)

$$i \frac{\partial A}{\partial t} + ic_g \alpha^{1/2} \frac{\partial A}{\partial x} + \frac{1}{2} \frac{\partial^2 A}{\partial x^2} + \rho \cos 2\theta |A|^2 = 0.$$  \hspace{1cm} (47)

Kath and Smyth [5] determined the radiation loss for the standard NLS equation for which $c_g = 0$ and $\rho = 0$ by solving

$$i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0$$

using Laplace transforms. Their results for the radiation loss can be transferred to the present NLS equation when $\rho \neq 0$ and $c_g \neq 0$ via the transformation

$$A = u(x - c_g \alpha^{1/2} t) e^{i \rho \cos 2\theta t}.$$  \hspace{1cm} (49)

Using this transformation and the results of [5], we find that the inclusion of loss to shed dispersive radiation modifies the variational equations in the following manner. Eq. (37) for $g$ becomes

$$g' = \frac{2\eta}{3\pi w^2} + \frac{\rho \eta}{\pi} \left[ \cos 2\theta_3 - \cos 2\theta_2 \right] \frac{\xi - y}{w} \text{sech}^2 \frac{\xi - y}{w} + \frac{\lambda \eta^3}{2\pi} \left[ \sin 2\theta_3 \left( \frac{2}{3} + T - \frac{1}{3} T^3 \right) \right.

\left. + \sin 2\theta_2 \left( \frac{2}{3} - T + \frac{1}{3} T^3 \right) \right] + \left( \sin 2\theta_3 - \sin 2\theta_2 \right) \frac{\xi - y}{w} \text{sech}^4 \frac{\xi - y}{w} - 2\delta g,$$

\hspace{1cm} (50)

where

$$\delta = \frac{3\dot{\eta} \sqrt{\lambda |\sin 2\theta_2|}}{8r(t_0) \sqrt{\pi} t}.$$  \hspace{1cm} (51)

The variable $r(t)$ is a measure of the amplitude of the shed dispersive radiation and is given by

$$r^2 = \frac{3\dot{\eta} \sqrt{\lambda |\sin 2\theta_2|}}{8} \left( 2\eta^2 w - \frac{2\dot{\eta}}{\sqrt{\lambda |\sin 2\theta_2|}} + \xi g^2 \right)$$

\hspace{1cm} (52)

with $r(t_0)$ being the value of $r$ at some initial time $t_0$. This time $t_0$ is chosen to be the time at which the pulse passes the front, so that $y > \xi$, and the pulse is in the focusing region for which the radiation loss in (50) holds. The radiative loss is switched on at this time as in the defocusing region there is no adequate theory for the radiative collapse of a pulse.

The parameter $\dot{\eta}$ is the final steady soliton amplitude as $y \to \infty$. This amplitude is determined from the energy conservation equation for the NLS equation (26). It can be shown directly from the NLS equation (26) that the equation for conservation of energy is

$$\frac{1}{2} \frac{\partial}{\partial t} \left( |A|^2 + \lambda \sin 2\theta |A|^4 \right) + \frac{\partial}{\partial x} \left[ ic_g \alpha^{1/2} |A|^2 + \frac{1}{2} (A_x^* A_{xx} - A_x A_{xx}^*) + \lambda \sin 2\theta (ic_g \alpha^{1/2} |A|^4 + |A|^2 A_x^* A_x - |A|^2 A_x^* A_x) \right.$$

$$\left. - |A|^2 A_x^* + \rho \cos 2\theta (A A_x - A_x A) - i\lambda c_g \alpha^{1/2} \sin 2\theta |A|^4 \right]$$

$$= \rho \cos 2\theta \frac{\partial}{\partial x} (A A_x^* - A_x A) - i\lambda c_g \alpha^{1/2} \sin 2\theta \frac{\partial}{\partial x} |A|^2.$$  \hspace{1cm} (53)
Substituting the approximate solution (32) into this energy equation and integrating from $x = -\infty$ to $x = \infty$ gives the energy conservation equation

\[
\frac{dE}{dt} = \frac{d}{dr} \left[ \frac{2\eta^2}{3w} + (2\eta^2 w + \ell g^2) V^2 + \lambda \eta^4 w \left( \sin 2\theta_3 \left( \frac{2}{3} T - \frac{1}{3} T^3 \right) + \sin 2\theta_2 \left( \frac{2}{3} T + \frac{1}{3} T^3 \right) \right) \right]
\]

\[
= 4\rho \left( \cos 2\theta_3 - \cos 2\theta_2 \right) V \eta^2 \sech^2 \frac{\zeta - y}{w} - \lambda \left( \sin 2\theta_3 - \sin 2\theta_2 \right) c_g a^{1/2} \eta^4 \sech^4 \frac{\zeta - y}{w}.
\]  

(54)

Then using the soliton fixed point relation (44) between $\hat{\eta}$ and $\hat{w}$, we find that as $y \to \infty$, the fixed point is given by

\[
\hat{\eta}^3 = -\frac{3E}{2\sqrt{\lambda} \sin 2\theta_2}.
\]  

(55)

This expression was used to calculate $\hat{\eta}$ for all values of $y$.

As the energy conservation equation was used to calculate the soliton fixed point, the variational equation (40) for conservation of mass was replaced by the energy conservation equation (54), as was done by [5]. In this regard, we note that the variational equations (36)–(41) and the energy equation (54) are not independent, as shown by Nöther’s theorem. The final set of approximate equations for the pulse evolution used in the present work are the variational equations (36), (38) and (39), the energy equation (41), the energy equation (54) and Eq. (50) for $g$, which includes mass loss to dispersive radiation.

In principle the variational equations (36)–(41) together with the front equation (31) give a complete description of the evolution of the pulse. However, to understand the evolution of the pulse and front in simpler terms, we shall first consider a simple special case. If we take the amplitude $\eta$ and width $w$ of the pulse to be constant and ignore the shelf of radiation, so that $g = 0$, we see from the variational equations (36)–(41) and the front equation (31) that the evolution of the pulse is then governed by

\[
\frac{dy}{dt} = c_g a^{1/2} + V,
\]  

(56)

\[
4\eta^2 w \frac{dV}{dt} = 2\rho \eta^2 \sech^2 \frac{\zeta - y}{w} \left( \cos 2\theta_3 - \cos 2\theta_2 \right) + \lambda \eta^4 \sech^4 \frac{\zeta - y}{w} \left( \sin 2\theta_3 - \sin 2\theta_2 \right),
\]  

(57)

\[
\frac{d\zeta}{dt} = \frac{1}{\sqrt{2}} \left( \frac{\pi}{6} - 5.2 \beta \eta^2 \sech^2 \frac{\zeta - y}{w} \right).
\]  

(58)

Eqs. (56) and (57) for the motion of the centre of mass $y$ of the pulse can be easily examined on assuming $\dot{\zeta} \ll V$. In this limit, these equations can be examined in a slowly varying phase plane with $\zeta$ given as a slowly varying function of time. The fixed points in the phase plane are the solutions of

\[
\sech^2 \frac{\zeta - y}{w} = -\frac{2\rho}{\lambda \eta^2} \cos 2\theta_3 - \cos 2\theta_2 \frac{\sin 2\theta_3 - \sin 2\theta_2}{2} \leq 0.
\]  

(59)

This shows that there are no fixed points $y \approx \zeta$. This is due to the fact that when $\theta_3 > 0$ satisfies $\theta_3 > \theta_2$ the medium is defocusing in the region behind the transition of the re-orientation front. This behaviour is very different from the focusing case discussed in [6] for a fixed front.

The preceding analysis also shows that a soliton cannot bounce back from the front. The non-existence of fixed points also indicates the impossibility of the locking of a soliton to the transition layer. It is also expected that the pulse will disintegrate into radiation if its velocity is comparable to that of the front. These various cases will be considered in more detail in the next section.
Fig. 3. Case for which the initial pulse disintegrates. Initial conditions are $\eta = 1.0$, $w = 1.0$, $\xi = 0$, $y = -5.0$, $c_g = 1.0$ and $V = 0.1$. The parameter values are $\nu = 0.1$, $\alpha = 0.1$, $\rho = 0.1$, $\lambda = 1.0$ and $\beta = 0.1$. (a) (---) Initial pulse; (---) initial $\theta$. (b) Numerical solution at $t = 50$. (---) Pulse; (---) $\theta$. (c) Numerical solution at $t = 80$. (---) Pulse; (---) $\theta$. (d) Numerical solution at $t = 120$. (---) Pulse; (---) $\theta$. (e) Pulse amplitude $\eta$. (---) Numerical solution; (---) approximate solution.

4. Approximate and numerical solutions

The approximate equations (31), (36), (38), (39), (41), (50) and (54) were solved using a fourth order Runge–Kutta scheme. The numerical solution of the diffusion equation (28) was obtained using the Crank–Nicolson scheme, while the numerical solution of the NLS equation (26) was obtained using a pseudo-spectral method similar to that of [7] with the $t$ integration carried out in Fourier space using a fourth order Runge–Kutta scheme.
The first case we shall consider is that of a pulse starting in the defocusing region at $y = -5.0$ with $\eta = 1.0$, $w = 1.0$ and $V = 10.0$ at $t = 0$. This initial condition is shown in Fig. 2(a) and the numerical solution at $t = 20$ is shown in Fig. 2(b), at which time the pulse is in the focusing region. It can be seen that only a single soliton forms in the focusing region and that the front has undergone very little deformation or movement. It can also be seen that the optical axis undergoes a slight re-orientation as the soliton passes. Fig. 2(c) shows a comparison between the time evolution of the amplitude $\eta$ of the pulse as given by the numerical and approximate solutions. Finally, Fig. 2(d) shows a comparison between the position $y$ of the pulse maximum as given by the numerical and approximate solutions. It can be seen that the agreement for the pulse position is excellent, while that for the pulse amplitude is only fair. From Fig. 2(d) it is clear that there is very little change in the pulse velocity. The high initial velocity of the
pulse means that it rapidly passes through the front and so its velocity is not greatly affected by it. The major cause of the difference between the numerical and approximate pulse amplitudes is the initial sharp drop in the numerical pulse amplitude which is not reflected in the approximate amplitude. The reason for this is that the approximate equations of the previous section are not an adequate description of the pulse evolution in the defocusing region as there is no coherent structure for the defocusing NLS equation. However, when the pulse passes into the focusing region, the numerical and approximate amplitudes become closer. If the pulse is started too far from the front or with too small an initial velocity, then the approximate and numerical solutions are in poor agreement as the pulse spends too long in the defocusing region.

The next case considered was that for the initial condition shown in Fig. 3 (a). In this case, the initial pulse velocity $V = 0.1$ was chosen to be close to the front velocity and the pulse disintegrates into radiation in the focusing region. Fig. 3(b)–(d) show the evolution of the pulse and the optical axis, at times $t = 50$, $t = 80$ and $t = 120$. It can be seen that as well as the disintegrating pulse in the focusing region, there is also a disintegrating reflected pulse in the defocusing region. The evolution of the pulse amplitude as given by the numerical and approximate solutions is shown in Fig. 3(e). The approximate equations predict that the pulse rapidly disintegrates with the pulse amplitude vanishing at about $t = 1.2$. This is not in quantitative agreement with the numerical solution, which gives a much slower decay of the pulse amplitude. However, good quantitative agreement is not expected as the pulse spends a large amount of time in the defocusing region where the approximate equations are not a good approximation to the pulse dynamics. As a general comment, the solutions shown in Fig. 3 confirm the qualitative picture obtained in the previous section from the simplified approximate equations in that the pulse cannot lock onto the front, but instead disintegrates into radiation.

The final case considered was that in which an initial pulse splits into two solitons in the focusing region. Fig. 4(a) shows the initial condition for this case, while Fig. 4(b) shows the solution at $t = 20$, at which time the pulse has propagated into the focusing region and has split into two pulses. The two solitons in the focusing region can be clearly seen. It can also be seen that there is a small wave in the defocusing region which has been reflected from the front and that there is again a re-orientation of the optical axis as the solitons pass. Fig. 4(c) shows a
Fig. 4. Case for which two solitons form in focusing region. Initial conditions are $\eta = 1.0$, $w = 2.0$, $\xi = 0$, $y = -10.0$, $c_g = 1.0$ and $V = 10.0$. The parameter values are $\nu = 0.1$, $\alpha = 0.1$, $\rho = 0.1$, $\lambda = 1.0$ and $\beta = 0.1$. (a) (---) Initial pulse; (- - -) initial $\theta$. (b) Numerical solution at $t = 20$. (---) Pulse; (- - -) $\theta$. (c) Pulse amplitude $\eta$. (---) Numerical solution; (- - -) approximate solution. (d) Pulse position $y$. (---) Numerical solution; (- - -) approximate solution.

Comparison between the numerical and approximate amplitudes of the largest pulse. The agreement is good, which could be considered surprising as the initial pulse has broken up into multiple pulses. However, it has been previously found that the approximate method of Section 3 can accurately predict the amplitude of the largest pulse when an initial pulse breaks up into multiple pulses [8]. Again the main difference between the numerical and approximate amplitudes is that the initial drop in the numerical amplitude is not reflected in the approximate amplitude. This is again due to the approximate solution (32) not being an accurate description of the pulse in the defocusing region. The main reason that there is better agreement between the numerical and approximate amplitudes in Fig. 4(c)
than in Fig. 2(c) is that the pulse has been started closer to the front at $y = -5.0$. Fig. 4(d) shows a comparison between the pulse position as given by the full numerical and approximate solutions. It can be seen that there is good agreement between the two solutions, except for large times where the two positions start to deviate due to a velocity difference between the numerical and approximate solutions.

5. Conclusions

The possibility for the focusing of a localised optical modulation into a soliton due to the basic mechanical distortion of the optical director of a liquid crystal has been shown. The modulation approach of the present work
gives a relatively simple understanding of the focusing to form a soliton and the propagation of the pulse in the distorted medium.

References