Abstract

The evolution lump and ring solutions of a Sine-Gordon equation in two-space dimensions is considered. Approximate equations governing this evolution are derived using a pulse or ring with variable parameters in an averaged Lagrangian for the Sine-Gordon equation. It was found by Neu [Physica D 43 (1990) 421] that angular variations of the pulse shape may stabilise it. However, no study of the radiation produced by the pulse was available. In the present work, the coupling of the pulse to the shed radiation is considered. It is shown both asymptotically and numerically that the angular dependence produces spiral waves which shed angular momentum, leading to the ultimate collapse of the pulse. Good quantitative agreement between the asymptotic and numerical solutions is found. In addition, it is shown how the results of the present work can be applied to the Baby Skyrme model. In this regard, it is shown how the non-zero degree of solutions of the Baby Skyrme model prevents the collapse of a non-zero degree pulse shedding zero degree radiation. It is also indicated how the present results could be applied to the study of vortex models. The analysis presented in this work shows how complicated behaviour due to radiation of angular momentum can be captured in simple terms by approximate equations for the relevant degrees of freedom. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Modulation theory; Sine-Gordon equation; Skyrmion

1. Introduction

The Sine-Gordon equation

\[ \frac{\partial^2 u}{\partial t^2} - \nabla^2 u + \frac{1}{\epsilon^2} \sin u = 0 \]  

arises in a large number of areas of physics, for example, crystal dislocation theory [2], self-induced transparency [2], laser physics [2] and particle physics [3,4]. It is also a special case of the Baby Skyrme model which describes baryons in a nonlinear manner [5].
In previous work, in one-space dimension [6], the effect of quantum fluctuations on the Sine-Gordon soliton were studied using the approximate method of Smyth and Worthy [7] which includes the stabilising effect of dispersive radiation. It was found by Cruz et al. [6] that the complete integrability of the one-dimensional case precluded effects such as fusion of solitons due to quantum fluctuations. It is, therefore, of interest to investigate: (i) the Sine-Gordon equation in two-space dimensions without the assumption of radial symmetry; (ii) systems of Sine-Gordon type equations, such as the Baby Skyrme model [4,5,8], in order to understand the existence and behaviour of lump-type solutions as models of elementary particles. For such equations and systems of equations, there is no exact inverse scattering solution as for the one-dimensional Sine-Gordon equation and so approximate methods must be used.

In the present work, the behaviour of lump-type solutions, such as those considered by Neu [1] in the long wave modulation limit, is studied. A more general approximate solution than that of Neu [1] is considered, in the same spirit as [7]. Within this approximation, the effect of the shedding of dispersive radiation on the evolution of the lump is studied. Approximate equations are derived which show the existence of collapsing radially symmetric solutions and of solutions which have waves propagating in the angular direction. It is shown that the radially symmetric solutions collapse, as found by Neu [1]. Furthermore, in the present work, more general trial functions are used which allow the exploration of the details of the “microscopic structure” [1] of the ripple solutions. It is found that initial conditions which are flat in the interior of the pulse are unstable and eventually collapse. These flat initial conditions were not considered by Neu [1]. For such initial conditions, since the basic configuration is unstable, radiation plays no major role in the pulse evolution and the approximate equations, which do not include the effect of shed dispersive radiation, accurately describe the evolution of the pulse.

On the other hand, rippled initial conditions, such as the ones considered in [1], are stable for longer time periods. However, they develop spiral arms, since the basic stable centre configuration can shed angular momentum. Eventually, this shedding of angular momentum leads to the collapse of the pulse. Since the effect of the shed dispersive radiation is important in this case, the approximate equations obtained from an averaged Lagrangian are supplemented by an equation for the loss of angular momentum from the pulse into dispersive radiation. The agreement between the solutions of the so-obtained approximate equations and full numerical solutions of the governing Sine-Gordon equation is excellent, and even remarkable in view of the dominant role of the shed dispersive radiation. It is also shown that as the number of ripples increases, the life-time of the rippled pulse increases. However, these rippled pulses are still ultimately unstable due to the shedding of dispersive radiation.

It is, therefore, apparent that for a single equation ripples are not sufficient to ensure stability of a lump. However, it is shown that for the Baby Skyrme model in two variables, the static ripples in a new dependent variable suffice to produce a lump which cannot decay into dispersive radiation. This is because a lump in the Baby Skyrme model has a non-zero degree which is preserved by the governing equation, while radiation has zero degree.

2. Approximate equations

In two dimensions, the Sine-Gordon equation (1) has the Lagrangian

\[ L = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - \frac{1}{2} u_y^2 + \frac{1}{\epsilon^2} \cos u. \]  

(2)

An approximate solution of the Sine-Gordon equation (1) will now be sought in the form

\[ u = 4 \tan^{-1} \left\{ \frac{\epsilon w}{e^w} \right\} \left( r - R - a \cos(n(\theta - \xi)) \right). \]  

(3)

This approximate solution is shown in Fig. 1(a). Here \((r, \theta)\) are the usual plane polar coordinates and the parameters \(R, a, \xi\) and \(w\) are functions of \(t\) so that the pulse can evolve. This approximate solution is similar to that used
Fig. 1. Numerical solution of Sine-Gordon equation (1) for pulse initial condition (3) with $R = 2$, $\alpha = 0.0$ and $w = 1$ at $t = 0$ with $\epsilon = 0.2$: (a) $t = 0$, (b) $t = 2.7$ (just before collapse).

by Smyth and Worthy [7] in one-space dimension. Indeed for $\alpha = 0$ and $r$ replaced by $x$, the pulse (3) is the one-dimensional soliton solution of the Sine-Gordon equation.

An averaged Lagrangian is now calculated by substituting the approximate solution (3) into the Lagrangian (2) and integrating over all space. The averaged Lagrangian $\mathcal{L}$ is then

$$\mathcal{L} = \int_0^{2\pi} \int_0^{\infty} Lr \, dr \, d\theta. \quad (4)$$

To produce a pulse with a sharp front, it is now assumed that $0 < \epsilon \ll 1$. The integrals involved in the averaged Lagrangian (4) are then evaluated using this assumption of small $\epsilon$. A typical integral is

$$I = \int_0^{2\pi} \int_0^{\infty} \frac{1}{\epsilon^2} \frac{u^2}{2u^2} \, dr \, d\theta = \int_0^{2\pi} \int_0^{\infty} \frac{2u^2}{\epsilon^2 w^2} \sin^2(\theta - \xi) \, \text{sech}^2 \Psi \, dr \, d\theta, \quad (5)$$
where
\[ \Psi = r - R - a \cos n(\theta - \xi) \quad (6) \]
Since \( \epsilon \) is small, the sech term decays rapidly away from \( \Psi = 0 \). Hence all the other terms containing \( r \) in the integral can be approximated with \( r = R + a \cos n(\theta - \xi) \). Using this approximation, we obtain
\[ I \approx \frac{2\pi a^2 n^2}{\epsilon w} \int_0^\infty \frac{R \, e^{(r - R) \epsilon w}}{\cosh^{2} \frac{r - R}{\epsilon w}} \, dr = \frac{4\pi a^2 n^2 R \epsilon w}{\cosh^{2} \frac{R \epsilon w}{\epsilon w}} \quad (7) \]
for \( \epsilon \) small.

The other terms in the averaged Lagrangian (4) are evaluated in a similar manner and we find that
\[ L = \frac{8\pi R R'}{\epsilon w} \left( 1 + \frac{3}{2} a^2 \right) + \frac{24\pi a R}{\epsilon w} R' + \frac{4\pi R^3}{\epsilon w} + 6a R^2 \frac{a' w'}{w} \]
\[ - \frac{8\pi a^2 R^2}{\epsilon w} - \frac{8\pi a R}{\epsilon} \left( 1 + \frac{3}{2} a^2 \right) \quad (8) \]
The approximate equations governing the evolution of the lump (3) are now obtained by taking variations of the averaged Lagrangian (8) to obtain the equations for \( R \) and \( a \):
\[ \delta R : 4R \left( 1 + \frac{3}{2} a^2 \right) R'' + 6a R a' = -2 \left( 1 + \frac{3}{2} a^2 \right) R^3 - 12a R R' a' + \frac{4R}{w} \left( 1 + \frac{3}{2} a^2 \right) R' w' - 6R^2 a^2 + 6a R^2 \frac{a' w'}{w} \]
\[ + 3R^2 (a^2 + a'^2 V^2) - 2 - a^2 w^2 - 2 \left( 1 + \frac{3}{2} a^2 \right) w^2, \quad (9) \]
\[ \delta a : 3R a R' + R' a'' = -3a R^2 + \frac{3}{2} R \frac{R'}{w} - 3R R' a' + \frac{R^3}{w} a' w' + n^2 a R V^2 - n^2 a - 3aw^2 \]
\[ + 3a R R' + R a a' + n^2 a R V^2 - n^2 a - 3aw^2, \quad (10) \]
plus the equations for \( w \) and \( V = \xi' \)
\[ \delta w : 2R \left( 1 + \frac{3}{2} a^2 \right) a = 2R + n^2 a R - 2R \left( 1 + \frac{3}{2} a^2 \right) R^2 - 6a R^2 R' - R' (a^2 + n^2 a V^2), \quad (11) \]
\[ \delta \xi : \frac{d}{dt} \left( \frac{a^2 R R'}{w} \right) = 0. \quad (12) \]
Solutions of the system (9)-(12) will be examined in the following sections and compared with full numerical solutions of the governing Sine-Gordon equation (1).

3. Solutions of approximate equations

In this section, the behaviour of solutions of the approximate equations (9)-(12) will be examined and compared with full numerical solutions of the Sine-Gordon equation (1) obtained by using a finite difference scheme based on centred, second-order differences for \( u_{tt} \) and \( \nabla^2 u \). In this regard, we distinguish two cases: (i) \( a = 0 \); (ii) \( a \neq 0 \). Let us first consider the simpler case \( a = 0 \).
For $a = 0$, the averaged Lagrangian (8) reduces to
\[ L = \frac{8\pi R R'^2}{e w} \left[ \frac{8\pi R}{e w} - \frac{8\pi w R}{e} \right]. \] (13)
Taking variations of this averaged Lagrangian then gives ordinary differential equations for $R$ and $w$ as
\[ \delta w : \quad R'^2 = 1 - w^2, \] (14)
\[ \delta R : \quad \frac{d}{dt} \left( \frac{2RR'}{w} \right) = \frac{R'^2}{w} - \frac{1}{w} - w. \] (15)
Using the initial conditions $R(0) = R_0$ and $w(0) = w_0$, these equations can be integrated to give the solutions for $R$ and $w$:
\[ R = \frac{R_0}{w_0} \sin \left( \frac{u_0}{R_0} + \sin^{-1} w_0 \right), \] (16)
\[ w = \sin \left( \frac{u_0}{R_0} + \sin^{-1} w_0 \right). \] (17)
The radially symmetric lump solution is then given by
\[ u = 4\tan^{-1} \exp \left\{ \frac{r - \bar{R}(t)}{\epsilon \sin((u_0/R_0)t + \sin^{-1} w_0)} \right\}. \] (18)
It is apparent that when
\[ t = t_c = \frac{R_0}{w_0}(\pi - \sin^{-1} w_0), \] (19)
$R = 0$ and $w = 0$. The radially symmetric lump solution then collapses in the finite time $t_c$, which reproduces the collapse result of [1]. However, Neu [1] did not include changes in the lump width $w$. We, therefore, see that changes in the lump width have no influence on the collapse of the lump in finite time. Fig. 1(a) and (b) show full numerical solutions of the Sine-Gordon equation (1) which illustrate the collapse of the radially symmetric lump in finite time. Fig. 1(a) shows an initially radially symmetric lump with $R_0 = 2$ and Fig. 1(b) shows the lump at $t = 2.7$, just before the theoretical collapse time (19), which in this case is $t = \pi$.

Fig. 2 shows a comparison between the average radii $\bar{R}$ of the lump as given by the full numerical solution of the Sine-Gordon equation (1) and by the approximate solution (16). The average radius of the full numerical solution was obtained via the integral
\[ \bar{R}(t) = -\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \frac{\partial u}{\partial r} r \, dr \, d\theta. \] (20)
It can be easily verified that for the approximate solution (3), this integral gives $R$. For the full numerical solution, it is expected that the integral (20) will be a good approximation to the average lump radius $\bar{R}$ until the lump is near total collapse. It can be seen that there is an excellent comparison between the numerical and approximate radii right up to the time of collapse. At this point, it should be noted that the approximate solution (18) does not include the dispersive radiation shed by the lump as it evolves. The excellent agreement between the numerical and approximate solutions then shows that this radiation does not play a large role in the collapse of the lump, as would be expected as the dominate behaviour is the self-similar collapse.
Let us now consider the second case $a \neq 0$. The approximate equations (9)-(12) then have the family of fixed points

$$w_0^2 = \frac{-n + \sqrt{n^2 + 24}}{6n}, \quad (21)$$

$$R_0^2 = \frac{A w_0}{n^2 + 3 w_0^2}, \quad (22)$$

$$w_0^2 = \frac{1}{1 + 3w_0^2}, \quad (23)$$

$$V_0 = \frac{A w_0}{n^2 w_0^2 R_0^2}, \quad (24)$$

where $A$ is a constant of integration. This family of fixed points corresponds to constant values of the parameters $R$, $w$ and $a$ and to a linear increase in the phase $\xi = V_0 t$ of the wave propagating around the boundary layer

$$r = \frac{R_0}{1 + a_0 \cos(n(\theta - V_0 t))}, \quad (25)$$

at the edge of the lump.

To determine the stability of the family of fixed points (21)-(24), the approximate equations (9)-(12) were solved numerically for several values of $n$ using a fourth order Runge-Kutta scheme. For small values of $n$, it was found that the lump was unstable as its radius decreased to zero in finite time. Fig. 3(a) shows the lump (3) with $R = 2$, $a = 0.1$, $V = 0.01$ and $w = 1$ at $t = 0$ with $\epsilon = 0.2$ and $n = 3$, while Fig. 3(b) shows the full numerical solution of the Sine-Gordon equation (1) for this initial condition at $t = 2.5$, which is just before the lump collapses. It can be

Fig. 2. Comparison between pulse radius $R$ as given by the full numerical solution of the Sine-Gordon equation (1) and by the solution (16) of the approximate equations. The initial condition is (3) with $R = 2$, $a = 0.0$ and $w = 1$ at $t = 0$ and $\epsilon = 0.2$. Numerical solution —; approximate solution ——.
Fig. 3. Numerical solution of Sine-Gordon equation (1) for pulse initial condition (3) with $R = 2$, $a = 0.1$, $V = 0.01$ and $u = 1$ at $t = 0$ with $\epsilon = 0.2$ and $n = 3$: (a) $t = 0$; (b) $t = 2.5$ (just before collapse).

seen that there is little dispersive radiation shed as the lump collapses and overall the solution is very similar to the radially symmetric case shown in Fig. 1(b). Fig. 4 shows a comparison between the radii $R$ of the lump as given by the solution of the approximate equations and as given by the full numerical solution of the Sine-Gordon equation (1) via the integral (20) for the average radius. It can be seen that there is again excellent agreement between the numerical and approximate solutions. It can further be noted that the propagating wave around the edge of the lump has increased its lifetime over that for the radially symmetric case.

Let us now consider the evolution of the lump for large $n$. Figs. 5(a) and (b) show the full numerical solution of the Sine-Gordon equation (1) for $n = 10$ at $t = 0$ and $t = 3$, respectively, for the initial condition
Fig. 4. Comparison between pulse radius $R$ as given by the full numerical solution of the Sine-Gordon equation (1) and by the solution of the approximate equations (9)–(12). The initial condition is (3) with $R = 2$, $a = 0.1$, $V = 0.01$ and $w = 1$ at $t = 0$ with $\epsilon = 0.2$ and $n = 3$.

Numerical solution: ——, approximate solution: ——–.

(3) The lump again collapses, but it can be seen that in contrast to the radially symmetric solution shown in Fig. 1(b) and the solution for small $n$ shown in Fig. 3(b), the collapsing pulse sheds a significant amount of dispersive radiation. It is this shed radiation that dominates the collapse process. Fig. 6 shows a comparison between the radii $R$ of the lump as given by the approximate equations and as given by the full numerical solution via the radius integral (20). As for the radially symmetric case shown in Fig. 2 and the $n = 3$ case shown in Fig. 4, there is again excellent agreement between the full numerical and approximate radii, right up to near collapse.

Table 1 lists the pulse lifetime as predicted by the approximate equations for a range of $n$ from 0 to 500. It can be seen that the lifetime increases for small $n$ over the value $\pi$ for $n = 0$ (see (19)). However, this increase is not large, and the lifetime decreases for large $n$. It is noted, however, that the initial increase in lifetime, of the order of 7%, is not large. The reason for this is that the amplitude $a$ of the ripples rapidly decays, whereupon the pulse evolution is as for a radially symmetric pulse ($n = 0$). Indeed, linearising the approximate equations (9)–(12) about the fixed

<table>
<thead>
<tr>
<th>$n$</th>
<th>Pulse lifetime, $t_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pi$</td>
</tr>
<tr>
<td>10</td>
<td>3.357</td>
</tr>
<tr>
<td>20</td>
<td>3.187</td>
</tr>
<tr>
<td>100</td>
<td>2.122</td>
</tr>
<tr>
<td>300</td>
<td>2.102</td>
</tr>
<tr>
<td>500</td>
<td>2.080</td>
</tr>
</tbody>
</table>

*The initial condition is (3) with $R = 2$, $a = 0.1$, $V = 0.01$ and $w = 1$ at $t = 0$ with $\epsilon = 0.2$. 
point (21)-(24), it can be found that the lifetime of an initial pulse near this fixed point is
\[ t_c = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{a(0)}{\sqrt{\nu}} \]  (26)
noting that this expression is not correct as \( a \to 0 \), which is why it does not agree with (19). The independence of \( n \) of this lifetime is consistent with the small increase in \( t_c \) for small \( n \) shown in Table 1. The reason for the decrease in \( t_c \) for large \( n \) is not clear, but must be due to the nonlinear details of the approximate equations.
Fig. 6. Comparison between pulse radius $R$ as given by the full numerical solution of the Sine-Gordon equation (1) and by the solution of the approximate equations (9)-(12). The initial condition is (3) with $R = 2$, $u = 0.1$, $V = 0.01$ and $w = 1$ at $t = 0$ with $\epsilon = 0.2$ and $a = 10$.

Numerical solution: —; approximate solution: – – –.

Smyth and Worthy [7] considered the evolution of a pulse to a soliton for the one-dimensional Sine-Gordon equation. It was found that in order to obtain good agreement with numerical solutions, the effect of the dispersive radiation shed as the pulse evolves had to be included. For the present two-dimensional Sine-Gordon equation, the effect of shed dispersive radiation has not been included since, as is clear from Figs. 2, 4 and 6, the inclusion of this radiation is not necessary in order to obtain good agreement with numerical solutions.

4. Generalised approximate equations

In order to directly compare the present results with those of Neu [1], we observe that the fixed point (21)-(24) is different to the one obtained by Neu [1]. In the present work, the fixed point amplitude $a_0$ is a function of the number of ripples, while the fixed point radius $R_0$ is arbitrary as it depends on the initial condition. In [1], the opposite is the case as at the fixed point the ripple amplitude is arbitrary and the radius is fixed by the number of ripples and their amplitude. To understand this difference, it is noted that the trial function (5) becomes flat as $r \to 0$. On the other hand, the trial function used in [1] has non-trivial behaviour only in a boundary layer. Thus in order to interpolate between these two types of trial functions, we consider the new trial function

$$\psi = 4 \tan^{-1} \exp \left\{ \frac{r - R(t) - a \delta \cos n(\theta - \xi(t))}{\epsilon} \right\},$$

where $\delta$ is a free parameter. The width $w$ has been taken to be 1, since in the previous section, it was found that it plays no major role in the evolution of the pulse. The variations of the amplitude $a$ are assumed to be on a slow scale.

With this new trial function, the averaged Lagrangian (4) is

$$\mathcal{L} = \frac{8 \pi}{\epsilon} R \dot{R}^2 + \frac{4 \pi}{\epsilon} a^2 \eta^2 \dot{\xi}^2 R^{1+2\delta} - \frac{16 \pi}{\epsilon} R - \frac{4 \pi}{\epsilon} a^2 \eta^2 R^{-1+2\delta}.$$ (28)
The variational equations with respect to $a$, $\xi$, and $R$ are then

\[ \delta a : 8\pi a^2 \left( \dot{\xi}^2 R^{1+2\delta} - R^{-1+2\delta} \right) = 0, \]  
\[ (29) \]

\[ \delta \xi : \frac{d}{d\tau} (a^2 \dot{\xi} R^{1+2\delta}) = 0, \]  
\[ (30) \]

\[ \delta R : \frac{d}{d\tau} (4R \dddot{R}) - 2a^2 n^2 R^{-2+2\delta} + 4 = 0. \]  
\[ (31) \]

From the $\delta a$ equation (29), we see that

\[ \dot{\xi} = R^{-1}. \]  
\[ (32) \]

The variational equation (30) expresses conservation of angular momentum. Note that for $\delta \geq 1$, which is the case for a flat initial condition as $t \to 0$, the critical point of the variational equations is an (unstable) saddle point. This observation explains the instability observed for the original trial function (3). On the other hand, when $\delta \ll 1$, the trial function (27) is the boundary layer trial function considered by Neu [1]. In this case, the fixed point of the variational equations (29)–(31) is a centre. It is also noted that for $\delta = 0$, we recover the results of [1]. In our formulation (27) of the trial function, relativistic invariance is not explicitly taken into account as in [1], so that a factor of $1/\sqrt{1 - \dot{\xi}^2}$ is missing in the exponential of (27). This relativistic factor limits the velocity of the collapse of the pulse [1]. This factor is, however, not needed as the collapse of the pulse is always subsonic.

To relate the present work to that of [1], let us consider the solution of the variational equations (29)–(31) in the limit as $\delta \to 0$. In this limit, the equations become

\[ \dot{\xi} = R^{-1}, \]  
\[ (33) \]

\[ \frac{d}{d\tau} a^2 = 0, \]  
\[ (34) \]

\[ \frac{d}{d\tau} (4R \dddot{R}) + 4 - 2a^2 n^2 R^{-2+2\delta} = 0. \]  
\[ (35) \]

The fixed point of these equations is a centre at

\[ R_0 = \frac{an}{\sqrt{2}}. \]  
\[ (36) \]

We note that conservation of angular momentum (34) gives that $a$ is constant, so that the ripple amplitude is arbitrary.

As in the one-dimensional Sine-Gordon equation studied by Smyth and Worthy [7], perturbations in $R$ will result in the pulse shedding dispersive radiation. It is, therefore, necessary to modify the variational equations (29)–(31) to incorporate the effect of this shed dispersive radiation, which is done in the next section. The calculation of the effect of the shed dispersive radiation closely follows that of [7].

5. Radiation

Since the shed dispersive radiation is of small amplitude relative to the pulse, it satisfies the linearised Sine-Gordon (Klein-Gordon) equation

\[ \frac{\partial^2 v}{\partial \tau^2} - \nabla^2 v + \frac{1}{\tau^2} v = 0, \]  
\[ (37) \]
subject to appropriate boundary conditions at the pulse. Let us denote the position of the boundary layer of the pulse by

$$B(\theta, t) = R + a(t) \cos n(\theta - \xi(t)).$$

(38)

As in the one-dimensional case considered by Smyth and Worthy [7], we note that a shelf in $|\nabla u|$ develops at the boundary $B(\theta, t)$. Then both the boundary and the pulse have an angular dependence. Since the equation for the radiation (37) is linear, we can independently treat the radiation due to the variation of the pulse radius and the radiation due to the angular dependence of the pulse boundary. The use of the Klein-Gordon equation to describe the shed dispersive radiation can be further justified as follows. It was shown in [7] that the Klein-Gordon equation gave an excellent description of the shed radiation for the one-dimensional Sine-Gordon equation, since the shelf in $\nabla u$ is small. We note that this radiation for the one-dimensional Sine-Gordon equation is the same as the radial radiation for the present two-dimensional equation. Additionally, for the angular radiation, we have assumed that the ripples are small, and, therefore, the angular variations of the radiation will be small. Nonlinear interactions in the shed radiation can thus be neglected and the Klein-Gordon equation (37) is appropriate to describe the radiation and its coupling with the nonlinear pulse.

To determine the shed dispersive radiation, we shall need the energy and angular momentum conservation equations for the linearised Sine-Gordon equation (37), which are

$$\frac{\partial}{\partial t} \left( \frac{1}{2} (v_t^2 + v_r^2 + \frac{1}{r^2} v_\theta^2 + \frac{1}{r^2} \epsilon^2 v_\phi^2) \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( rv_t v_r \right) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( rv_\theta v_\theta \right) = 0,$$

(39)

and

$$\frac{\partial}{\partial t} \left( rv_\phi v_\phi \right) - \frac{\partial}{\partial r} \left( rv_r v_\phi \right) - \frac{r}{2} \frac{\partial}{\partial \theta} \left( v_t^2 + v_r^2 + \frac{1}{r^2} v_\theta^2 + \frac{1}{r^2} \epsilon^2 v_\phi^2 \right) = 0,$$

(40)

respectively.

The average of the pulse $u$ at the boundary layer will be matched to the radiation via an energy argument, as was done by Smyth and Worthy [7] for the one-dimensional Sine-Gordon equation. This gives the energy lost by the pulse to dispersive radiation. On the other hand, the angular variation of the pulse at the boundary layer will be matched pointwise to the radiation field to give the angular momentum lost by the pulse to dispersive radiation. Therefore, in detail, we solve the linearised equation (37) subject to

$$\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} \text{ on } B(\theta, t),$$

(41)

$$v = 0 \text{ at } t = 0,$$

(42)

where $n$ is the normal to $B(\theta, t)$. Let us now approximate the boundary $B(\theta, t)$ by the circle $r = R(r) + \mu$, since $a \ll R$, noting that we have assumed that $\epsilon \ll a$. At this point, we note that in the examples shown in the figures, we have taken $a = 0.1, 0.2$ and $\epsilon = 0.2$. The reason that this relatively large value of $\epsilon$ was taken was to get adequate numerical resolution of the boundary layer. It is noted that good agreement was obtained between the numerical and approximate solutions even though the condition $\epsilon \ll a$ was not strictly satisfied. The matching is, therefore, to be done on a circle of radius $R(t) + \mu$, which will be determined as part of the matching procedure. Expanding $\partial u$/$\partial n$ in a Fourier series, we obtain

$$\frac{\partial u}{\partial n} = \sum_{k} c_k \cos k(\theta - \xi(t)),$$

and

$$\frac{\partial u}{\partial n} = \sum_{k} c_k \cos k(\theta - \xi(t)),$$

(43)
where
\[ c_k(a) = \frac{1}{\pi} \int_0^{2\pi} \text{sech} \left( \frac{\mu - R a \cos \theta}{\epsilon} \right) \cos k \theta \, d\theta \] (44)
after a change of variable of integration.

Let us now decompose the radiation into two components \( v = v_1 + v_2 \), where \( v_1 \) is purely radial and \( v_2 \) is purely circumferential. The radial component \( v_1 \) then satisfies
\[ \frac{\partial^2 v_1}{\partial t^2} - \nabla^2 v_1 + \frac{1}{\epsilon^2} v_1 = 0 \] (45)
with the boundary and initial conditions
\[ \frac{\partial v_1}{\partial r} = h(t) \quad \text{on} \quad r = R(t), \] (46)
\[ v_1 = \frac{\partial v_1}{\partial t} = 0 \quad \text{at} \quad t = 0. \] (47)

As for the one-dimensional Sine-Gordon equation considered by Smyth and Worthy [7], the function \( h(t) \) represents the shelf of radiation in \( v_r \) at the edge of the pulse. Also, as in [7], the function \( h(t) \) is determined as part of the calculation of the effect of the shed dispersive radiation. From the Fourier series (43), it can be seen that \( h(t) \) will contain \(-c_0/\epsilon\), but it will also contain an unknown component due to the shelf formation. We have also taken \( \delta = 0 \), as it can be absorbed into \( h(t) \).

The equation for the circumferential component \( v_2 \) of the dispersive radiation takes the form
\[ \frac{\partial^2 v_2}{\partial t^2} - \nabla^2 v_2 + \frac{1}{\epsilon^2} v_2 = 0 \] (48)
with the boundary condition
\[ \frac{\partial v_2}{\partial r} = -2 \epsilon \sum_{k=1}^{\infty} c_k(a) \cos k(n \theta - \xi(t)) \quad \text{on} \quad r = R(t) + \mu. \] (49)

It is clear that \( c_k(a) \to 0 \) as \( a \to 0 \) for \( k \geq 1 \), so that the linearised Sine-Gordon (Klein-Gordon) equation is appropriate to describe the circumferential radiation. The radiation \( v_2 \) is taken to be an outgoing wave. The initial conditions for \( v_2 \) are also neglected as we are interested in the long-term influence of the circumferential radiation.

Let us first consider the radial radiation governed by the linearised Sine-Gordon equation (45). The calculation of this radiation mirrors that for the one-dimensional Sine-Gordon equation considered by Smyth and Worthy [7], except for changes due to the present two-space dimensions. Therefore, only the main steps will be outlined with the details given in [7].

Let us first assume that the motion of the radius \( R(t) \) is slow relative to the motion of the dispersive radiation. Then using Laplace transforms and expanding in small \( \epsilon \), the solution of (45) for the radial radiation is
\[ u(r, t) = -\left( \frac{k}{\tau} \right)^{1/2} \int_{t-\tau}^{t} h(t - \tau) J_0 \left( \frac{\sqrt{\tau^2 - (t - \tau)^2}}{\epsilon} \right) \, d\tau, \] (50)
where \( J_0 \) is the Bessel function of order zero.
The final thing to determine for the radial radiation is the shelf height \( h(t) \), which as in [7] is determined via the energy in the radiation. By an application of Noether’s theorem on the Lagrangian (2) for the Sine-Gordon equation, it can be found that the energy of the pulse (27) is

\[
H = \frac{1}{\sqrt{c}} \int_0^{\infty} \left( \frac{u^2}{2} + \frac{1}{2} R^2 \frac{R^2}{R^2} - \frac{2}{c^2} \cos u \right) R \, dr \, d\theta = \frac{8\pi}{\sqrt{c}} \left( R R^2 + 2 R + \frac{a^2 R^2}{R} \right). \tag{51}
\]

Let us now linearise this energy around the fixed point (36) of the approximate equations (33)–(35). As in [7], the excess of the pulse energy over the fixed point energy \( H_0 \) is taken to be shed into the radial dispersive radiation \( e_1 \).

Expanding the pulse energy \( H = H_0 + H \) around the fixed point, we have to quadratic order

\[
H = H_0 + \frac{8\pi}{\sqrt{c}} \left( \frac{a}{\sqrt{2}} R + \frac{4\sqrt{2}}{a} R \right) = H_0 + \Delta H. \tag{52}
\]

Now matching to the radiation, we have from the linearised Sine-Gordon equation (45)

\[
\Delta H = \frac{8\pi}{\sqrt{c}} \left( \frac{a}{\sqrt{2}} R + \frac{4\sqrt{2}}{a} R \right) R \, dt = H_{rad}. \tag{53}
\]

At the radiation front \( r = R + t, \ v_1 = v_2 = 0 \). Then evaluating the integral in (53) using the trapezoidal rule and matching the radiation energy (53) to \( \Delta H \) in the pulse energy (52), we have

\[
\frac{8\pi}{\sqrt{c}} \left( \frac{a}{\sqrt{2}} R + \frac{4\sqrt{2}}{a} R \right) = \pi R \left( \frac{\dot{h}^2(t)}{2} + h^2(t) \right). \tag{54}
\]

As for the one-dimensional Sine-Gordon equation [7], the function \( h(t) \) is taken to be proportional to \( R \) so as to avoid singularities produced by division by \( R \). We further note that \( R \sim h \sim R \). In this manner, we find that

\[
h(t) = \frac{\dot{h}(t)}{\sqrt{c}} \frac{\sqrt{am}}{R}. \tag{55}
\]

To finally determine the terms that need to be added to the approximate equations (33)–(35) to incorporate the effect of the shelf dispersive radiation, let us re-visit total energy conservation. The total energy is given by \( E = H + H_{rad} \). Therefore, as energy is conserved

\[
0 = \frac{dE}{dt} = \frac{dH}{dt} + \frac{dH_{rad}}{dt}. \tag{56}
\]

The linearised energy conservation equation (39) then gives the energy flux in the radial direction as

\[
\frac{dH}{dr} = -\frac{dH_{rad}}{dr} = -2\pi R u_1 u_2 R. \tag{57}
\]

Finally, using the radial radiation solution (50) to determine the radial radiation loss given by (57), the pulse energy conservation equation (51) becomes

\[
2 RR + R^2 - \frac{a^2 R}{R^2} + 2 = -\frac{a}{\sqrt{2}} R + \frac{\sqrt{am}}{\sqrt{R}} \int_{\theta_0}^{\theta_1} \frac{\sqrt{am}}{\sqrt{R}} R(\tau) J_1 \left( \frac{\tau - \tau_0}{\sqrt{c}} \right) \, d\tau. \tag{58}
\]

In this energy equation which incorporates loss to dispersive radiation, the centre of the lossless approximate equations at \( R_0 \) given by (36), is changed to a slowly varying spiral point. The approximate equations (33) to (34) and (58) give that any amplitude \( a \) is an allowed fixed point. However, this is not the case as in fact the pulse will
now be shown to lose angular momentum into a spiral wave. This loss of angular momentum causes $a$ to decrease and the pulse to collapse.

A further application of Noether’s theorem to the Lagrangian (2) for the Sine-Gordon equation gives the total angular momentum as

$$P = \int_0^{2\pi} \int_0^\infty a(t,r) dr d\theta.$$  

(59)

Angular momentum conservation gives that $\dot{P} = 0$. As for the radial component of the radiation, let us decompose the angular momentum as $P = P_p + P_r$, where $P_p$ is the angular momentum of the pulse and $P_r$ is the angular momentum of the radiation.

From the trial function (27), the angular momentum of the pulse is

$$P_p = -\frac{8\pi}{\epsilon} a^2 n^2 R \dot{\xi}.$$  

(60)

On differentiating the total angular momentum (59) with respect to time, using the linearised Sine-Gordon equation (48) for the circumferential radiation and then integrating by parts, we have the angular momentum conservation relation

$$\frac{dP_p}{dt} = -\frac{dP_r}{dt} = \int_0^{2\pi} R v^2 v_\theta^2 \theta dr d\theta.$$  

(61)

Angular momentum conservation, therefore, gives

$$\frac{d}{dt} \left( \frac{8\pi}{\epsilon} a^2 n^2 \right) = -\int_0^{2\pi} R v^2 v_\theta^2 \theta dr d\theta.$$  

(63)

To close the equations for the circumferential radiation, we need to find the solution for $v^2$ and from this compute the angular momentum flux to the circumferential radiation. In order to do this, we take $\mu = 1/2a(0)$ for the matching between the radiation and the pulse, since the region where the angular momentum of the radiation is generated is a strip between $R$ and $R + a(0)$. Eq. (48) for $v^2$ can be solved in the geometric optics approximation since $n$ is taken to be large and $n \approx 1/\epsilon$ was originally assumed. We, therefore, take

$$v^2 = \sum_{k=1}^{\infty} A_k(r, \theta, t) \sin(nkS_k(r, \theta, t)).$$  

(64)

The $S_k$ will then satisfy eikonal equations. The transport equations for the $A_k$ are not needed in the present work since we just need their values at $r = R(t) + \mu$.

Substituting the geometric optics expansion (64) into Eq. (48) for $v^2$, we find that the eikonal equations are

$$S_k^2 - S_0^2 - 1 + \frac{1}{n^2 c^2} = 0,$$  

(65)

where

$$S_k(R(t) + \mu, \theta, t) = \theta - \xi(t).$$  

(66)

$S_0$ outgoing as $r \to \infty$.  

(67)
An approximate solution of the eikonal equations can be found by separation of variables since $\mu \ll R$ and $\dot{\xi} = 1/R$.

In this manner, it is found that

$$S_k = f_k(r) + \theta - \xi(t), \quad (68)$$

where

$$f_k^2 = \frac{1}{\epsilon k^2 n^2}. \quad (69)$$

For an outgoing wave, we take

$$f_k(r) = R(t) - \frac{r}{\epsilon k n}. \quad (70)$$

The phase given by (68) is a spiral wave whose arms are $r = \theta + \xi(t) + R(t) + \text{constant}$. We, therefore, see that unlike the situation for the $\delta = 1$ case of Section 2 for which the radiation played no major role, the $\delta \ll 1$ case produces an outgoing, rotating radiation pattern. The amplitudes $A_k$ of this spiral wave are determined from the boundary condition (49) for $v_2$, giving

$$A_k = 2nk c_k(a). \quad (71)$$

The solution for the circumferential radiation has thus been determined.

Using the geometric optics solution (64) for $v_2$, with the coefficients of the expansion given by (71), the angular momentum balance equation (63) becomes

$$\frac{d}{dt} a^2 = -\frac{R(t)}{2\pi n} \sum_{k=1}^{\infty} f_k^2 c_k(a). \quad (72)$$

The energy and angular momentum loss to the dispersive radiation shed by the pulse has, therefore, been determined and the system of equations incorporating energy and angular momentum loss are then (58) and (72). The functions $c_k(a)$ are determined via the Fourier coefficients given by (44).

6. Comparison with numerical solutions for $\delta$ small

The Sine-Gordon equation (1) was numerically solved with the initial condition (27) using centred, second-order differences. Also the loss equations (58) and (72) were integrated numerically using a fourth-order Runge–Kutta method with the energy loss integral in Eq. (58) and the Fourier coefficients $c_k$, given by (44), for the angular momentum loss in Eq. (72) evaluated using the trapezoidal rule. Fig. 7 shows the full numerical solution of the Sine-Gordon equation for $R = 2$, $\dot{\xi} = 0.5$ and $a = 0.2$ at $t = 0$ with $n = 18$, $\delta = 0.1$ and $\epsilon = 0.2$. The initial condition is shown in Fig. 7(a) and the solution at $t = 4$ is shown in Fig. 7(b). It can be seen by comparing Figs. 1(a), 3(a), 5(a) and 7(a) that the gradient of the initial condition is less uniform than for $\delta = 1$. The spiral wave of shed radiation can be clearly seen in Fig. 7(b). The presence of this distinct dispersive radiation is again to be contrasted with the solutions for $\delta = 1$ shown in Figs. 1(b) and 3(b) for which there is little shed radiation.

The solution of the approximate equations (58) and (72) possesses two forms of collapse. The first type is one of pure collapse. This form of collapse occurs when the slowly varying fixed point reaches the origin before the orbit of the original centre, which exists when there is no radiative loss, has a chance to circle it. The second type is an oscillatory decay to collapse. Which type of collapse occurs depends on the strength of the radiative damping. For strong damping, pure collapse occurs, while for weak damping for $n \to \infty$, oscillatory collapse results. Fig. 8 shows the
radius $R$ of the pulse as a function of time as given by the full numerical solution of the Sine-Gordon equation (1) and by the solution of the approximate equations (58) and (72) for $n = 18$. In calculating the angular momentum loss to radiation as given by (72), $M$ terms were included in the Fourier series, where $M = 3, 4, 5$. The agreement between the numerical and approximate solutions is remarkable given the number of assumptions made to calculate the loss to dispersive radiation. This dispersive loss from the pulse is clearly a complicated process, as seen from Fig. 7(b), and the radiation analysis of the previous section involved a number of simplifications. Other choices of initial values for the initial condition (27) showed similar agreement between the numerical and approximate solutions. On the basis of the solution of the approximate equations, large values of $n$ of the order of 100 are needed to obtain oscillatory collapse.
An interesting feature of the solution of the approximate equations without radiation damping is that they allow $a$ to be arbitrary at the fixed point. This suggests the possibility of looking for more general periodic waves travelling around the boundary layer of the pulse. The aim of this is to determine whether there exists a non-radiating solution with a constant amplitude and radius. To this end, we take the trial function

$$ u = -4 \tan^{-1} \exp \left\{ -\frac{r - p(\theta, t) - \delta}{\epsilon} \right\}, \quad (73) $$

where $p(\theta, t)$ is a $2\pi/n$ periodic function of $\theta$ which is to be determined. Averaging in the $r$ variable gives the averaged Lagrangian

$$ L = p \left( \frac{V^2}{p} - \frac{1}{p^2} \right). \quad (74) $$

This averaged Lagrangian is similar to the Born-Infeld Lagrangian [1].

A steady periodic wave of permanent form for $p$ is then $p = p(\xi) = q(\theta - Vt)$, where $q$ and $V$ are to be determined. For this choice of $p$, the averaged Lagrangian (74) takes the form

$$ L = p \left( V^2 \frac{1}{p^2} \right) \left( p^2 - 1 \right). \quad (75) $$

The periodic solutions are then the curves $H = E$, where $H$ is the Hamiltonian for this averaged Lagrangian, so that

$$ \frac{1}{p} \left( p^2 V^2 - 1 \right) = E - p. \quad (76) $$
Therefore, in the phase plane the periodic solutions are the curves
\[ p_2^2 = \frac{p(E - p)}{p^2 V^2 - 1} \]  
(77)

These solutions are not closed orbits. However, it is possible to construct a singular solution with square integrable derivative which joins the points \( p = \pm 1/V \). The prescribed period is then obtained by adjusting \( V \). Furthermore, the amplitude parameter \( E \) for this singular solution is free. In this manner, a one parameter family of steady, singular approximate solutions of the Sine-Gordon equation (1) is obtained. Since this solution has a cusp, it is not physically relevant. However, it has been shown by Padilla\(^1\) that a variational argument, based in turn on a mountain pass argument, gives that these singular solutions are saddle points and thus are unstable. Preliminary numerical results indicate that approximations to these singular solutions behave qualitatively in the same manner as the solutions studied in detail in the present work.

7. Relation to Skyrme model

There has been much recent work on the so-called Baby Skyrmion model [4,5,8]. This Baby Skyrmion model is a reduction to two-space dimensions of the full Skyrme model accompanied by a reduction in the number of internal degrees of freedom from 3 to 2. In the full Skyrme model, the field is matrix valued. In the reduced model, the field is constrained to lie on the unit sphere. Thus to every point \((x, t)\) there corresponds a unit vector on the sphere which is conveniently parameterised in the form
\[ \Phi = \left( \sin u(x, y, t) \cos \psi(x, y, t), \sin u(x, y, t) \sin \phi(x, y, t), \cos u(x, y, t) \right) . \]  
(78)

The Lagrangian for the corresponding field equations is
\[ L = \frac{1}{2}(u_t^2 + \nabla^2 u^2) + \frac{1}{2} \sin^2 f[(u_x \psi_y - u_y \psi_x)^2 + (u_x \psi_y - u_y \psi_x)^2 - (u_x \psi_y - u_y \psi_x)^2] - \mu^2 \sin^2 u , \]  
(79)

Here \( \mu \) is related to the mass of the field and \( u \) and \( \psi \) are the polar angles of the vector field \( \Phi \). It is well established [3,4] that when \( \psi = n \theta \), where \( \theta \) is the two-dimensional polar angle, there is a static solution \( u = f(r) \), which is called the Hedgehog solution. In this case, the action is the reduction of the full Skyrme action [3] and \( f \) is a decreasing function with \( f(0) = \pi \) and \( \lim_{r \to \infty} f(r) = 0 \). For this solution, the field is characterised by \( n \) ripples in the angular variable and when \( \mu \) is large, has a layered structure in the radial variable.

It is now apparent that the problem for the Sine-Gordon equation studied in the previous sections is a special case of the Baby Skyrmion model for \( n = 0 \) which keeps the ripple structure in one of the dependent variables. In fact for \( n = 0 \), the Lagrangian (79) takes the form
\[ L = \frac{1}{2}(u_t^2 - |\nabla u|^2) - \mu^2 \sin^2 u , \]  
(80)

which is the Sine-Gordon Lagrangian.

It was found in Section 3 that for a representative choice of parameter values the periodic solutions were unstable. However, it was found that they could last a long time provided that the velocity of the waves propagating around the boundary of the lump was small. The typical instability due to angular momentum loss discussed in the previous two sections shows that ripples in the independent variables are not sufficient to stabilise the lump configuration.

\(^1\) Private communication from P. Padilla.
In this regard, it is to be noted that the degree of the map for the field (78) which is conserved by the evolution equation derived from the Lagrangian (79) is given by

$$\deg(\phi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \cdot (\partial_x \phi \times \partial_y \phi) \, dx = 0. \quad (81)$$

This shows that there is no constraint on the lump to stop it evolving into radiation, since radiation also has degree zero as for small $\phi$, the integral in the degree equation (81) can be made small. Since the degree is an integer, the integral must then vanish.

In contrast, the Hedgehog lump solution has non-zero degree and thus cannot decay into pure radiation. This qualitative difference between the two solutions is also reflected in the approximate solution for the Baby Skyrmion model. Indeed, to find the approximate solution of the Baby Skyrmion model, we consider the one-dimensional boundary layer solution associated with

$$\psi_y = ny. \quad (82)$$

For a stationary solution, the Lagrangian (80) becomes

$$L = -\frac{1}{2} u_x^2 + \frac{n^2}{2} (\sin^2 u) u^2_x - \frac{\mu^2}{2} \sin^2 u. \quad (83)$$

The solutions are then the level lines which join the critical points $u = \pi$ to $u = 0$ of the Hamiltonian related to the Lagrangian (83). These solutions, therefore, satisfy

$$H = u_x \frac{\partial L}{\partial u_x} - L = -\frac{1}{2} + \frac{n^2}{2} \sin^2 u - \frac{\mu^2}{2} \sin u = 0, \quad (84)$$

and the required solution is of the form

$$u = g(\mu (x - x_0)), \quad g(-\infty) = \pi, \quad g(\infty) = 0. \quad (85)$$

To find a two-dimensional approximation, we assume

$$\psi = n\theta, \quad u = g(\mu (r - R(t))), \quad (86)$$

where $R(t)$ is to be determined. In polar coordinates, the Lagrangian (83) takes the form

$$\mathcal{L} = \int_0^{2\pi} \int_0^\infty \left[ \frac{1}{2} u^2 - \frac{1}{2} u^2 \sin^2 \theta - \frac{n^2}{2} u^2 \theta^2 - \frac{\mu^2}{2} \sin^2 u \right] \, r \, dr \, d\theta \quad (87)$$

for the special solution (86). Using the assumed form (86), this averaged Lagrangian (87) becomes

$$\mathcal{L} = \mu \pi \left[ I_1 R^2 - (I_1 + I_2) R + \frac{n^2}{R} I_1 \right], \quad (88)$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{(g'(\xi))^2}{R} \, d\xi, \quad (89)$$

and

$$I_2 = \int_{-\infty}^{\infty} \sin^2 g(\xi) \, d\xi. \quad (90)$$
Variations of the averaged Lagrangian (88) with respect to $R$ then gives the equation for the radius $R$ as

$$I_1 \frac{d}{dt}(R^2) + (I_1 + I_2) - \frac{n^2}{R^2} I_1 = 0. \quad (91)$$

The fixed point of this radius equation is

$$R_0^2 = \frac{n^2 I_1}{I_1 + I_2}. \quad (92)$$

Linearising about this fixed point with $R = R_0 + \tilde{R}$ gives the linearised equation

$$\ddot{\tilde{R}} + \frac{3n^2}{R_0^2} \tilde{R} = 0, \quad (93)$$

so that the fixed point $R_0$ is a stable centre. It is, therefore, again apparent that the ripples stabilise the fixed point. This is consistent with the fact that the Hedgehog solutions are known to be minima of the corresponding Hamiltonian [3]. It is to be further noted that the assumed solution (86) is static in time and that the integer $n$ which produces the fixed point (92) in the radial equation (91) is the degree of the map. It can, therefore, be seen that the stabilisation mechanism is robust since a solution with non-zero degree cannot decay into pure radiation.

It is known that the full Skyrme model has toroidal solutions of degree 2 [9]. What is not known is if the Baby Skyrme model has such solutions. In this regard, let us now consider ring solutions of the Sine-Gordon equation (1). We then seek approximate solutions of the Sine-Gordon equation of the form

$$u = 4 \tan^{-1} \exp \left\{ \frac{r - R_1 - a_1 r \cos \eta_1(\theta - \xi_1)}{\epsilon w_1} \right\} - 4 \tan^{-1} \exp \left\{ \frac{r - R_2 - a_2 r \cos \eta_2(\theta - \xi_2)}{\epsilon w_2} \right\}. \quad (94)$$

This approximate solution corresponds to a ring of inner radius $R_1$ and outer radius $R_2$ and is valid for $n_2 > n_1$.

Approximate equations governing the evolution of the ring can be derived from the averaged Lagrangian (95),

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12}, \quad (95)$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the averaged Lagrangians for a lump (8) with indices 1 and 2, respectively. For small $\epsilon$, the interaction term $\mathcal{L}_{12}$ is exponentially small as

$$\mathcal{L}_{12} = O \left( \epsilon^{-1} \text{sech}^{2} \frac{R_1 - R_2}{\epsilon} \right). \quad (96)$$

The fixed points of the variational equations obtained from the averaged Lagrangian (95) give ring type solutions. We note that these fixed points exist by an application of the implicit function theorem, since the linearised equations corresponding to the saddle points of $\mathcal{L}_1 + \mathcal{L}_2$ have non-zero eigenvalues and the perturbation term $\mathcal{L}_{12}$ is small as $\epsilon \to 0$.

As for the lump solution, the ring solution is unstable and again its lifetime increases with $n_1$ and $n_2$. In Fig. 9(a) and (b), the evolution of a ring solution as obtained from the full numerical solution of the Sine-Gordon equation (1) is shown. The instability of the ring solution is clearly visible. The extension of the instability of the ring solution from the Sine-Gordon equation (1) to the Baby Skyrme model is not possible and the existence of a stable ring solution of the Baby Skyrme model is currently under investigation.

We further note that simpler models, such as

$$\frac{1}{\hbar_0} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u + \frac{1}{\epsilon^2} u(1 - |u|^2), \quad (97)$$
Fig. 9. Numerical solution of Sine-Gordon equation (1) for ring initial condition (94) with $R_1 = 5$, $R_2 = 2$, $a_1 = a_2 = 0.1$, $V_1 = V_2 = 0.01$ and $w_1 = w_2 = 1$ at $t = 0$ with $s_1 = s_2 = 0.2$, $n_1 = 10$ and $n_2 = 5$. (a) $t = 0$, (b) $t = 5$.

have been considered, where $u$ is complex [10]. Equations such as this have solutions which represent non-zero degree localised disturbances. It was shown rigorously in [10] that the dynamics of a collection of localised solutions (vortices) can be approximated by finite-dimensional dynamics controlled by the re-normalised energy and a function which is related to the effect on the vortices of the non-vortex part of the solution. In this regard, we note that the energy was re-normalised, since each vortex has infinite self-energy. The detailed behaviour of the non-vortex part, which is related to the radiation, was not studied in detail as far as its influence on the motion of the vortices was
concerned. In particular, the evolution of an initially disturbed vortex was not studied. The results of the present work suggest that distorted vortices will radiate zero degree waves and settle onto a non-zero degree configuration. This result is contained in principle in the solutions of the asymptotic equations of [10], but no explicit results have been obtained in this respect as yet.

8. Conclusions

The behaviour of lump initial conditions with an oscillatory boundary layer for the two-dimensional Sine-Gordon equation has been studied in detail. It was found that initial lumps with flat tops eventually collapse, emitting a small amount of dispersive radiation in the process which does not have a significant influence on the evolution of the lump. Approximate equations based on an averaged Lagrangian give a simple explanation for the collapse of the lump. Furthermore, it was found that the solutions of these approximate equations were in excellent agreement with full numerical solutions of the Sine-Gordon equation. On the other hand, initial lumps with ridged tops take a much longer time to collapse and they radiate a significant amount of dispersive radiation in the form of a spiral wave in the process. This radiative loss from the pulse needs to be taken into account in the approximate equations and it was shown how radiative loss can be a destabilising mechanism. The agreement between the full numerical and approximate solutions when radiative loss was taken into account was found to be excellent. This agreement is indeed remarkable given the complexity of the radiative processes involved. The approximate solutions suggested the possibility of the existence of steady, singular (weak) solutions with square integrable gradients. However, these solutions were found to be unstable.

It has, therefore, been shown that lump initial conditions with wavy boundaries for the Sine-Gordon equation will eventually collapse due to the shedding of angular momentum into dispersive radiation. This collapse process is very general and other systems could in principle show the same collapse mechanism for wavy initial conditions. It is to be further noted that as the number of oscillations of the wavy boundary approach infinity with \( n \rightarrow \infty \) and \( \alpha \) constant, the lifetime of the lump becomes \( O(n) \). This recovers the leading order result of Neu [1]. The instability of lumps for the Sine-Gordon equation is intimately related to the stability of lump solutions for the Baby Skyrme model [4,5]. The explanation of this difference in stability is the different values of the conserved degree of the map in each case. This shows in a very simple example the significance of the topological nature of the solution and its dynamics.

Acknowledgements

A.A. Minzoni and N.F. Smyth acknowledge support from the grant CONACYT G25427–E from Consejo Nacional de Ciencia y Tecnologia Mexico. They also would like to acknowledge computing support from Ana Cecilia Perez.

References