Formation of One-dimensional Waves for Resonant Flow in a Channel

N. F. Smyth

Abstract—When a forcing moves in a shallow channel at a velocity near the phase velocity for linear long waves, energy cannot escape from the forcing at the linear group velocity and nonlinear effects become important in describing the resulting flow. This flow is termed resonant or transcritical. It has been found both experimentally and numerically that large amplitude upstream propagating waves are generated by the forcing. These waves are straight crested, even though the forcing is two-dimensional. It is shown that these upstream waves become straight crested due to geometrical effects aided by the presence of side walls. Using energy conservation, approximate values of the amplitude of the upstream waves are obtained which are compared with recent experimental and numerical results.

Key words: Solitons, resonant flow, shock dynamics, transcritical flow, nonlinear waves, KdV theory.

1. Introduction

The study of the wavefield produced by the flow of a fluid over topography (or equivalently, the motion of a pressure field in, or on the surface of, a fluid) is a classical problem in linear water wave theory (Stoker, 1957). However, this linear wavefield is singular in the limit of the Froude number approaching one as in this limit, the linear group velocity of the waves equals the flow velocity and hence energy cannot escape from the topographic forcing. Linear theory then predicts that the amplitude of the waves grows, so that nonlinear effects become important and, indeed, dominant in the resulting flow. Experimental studies by Thews and Landweber (1935), Graff (1962), Schmidt-Stiebitz (1966), Huang et al. (1982), Ertekin (1984) and Ertekin et al. (1984) of forcings (such as ship models) moving in shallow channels near Froude number one have found that large amplitude waves are generated ahead of the forcing which propagate away from the forcing. These waves have the remarkable property, even though the forcing is two-dimensional, of becoming straight crested after leaving the region of the

1 Department of Mathematics, University of Wollongong, P.O. Box 1144, Wollongong, N.S.W., 2500, Australia.
forcing. Furthermore, these upstream waves are continuously generated and no steady state is reached upstream. Waves are also generated downstream of the forcing; however, these downstream waves remain highly two-dimensional in contrast to the upstream waves. Here one-dimensional means that the waves depend on the along-channel coordinate alone and two-dimensional means that they depend on both the cross-channel and along-channel coordinates. A distinction is made between the dimension of the wavefield and the dimension of the corresponding flow, the dimension of the flow always being one more than the dimension of the wavefield. This is because the wavefield is a surface phenomenon and its dimension refers to the number of spatial coordinates on which it depends. A straight crested wavefield which spans the channel is then referred to as one-dimensional even though the fluid flow is two-dimensional. Similar behaviour was also found in numerical studies by Ertekin et al. (1986) using the Green-Naghdi equations, Katsis and Akylas (1987) using the forced Kadomtsev-Petviashvili equation and Wu and Wu (1987) and Pedersen (1988) using Boussinesq type equations. Mei (1986) and Mei and Choi (1987) considered resonant flow in a channel for channel widths and forcing shapes for which the entire wavefield is essentially one-dimensional. This flow, characterised by large, unsteady upstream disturbances, has been termed resonant or transcritical. It is the mechanism by which these upstream waves become straight crested that is the central topic of the current work. This mechanism also explains why the downstream waves remain fully two-dimensional.

The formation of straight crested waves upstream of the forcing will be shown to be a consequence of geometrical shock dynamics as applied to solitons. Geometrical shock dynamics was developed by Whitham (1957, 1959, 1974) and is an extension of geometric optics to nonlinear wave propagation. Under this theory, a shock is propagated along rays normal to the shockfront at a local Mach number $M$, which is related to the local ray tube area $A$ by the so-called $A - M$ relation

$$A = A_0 \frac{f(M)}{f(M_0)},$$

(1.1)

where $A_0$ and $M_0$ are the initial ray tube area and Mach number, respectively. The $A - M$ relation is derived from the gas equations plus the shock jump conditions by considering a shock propagating in a slowly varying tube (Whitham, 1957, 1974). The idea of propagating a wave along rays normal to the wavefront at a local speed is not restricted to shock waves and has been applied by Miles (1977) to solitons. It was shown by Grimshaw and Smyth (1986) and Smyth (1987) that for resonant flow due to a one-dimensional forcing, which is governed by a forced Korteweg-de Vries equation, the upstream waves can be approximated by solitons for Froude numbers sufficiently near one (i.e., they are cnoidal waves with modulus near 1). The upstream waves for three-dimensional flow will then be approximated by solitons and geometrical shock dynamics will be used to study their development. To apply geometrical shock dynamics to solitons, the equivalent of the $A - M$
relation for shocks is needed. This relation, giving the local soliton speed in terms of the local ray tube area, was derived using energy conservation for the Korteweg-de Vries equation by OSTROVSKY (1976). By equating the initial energy of an upstream soliton to its energy when it is straight, approximate values of the steady amplitude of the upstream waves are obtained. These values are found to be in reasonable agreement with experimental and numerical results.

2. Geometrical Shock Dynamics Applied to Solitons

The solution for two-dimensional resonant flow will now be summarised as some details of this solution will be used to discuss three-dimensional resonant flow. Let us consider the surface waves produced by the flow of an inviscid, irrotational, incompressible fluid over a localised one-dimensional topography in the weakly nonlinear, long wave limit. All flow quantities and variables will be non-dimensionalised by a lengthscale $h$, $h$ being the depth of the fluid away from the topography, a timescale $(hg^{-1})^{1/2}$, $g$ being the acceleration due to gravity and the density $\rho_0$ of the fluid. The flow is described by a horizontal coordinate $X$, a vertical coordinate $z$ and a time $T$. The vertical surface displacement is denoted by $\eta$. The bottom topography is given by

$$z = \delta g(x), \quad (2.1)$$

where

$$x = \varepsilon X. \quad (2.2)$$

Here $\delta \ll 1$ is a parameter measuring the height of the topography and $\varepsilon \ll 1$ is a parameter whose inverse measures the horizontal lengthscale of the topography. Also for the topography to be localised, we assume $g \to 0$ as $x \to \pm \infty$.

There is an imposed upstream flow $V$ in the positive $X$ direction, which we shall assume is such that $V \approx 1$, so that the Froude number is near one and the flow is resonant (or transcritical). It has been shown by AKYLAS (1984), COLE (1985), LEE (1985) and GRIMSHAW and SMYTH (1986) that this resonant flow is described by the forced Korteweg-de Vries equation

$$-u_t - \Delta u_x + 6u u_x + u_{xxx} + G_x(x) = 0, \quad (2.3)$$

where

$$\eta = \frac{3}{5} \sqrt{\delta u}$$

$$G = \frac{9}{5} g$$

$$V = 1 + \frac{1}{5} \sqrt{\delta \Delta}$$

$$t = \frac{1}{5} \varepsilon \sqrt{\delta T}. \quad (2.4)$$
Also to balance nonlinear and dispersive effects, we set $\epsilon^2 = \delta^{1/2}$. The parameter $\Delta$ in (2.3) is a detuning parameter measuring how close the flow is to exact linear resonance; as $\Delta$ decreases, the flow becomes subcritical and as $\Delta$ increases, the flow becomes supercritical. The initial condition

$$u(x, 0) = 0,$$

(2.5)
corresponding to switching on the topography at $t = 0$, is used. Equation (2.3) also describes the flow produced by a pressure distribution moving at Froude number near one (see AKYLAS, 1984; LEE, 1985), with $G$ being related to the non-dimensional pressure distribution $\delta p(x)$ by (2.4b) with $g(x)$ replaced by $p(x)$. For a pressure distribution forcing, $\delta$ is a measure of the amplitude of the pressure forcing with pressure being non-dimensionalised by $\rho_0 gh$.

SMYTH (1987) found a solution of (2.3) using the modulation theory for the Korteweg-de Vries equation developed by WHITHAM (1965, 1974). The details of this solution which will be used in the present work will now be summarised. The solution of (2.3) consists of three regions; an upstream modulated cnoidal wave of modulus squared varying between $m = 1$ at its leading edge to $m = m_0$ at the forcing, an essentially flat depression behind the forcing and a modulated cnoidal wave following this depression of modulus squared $m$ varying from 1 at its trailing edge to 0 at its leading edge, this modulated cnoidal wave bringing the solution back to $u = 0$. The modulus $m_0$ is chosen so that the waves generated at the forcing propagate upstream. In the limit of a broad forcing, the amplitude $a$ of the waves at the leading edge of the upstream wavetrain, which are essentially solitons as $m = 1$ there, is

$$a = \frac{\Delta + \sqrt{12 g_0}}{3 \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right]},$$

(2.6)

where $m_0$ is the solution of

$$0 = 3\Delta \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right] - (\Delta + \sqrt{12 g_0})$$

$$\times \left[ 1 + m_0 - \frac{2 m_0 (1 - m_0) K(m_0)}{E(m_0) - (1 - m_0) K(m_0)} \right],$$

(2.7)

$g_0$ being the amplitude of the forcing. $K(m_0)$ and $E(m_0)$ are complete elliptic integrals of the first and second kind of modulus squared $m_0$, respectively. In the limit of a $\delta$-function forcing, $G = g_0 \delta(x)$, this amplitude is given by

$$a = \frac{\Delta + 3 g_0^{2/3}}{3 \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right]},$$

(2.8)
where \( m_0 \) is the solution of

\[
0 = 3\Delta \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right] - (\Delta + 3 \frac{g_0^{2/3}}{2}) \times \left[ 1 + m_0 - \frac{2m_0(1 - m_0)K(m_0)}{E(m_0) - (1 - m_0)K(m_0)} \right].
\]

(2.9)

For \( \Delta = 0 \) (i.e., exact linear resonance), it can be found from (2.7) and (2.9) that \( m_0 = 0.64 \). Hence for \( \Delta \) near 0, the modulus of the waves in the upstream wavetrain is near 1 and so the upstream wavetrain is essentially a train of solitons and will be approximated as such in the work that follows. As \( \Delta \) decreases from zero, \( m_0 \) decreases to zero and the approximation of the upstream wavetrain by a train of solitons becomes less useful. For further details of this solution, the reader is referred to SMYTH (1987).

The propagation of the upstream waves in three-dimensional resonant flow will be discussed using geometrical shock dynamics applied to solitons. WHITHAM (1957, 1959) developed, by analogy with geometric optics, an approximate theory of shock propagation. In this theory, called geometrical shock dynamics, the shock propagates on rays normal to the shockfront, with the local speed of propagation depending on the local ray tube area. In two dimensions, this theory is most conveniently expressed in terms of an intrinsic orthogonal curvilinear coordinate system \((\alpha, \beta)\), where successive shock positions are described by curves \( \alpha = \text{constant} \) and rays by \( \beta = \text{constant} \). The shockfront is described by its local Mach number \( M(\alpha, \beta) \) and its ray inclination angle \( \theta(\alpha, \beta) \), referred to the \( x \)-axis say. It then follows from geometrical considerations (WHITHAM, 1974) that the system of equations describing the propagation of the shockfront is

\[
\frac{\partial \theta}{\partial \beta} - \frac{A'(M)}{M} \frac{\partial M}{\partial \alpha} = 0
\]

\[
\frac{\partial \theta}{\partial \alpha} + \frac{1}{A(M)} \frac{\partial M}{\partial \beta} = 0.
\]

(2.10)

The function \( A(M) \), the so-called \( A - M \) relation, incorporates the dynamics of the problem. This function is a relation between the local ray tube area \( A \) and the local Mach number \( M \). WHITHAM found this relation by considering a shock propagating in a slowly varying shock tube. The system of equations (2.10) is hyperbolic and represents a wave motion for disturbances propagating on the shock. Since (2.10) forms a system of nonlinear hyperbolic equations, shocks can develop in the shockfront, these being called shock-shocks. Shock-shocks correspond physically to the occurrence of Mach reflection, the shock-shock discontinuity being the position of the Mach triple point. Geometrical shock dynamics ignores the reflected shock and the contact discontinuity (vortex sheet) associated with Mach reflection.
The ideas of geometrical shock dynamics can be applied to nonlinear wave phenomena other than shock waves. In the present work, it will be applied to solitary waves, as was done by MILES (1977). Under the geometrical shock dynamics approximation, the propagation of two-dimensional solitary waves is governed by equations (2.10) with a new $A - M$ relation indicating how the speed of a soliton travelling in a channel (ray tube) varies with the breadth of the channel (ray tube). OSTROVSKY (1976) showed that the speed $M$ of a soliton in a slowly varying channel varies with the breadth $b$ of the channel as

$$\frac{M}{M_0} = \left(\frac{b}{b_0}\right)^{-2/3},$$

(2.11)

where $M_0$ and $b_0$ are initial values of the speed and channel breadth, respectively. This result is obtained from the energy conservation equation for the Korteweg-de Vries equation for motion in a slowly varying channel. The geometrical shock dynamics equations governing the motion of a two-dimensional soliton are then (2.10) with $A(M)$ replaced by

$$b(M) = b_0 \left(\frac{M_0}{M}\right)^{3/2}.$$  

(2.12)

For geometrical shock dynamics, the channel width $b$ is identified with the ray tube area.

The initial condition to be used for the propagation in a channel of a soliton generated by a topographic or pressure distribution forcing is a straight soliton spanning part of the channel. This initial soliton is an approximation to a wave generated by the forcing. Unfortunately, for such an initial condition, the system (2.10) cannot be solved analytically and numerical methods must be used. The numerical method to be used is that of HENSHAW et al. (1986) for geometrical shock dynamics. This method will be briefly described here and for a full discussion, the reader is referred to HENSHAW et al.

Let the soliton be discretised by $N$ points $x_j(t), j = 1, \ldots, N$ and let $M_i(t)$ and $n_i(t)$ be approximations to the soliton speed and normal respectively at the point $x_i(t)$. Then as the soliton propagates along its normals at the speed $M$,

$$\frac{dx_j}{dt} = M_i(t)n_i(t), \quad i = 1, \ldots, N.$$  

(2.13)

This system of equations is equivalent to (2.10). The system (2.13) is discretised using the leap-frog scheme

$$x_i(t + \Delta t) = x_i(t - \Delta t) + 2\Delta tM_i(t)n_i(t), \quad i = 2, \ldots, N - 1,$$

(2.14)

where $\Delta t$ is the time step. This scheme is explicit and second order accurate in time. The normal $n_i(t)$ is determined by differentiating two cubic splines fitted to the data.
formed by $(s_j(t), x_j(t))$ and $(s_j(t), y_j(t))$, $j = 1, \ldots, N$. Here $s_j(t)$ is the discrete arclength given by

\begin{equation}
    s_j(t) = \begin{cases} 
        0, & j = 1 \\
        s_{j-1}(t) + \left| x_j(t) - x_{j-1}(t) \right|, & j = 2, \ldots, N. 
    \end{cases}
\end{equation}

(2.15)

If we let $\tilde{x}(s)$ and $\tilde{y}(s)$ denote the cubic spline interpolants, then the curve $(\tilde{x}(s), \tilde{y}(s))$ is an approximation to the front at time $t$. The normal is then

\begin{equation}
    n_i(t) = \frac{(\tilde{y}'(s_i), -\tilde{x}'(s_i))}{\left[ (\tilde{x}'(s_i))^2 + (\tilde{y}'(s_i))^2 \right]^{1/2}}, \quad i = 1, \ldots, N,
\end{equation}

(2.16)

where primes denote differentiation with respect to $s$. Also, the soliton speed $M_i(t)$ is determined by (2.11) with

\begin{equation}
    b_i(t) = \frac{1}{2} (s_{i+1}(t) - s_{i-1}(t)), \quad i = 2, \ldots, N - 1.
\end{equation}

(2.17)

The boundary conditions on the channel walls are satisfied by constraining the points $x_1$ and $x_N$ to propagate along the channel boundaries. Their positions on the boundaries are chosen so that the line joining $x_1$ and $x_2$ and the line joining $x_{N-1}$ to $x_N$ are perpendicular to their respective boundaries. This ensures that the soliton moves normal to the channel boundaries and hence that the fluid velocity normal to the boundaries is zero. The use of this boundary condition is supported by the numerical solutions of Ertekin et al. (1986), Katsis and Akylas (1987) and Pedersen (1988) which show the soliton moving normal to the channel boundaries.

To maintain an adequate resolution of the soliton front and to fit in shock-shocks (Mach reflection), points are inserted in expansive regions and deleted in compressive regions of the front. To do this, the point spacing $\Delta s_i(t)$ is checked periodically and we require

\begin{equation}
    d < \frac{\Delta s_i(t)}{\Delta s_{\text{avg}}} = \sigma_i(t) \leq D, \quad (2.18)
\end{equation}

where $d < 1$ and $D > 1$. If $\sigma_i(t) < d$, the point $x_i(t)$ is removed and if $\sigma_i(t) > D$, a new point $x_{i-1/2}(t)$ is added using the cubic spline interpolants evaluated at $\frac{1}{2} (s_i(t) + s_{i-1}(t))$. The area ratio $b_i(t)/b_i(0)$ in (2.11) is also preserved by removing or adding the points at $t = 0$ as well. The addition of points maintains the soliton front resolution and the deletion of points fits a Mach triple point (shock-shock) into the soliton front. A shock-shock indicates the occurrence of Mach reflection.

To dampen high frequency errors in $x_i(t)$, a simple two-step smoothing procedure is added to the numerical scheme. After every $n_s$ time steps (usually about 20 to 50), we let

\begin{equation}
    \frac{1}{2} (x_{i+1}(t) + x_{i-1}(t)) \rightarrow x_i(t), \quad (2.19)
\end{equation}

where we scan $i$ even, then $i$ odd for $1 < i < N$. The numerical scheme is then restarted using the smoother front as the initial condition.
To discuss the propagation of one of the upstream waves for resonant flow in a channel using geometrical shock dynamics, an initial soliton must be specified. As an approximation to the initial upstream wave, we shall assume that each point of the forcing initially generates a one-dimensional soliton whose amplitude at each point is given by the solution of Smyth (1987) for two-dimensional resonant flow. This initial soliton then evolves according to geometrical shock dynamics. Since the upstream waves are generated by the forcing, it is reasonable to take the initial wave to be along the spanwise length of the forcing. It is less clear that the initial wave should be straight, especially since not all of the wave need be generated simultaneously. In the absence of any information as to the exact form of the initially generated soliton, the simplest assumption is that it is straight.

Ertekin et al. (1986) considered the flow produced by a rectangular pressure distribution moving at Froude number one in a shallow channel. The flow was found by numerically solving the Green-Naghdi equations. The forcing used has a height to length ratio of about 0.21, so it can be considered to be a broad forcing. The flow is described using an $x$ coordinate along the centreline of the channel and a $y$ coordinate transverse to the channel. In the normalisation of (2.3), the forcing used by Ertekin et al. has

$$g_0(y) = \begin{cases} \frac{3}{2} \bar{P} & 0 \leq |y| < 0.8 \\ \frac{3}{2} \bar{P} \cos^2\frac{\pi(|y| - 0.8)}{2.4} & 0.8 \leq |y| \leq 2 \\ 0 & 2 < |y| < 4, \end{cases}$$

(2.20)

where $\bar{P} = 1$. Using (2.6), the initial amplitude $a(y)$ of the soliton can be found from this expression for $g_0(y)$. Figure 1 shows the subsequent evolution of the soliton in the channel. It can be seen that the soliton rapidly spans the channel and becomes straight. At the channel boundaries, discontinuities (shock-shocks) in the soliton front form, which reflect off the walls and each other, becoming progressively weaker as time increases. These discontinuities correspond to the occurrence of Mach reflection of the soliton at the boundaries. This Mach reflection was also noted by Katsis and Akylas (1987) and Pedersen (1988) from their numerical solutions. Katsis and Akylas also used a soliton symmetrically spanning half the
channel as an initial condition and found that it became straight after spanning the entire channel.

Geometrical shock dynamics ignores any reflected waves, so that error could occur due to any reflected wave associated with the Mach reflection. However, it was noted by Katsis and Akylas (1987) and Pedersen (1988) from their numerical solutions that there appeared to be no reflected wave associated with the Mach reflection at the boundaries. While Mach reflection at the boundaries facilitates the formation of a straight crested soliton, it is not necessary for a straight soliton to eventually form. From (2.11), it can be seen that as the ray tube area $b$ increases, the soliton slows down and as the ray tube area decreases, it speeds up. Hence any curved soliton will eventually become straight crested due to this geometrical effect alone. This tendency of the initially curved soliton generated by the forcing to become straight even in an unbounded domain is supported by the work of Lee and Grimshaw (1990). In this work the forced Kadomtsev-Petviashvili equation was solved numerically for flow in an infinite domain. A slowly varying similarity solution of the Kadomtsev-Petviashvili equation was also found which corresponded to a parabolic crested soliton which became straight as time approached infinity. The waves upstream of the forcing in the numerical solution were found to approach this similarity solution for large time. This approach is slow since as well as straightening, the soliton is also spreading in the transverse direction.

### 3. Energy Conservation

Since the area-speed relation (2.11) for a soliton is derived from energy conservation for the Korteweg-de Vries equation for motion in a slowly varying channel (Ostrovsky, 1976), the amplitude of the final straight crested soliton can be easily found from the energy of the initial soliton. The soliton solution of (2.3) with $G \equiv 0$ is

$$u = a \text{sech}^2\left(\frac{a}{2}\right)^{1/2} (x + (2a - \Delta)t),$$

(3.1)

$a$ being the amplitude of the soliton. The energy per unit length of a one-dimensional soliton is then

$$E = \int_{-\infty}^{\infty} u^2 \, dx = \frac{4}{3} \sqrt{2} a^{3/2}.$$  

(3.2)

Since the initial soliton of Section 2 has an amplitude which varies across the channel, its energy is given by

$$E = \frac{4}{3} \sqrt{2} \int_{-B}^{B} a^{3/2}(y) \, dy,$$

(3.3)
where the channel is from $y = -B$ to $y = B$. If we let the amplitude of the final straight crested soliton be $A$, we have, on equating the energy of the initial soliton (3.3) to the energy of the final straight crested soliton, given by (3.2) times $2B$,

$$A^{3/2} = (2B)^{-1} \int_{-B}^{B} a^{3/2}(y) \, dy. \quad (3.4)$$

As the area-speed relation (2.11) was derived from energy conservation, the question naturally arises as to mass conservation. A soliton propagating in a slowly varying channel does not conserve mass. As the soliton propagates, a lengthening tail or shelf forms behind it which contains the mass shed by the soliton (see Knickerbocker and Newell, 1980 and Smyth, 1984). In the present situation, there is actually a train of solitons upstream of the forcing. The tail or shelf links these individual waves together, so that they do not entirely propagate independently as there is mass exchange between them. There is, however, no energy exchange between them under the present approximation.

An approximation to the initial soliton is found as in Section 2 and the amplitude $A$ of the straight crested soliton is found from (3.4) on using $a(y)$ as given by the solution of Smyth (1987). For most forcing distributions, the integral in (3.4) cannot be evaluated exactly. This integral is evaluated numerically using the trapezoidal rule in the examples of this section, with $m_0$ found from (2.7) or (2.9) using the secant method.

Ertekin et al. (1986) numerically solved the Green-Naghdi equations in a shallow channel with a broad rectangular pressure distribution as the forcing. The amplitude $g_0(y)$ of this forcing is given by (2.20). The amplitude $A$ of the final straight crested soliton can then be found from (3.4) on using (2.6) and (2.7). Table 1 shows a comparison of the values of $A$ calculated using (3.4) and the corresponding numerical results of Ertekin et al., where the values $\bar{P} = 1, B = 4$ and $\delta = 0.3$ have been used. The agreement is reasonable. Since there are only a few values for comparison, the discussion of these results will be delayed so that they can be discussed in conjunction with other numerical results.

Katsis and Akylas (1987) numerically solved the forced KdV equation in a channel with a pressure distribution forcing that

<table>
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<th>$\Delta$</th>
<th>$A$</th>
<th>Numerical amplitude</th>
</tr>
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<tr>
<td>-1.0954</td>
<td>1.09</td>
<td>1.40</td>
</tr>
<tr>
<td>0.0</td>
<td>1.37</td>
<td>1.71</td>
</tr>
<tr>
<td>1.0954</td>
<td>1.66</td>
<td>2.12</td>
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approximated a two-dimensional δ-function. In the present notation and scaling, this forcing has

\[ g_0(y) = \frac{3\pi}{8} \delta(y). \]  

Katsis and Akylas used \( \Delta = 0 \), so that from (2.9), \( m_0 = 0.64 \). The initial amplitude \( a(y) \) is then found from (2.8) and the integral (3.4) gives

\[ A = \left( 3\pi \right)^{2/3} \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right]^{-1} B^{-2/3} \approx 1.2159 B^{-2/3}. \]  

In Figure 2, the amplitude relation (3.6) is compared with some of the numerical results of Katsis and Akylas (1987). It can be seen that the comparison is quite good, considering the approximations made, with the difference between the values of \( A \) increasing as \( B \) decreases. Decreasing \( B \) corresponds to increasing amplitude and a partial explanation of why the approximate values of \( A \) become less accurate as the amplitude increases will be considered below.

Pedersen (1988) numerically solved a set of forced Boussinesq type equations for flow in a shallow channel for Froude numbers near one. The forcing used was

\[ g = \left( \frac{3\pi}{16} \right)^{2/3} \left[ m_0 - 1 + \frac{2E(m_0)}{K(m_0)} \right]^{-1} B^{-2/3} \approx 1.2159 B^{-2/3}. \]  

\[ 1 \]

Figure 2
Comparison of the values of \( A \) from (3.4), ---, with the numerical results of Katsis and Akylas (1987), *symbol*.
Figure 3
Comparison of values of $A$ from (3.4), —, with the numerical results of Pedersen (1988), symbol. (a) $B = 20, R = 5, p = 2.25$. (b) $B = 40, R = 10, p = 4.5$. 
Comparison of values of $A$ from (3.4), ---, with experimental results of ERTEKIN et al. (1984), symbol. Forcing is (3.7) with $p = 5.1615$, $R = 10$, $B = 24.404$.

A pressure distribution, which in the present scaling corresponds to

$$g_0(y) = \begin{cases} 
\frac{1}{2} p \left( 1 + \cos \frac{\pi |y|}{R} \right), & 0 \leq |y| < R \\
0, & R \leq |y| \leq B 
\end{cases}$$

(3.7)

The detuning parameter $\Delta$ is related to the Froude number $F$ by

$$F = 1 + \frac{(0.1)^{1/2} \Delta}{6}$$

(3.8)

on using (2.4c). Using this value of $g_0(y)$, the initial amplitude of the soliton is then obtained from (2.6) and (2.7) and thence the final amplitude $A$ is found from the integral (3.4). Figures 3(a) and (b) show a comparison of the values of $A$ given by (3.4) with some of the results of PEDERSEN as a function of $\Delta$, the detuning parameter. Again the comparison is reasonable for $\Delta$ near zero, but the difference between the values of $A$ increases as $\Delta$ increases (or amplitude increases). This increasing difference as the amplitude increases was also seen in the comparison with the results of KATSIS and AKYLAS (1987). Unfortunately, no results were reported by PEDERSEN for $\Delta$ negative, for which $A$ is smaller and better agreement is expected.
The final set of results which will be compared with the integral (3.4) are the experimental results of Ertekin et al. (1984). Their forcing, a pressure distribution (ship model), in the present scaling is equivalent to (3.7) with \( p = 5.1615 \), \( R = 10 \) and \( B = 24.404 \). With these values, the blockage coefficient of the ship model and (3.7) are the same. The amplitude \( A \) of the straight crested soliton is then obtained from (3.4) on using (2.6), (2.7) and (3.7). Figure 4 shows a comparison of these values of \( A \) with some of the experimental results of Ertekin et al. as a function of \( \Delta \). It can be seen that the comparison is quite good for \( \Delta \) negative and near zero, but becomes less good as \( \Delta \) increases from zero. This again corresponds to increasing disagreement as the amplitude becomes larger.

To summarise the above comparisons between the values of \( A \) from (3.4) and the various numerical and experimental results, agreement is reasonable to quite good, but becomes less good as the amplitude \( A \) increases. \( A \) increases as \( \Delta \) increases or the strength of the forcing increases, so the agreement is good for \( \Delta \) negative or near zero and for small forcing amplitudes. A possible explanation for the increasing difference between the results from (3.4) and the numerical and experimental results as \( A \) increases is found in an examination of the evolution of the initially curved soliton to a straight crested soliton. It was shown by Akylas (1984), Cole (1985), Lee (1985), Grimshaw and Smyth (1986), and Smyth (1987) for the two-dimensional case and Pedersen (1988) for the three-dimensional case that the larger the amplitude of the upstream waves, the longer the time required for their formation at the forcing. This time can become long enough so that the spreading parts of the soliton reach the boundaries and undergo Mach reflection while the soliton is still being generated at the forcing. The process by which the soliton becomes straight then occurs while the soliton is being generated. However, it was assumed in deriving (3.4) that the processes are separate. If these two processes occur together, then the amplitude of the initially generated soliton, and hence the straight crested soliton, will be larger than the value given by (3.4). This is consistent with the results shown in Figures 2 to 4.

The striking difference between the wavetrains upstream and downstream of the forcing is that while the upstream wavetrain becomes one-dimensional, the downstream wavetrain remains highly two-dimensional (Ertekin et al. 1986; Katsis and Akylas, 1987; Wu and Wu, 1987; Pedersen, 1988). An explanation for this can be found in the character of the upstream and downstream wavetrains for two-dimensional resonant flow. In that case the upstream wavetrain is a partial undular bore headed by solitons, while the downstream wavetrain is a full undular bore headed by linear waves. As was shown in Section 2, in three-dimensional flow solitons become straight while undergoing no apparent reflection at the channel boundaries. Hence there are no reflected waves present upstream to interact with other waves. On the other hand, linear and near linear waves undergo significant reflections at boundaries. These reflected waves will interact with the other downstream waves and lead to a highly two-dimensional wave pattern downstream of the forcing.
REFERENCES


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