Approximate solutions for magmon propagation from a reservoir

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A 1D partial differential equation (pde) describing the flow of magma in the Earth’s mantle is considered, this equation allowing for compaction and distension of the surrounding matrix due to the magma. The equation has periodic travelling wave solutions, one limit of which is a solitary wave, called a magmon. Modulation equations for the magma equation are derived and are found to be either hyperbolic or of mixed hyperbolic/elliptic type, depending on the specific values of the wave number, mean height and amplitude of the underlying modulated wave. The periodic wave train is stable in the hyperbolic case and unstable in the mixed case. Solutions of the modulation equations are found for an initial-boundary value problem on the semi-infinite line, these solutions representing the influx of magma from a large reservoir. The modulation solutions are found to consist of a full or partial undular bore. Excellent agreement with numerical solutions of the governing pde is obtained, except in the limit where the wave train becomes a train of magmons. An alternative approximation based on the assumption that the wave train is a series of uniform magmons is also derived and is found to be superior to modulation theory in this limit.

Keywords: initial-boundary value problem; magma flow; modulation theory; undular bore; solitary waves.

1. Introduction

Models of the upflux of magma through the Earth’s mantle which account for the compaction and distension of the surrounding matrix by the magma have been developed by Stevenson (1980), McKenzie (1984) and Scott & Stevenson (1984, 1986). These models treat the magma flow as a flow through a porous medium and involve a generalisation of Darcy’s Law for porous media flow to include matrix deformation. These models are thought to be a good model of magma transport beneath oceanic volcanic centres or in the subcontinental mantle. To model the deformation of the matrix by the magma, the solid matrix is approximated by a fluid with finite bulk viscosity. The flow of the magma in the matrix is then treated as a two-phase fluid flow with the matrix permeability assumed to have the power law form $k = k_0 f^n$, where $f$ is the fraction of liquid magma.

In one spatial dimension, the non-dimensional version of the magma flow model is (see McKenzie, 1984)

$$f_t = [(1 - f_0 f)w]_x, \quad w_{xx} = w f^{-n} + \frac{1 - f_0 f}{1 - f_0},$$

(1.1)
where \( f \) is the melt fraction, \( f_0 \) is the background melt fraction and \( w \) is the vertical velocity of the solid matrix. When the background melt fraction is small (\( f_0 \ll 1 \)) and \( n = 3 \), the single partial differential equation (pde)

\[
ft + [f^3(1 - ft_x)]_x = 0
\]  
(1.2)

for the melt fraction \( f \) is obtained. This equation was also derived by Scott & Stevenson (1984), who stated that \( n = 3 \) is physically reasonable for magma flow. Moreover, the value of \( n = 3 \) results in an analytically tractable form of governing equation (see Barcilon & Richter, 1986). If the dispersive term in (1.2) is ignored (which corresponds to matrix compaction), then Darcy’s Law for porous media flow is obtained. In this case, the relevant solution is a uniform shock with constant velocity.

Equation (1.2) possesses steady periodic travelling wave solutions similar to the cnoidal wave solution of the Korteweg–de Vries (KdV) equation. One limit of these travelling wave solutions is a solitary wave, called a magmon. This magmon solution corresponds to a region of high melt fraction which rises, due to buoyancy, at a constant speed without change of form. In general, the properties of periodic travelling waves, such as amplitude, frequency, wave number and mean height, vary with space and time. Whitham (1974) developed modulation theory for the KdV equation for the case in which a periodic wave train is slowly varying. The modulation equations were found to form a system of three first-order pdes for the mean height, wave number and amplitude of the modulated periodic wave solution. A useful, and easily obtained, solution of the modulation equations for the KdV equation, derived by Gurevich & Pitaevskii (1974) and Fornberg & Whitham (1978), is a simple wave solution, which corresponds physically to an undular bore. This undular bore solution represents the evolution, and smoothing out, of an initial step in mean height.

Barcilon & Richter (1986) solved the magma equation (1.1) numerically to obtain solitary wave profiles (magmons) and considered solitary wave interaction. They found that the collision of magmons is inelastic; a small dispersive wave train was generated after collision. They also performed a systematic search for conservation laws of (1.2) and found two laws only. Harris (1996) also performed a systematic search for conservation laws on a generalised version of (1.2). It was proved that only two conservation laws exist for (1.2). As integrable systems possess an infinite number of conservation laws, the fact that only two conservation laws exist for (1.2) implies that it is not integrable.

Takahashi et al. (1990) also considered (1.2) and showed that travelling wave solutions with compact support do not exist. They considered the numerical interaction of solitary waves with the background melt level very close to zero. They found that the collision had a very significant effect on the magmons, with the amplitudes of the large and small waves decreasing by 13 and 18%, respectively (see their Fig. 3).

The magma equation is also related to a modified magma equation

\[
ft + [f^3(1 + f_{xx})]_x = 0,
\]  
(1.3)

which is integrable. Takahashi et al. (1990) showed that the soliton solution of (1.3) does have compact support and illustrated the corresponding two-soliton solution. The highest order derivative in the magma equation (1.2) is \( f_{xxx} \), so it represents a (non-integrable) higher order Benjamin-Bona-Mahony (BBM) type equation. Similarly, the modified magma equation, in which the highest order derivative is \( f_{xxx} \), is a (integrable) higher order KdV-type equation. Takahashi et al. (1990) derived the KdV equation as a leading order approximation to the magma equation (1.2) by introducing long space and time-scales and by considering small disturbances. In the same vein, Nakayama & Mason (1999) found a series solution of (1.2) for small amplitude solitary waves; at the lowest order, the solution is the KdV soliton.
Barcilon & Lovera (1989) showed that the magmon solution of (1.2) is unstable to transverse (2D) perturbations. They assumed that the ratio of the vertical and horizontal length scales was characterised by a small parameter \( \epsilon \). They then expanded the 2D version of the magma equation in a series and showed that the \( O(\epsilon) \) contributions grow with time. Hence, when the 2D version of (1.2) is solved numerically, the 1D magmon breaks up into 2D cylindrical magmons. This behaviour is qualitatively similar to the Kadomtsev–Petviashvili (KP) equation for which 1D KdV solitons evolve into 2D lump solitons, see Infeld et al. (1995).

Wiggins & Spiegelman (1995) solved the 3D version of the full magma equation (1.1) numerically and found that both 1D and 2D magmons are unstable; they break up into sets of spherical 3D magmons. They also considered the numerical interaction of 3D magmons and found that the smaller wave breaks into two after repeated collisions.

Calculations using the expression (3.34) in Barcilon & Lovera (1989) show that the time-scale for transverse instability shortens in an approximately linear manner as the amplitude of the magmon increases. Also, a simple scaling of (1.2) indicates that a low background melt fraction increases the time for the instability to develop. Hence, small amplitude magmons propagating on a low background melt fraction, of which Figs 1 and 2 in this paper are examples, will be stable for relatively long times.

As a related problem to the present magma flow problem, Marchant & Smyth (1991) considered the initial-boundary value (IBV) problem for the KdV equation

\[
\begin{align*}
  u_t + 6u u_x + u_{xxx} &= 0, \quad x > 0, \\
  u(x, 0) &= u_b(x), \quad x > 0, \\
  u(0, t) &= u_r(t), \quad t > 0,
\end{align*}
\]

(1.4)

and found approximate and exact solutions. For the case of constant boundary and initial conditions, various types of steady and transient solutions were derived, the particular form of which depended on

\[\text{FIG. 1. The liquid magma fraction } f \text{ versus } x \text{ at } t = 60 \text{ for the parameter values } f_r = 0.4 \text{ and } f_b = 0.2. \text{ Shown are the modulation (dashed lines) and numerical (solid lines) solutions of (1.5). The modulation solution consists of the wave peak and trough envelopes and the mean height.}\]
Fig. 2. The liquid magma fraction $f$ versus $x$ at $t = 100$ for the parameter values $f_r = 0.4$ and $f_b = 0.3$. Shown are the modulation (dashed lines) and numerical (solid lines) solutions of (1.5). The modulation solution consists of the wave peak and trough envelopes and the mean height.

the relationship between $u_b$ and $u_r$. Of particular relevance to the present work, one of these transient solutions is described by the undular bore solution obtained from modulation theory. For all the cases considered, excellent comparisons were obtained between the approximate solutions and numerical solutions of the KdV equation. The IBV problem for magma flow is described by

$$f_t + [f^3 (1 - f_x)]_x = 0, \quad x > 0,$$

$$f(x, 0) = f_b(x), \quad x > 0, \quad f(0, t) = f_r(t), \quad t > 0. \tag{1.5}$$

Physically, a large reservoir of magma is located at $x = 0$ with liquid fraction $f_r$, and the magma then propagates into the semi-infinite domain, which has a background liquid fraction $f_b$. Hence, (1.5) represents a realistic model for magma rising to the Earth’s surface from a large reservoir.

Approximate solutions of the magma flow problem (1.5), for the physically relevant case of $f_r$ and $f_b$ both positive, are developed in the current paper. In Section 2, modulation theory to describe slowly varying travelling wave solutions of (1.2) is developed. In Section 3, solutions of the derived modulation equations are used to solve (1.5); it is found that both a full or partial undular bore can occur. Excellent comparisons between the approximate wave envelopes from modulation theory and numerical solutions of (1.5) are obtained. In Section 4, modulation theory and an alternative approximate theory, which assumes that a train of uniform magmons is generated, are used to estimate the amplitude of magmons generated by (1.5). Simple explicit and implicit expressions for the magmon amplitude are obtained in terms of the reservoir and background melt fractions $f_r$ and $f_b$; good agreement with numerical solutions is found. In Section 5, the longitudinal (1D) stability of the magma waves is considered. In contrast to the KdV equation, which has hyperbolic modulation equations, the modulation equations for (1.2) can be either hyperbolic or of mixed hyperbolic/elliptic type, depending on the parameter values of the modulated wave.
2. Modulation theory

In this section, the modulation equations for the magma equation (1.2) will be derived. Firstly, the steady periodic travelling wave solution of (1.2) will be determined, as this solution forms the basis of modulation theory. Periodic travelling wave solutions have the form

\[ f = f(\theta), \quad \theta = kx - \omega t, \]  

which is substituted into (1.2). Performing two integrations then gives

\[ k^2 f'(\theta)^2 = 2f - 2U^{-1} f^3 + Af^2 + B, \]  

where \( A \) and \( B \) are constants of integration and \( U = \omega k^{-1} \) is the wave speed. To evaluate the solution of (2.2) in terms of elliptic functions, the cubic polynomial is factorised to give

\[ k^2 f'(\theta)^2 = 2f - 2U^{-1}(p - f)(f - q)(f - r), \]  

where we assume \( p > q > r \). The zeros of this polynomial are related to \( U \) and the constants \( A \) and \( B \) by

\[ p + q + r = AU, \quad pq + pr + qr = U, \quad pqr = BU. \]  

These zeros play a fundamental role in the development of the modulation equations for (1.2). It can now be found (see Byrd & Friedman, 1971) that the periodic wave solution is given implicitly by

\[ \left( \frac{2}{U k^2} \right)^{\frac{1}{2}} \theta = \frac{2p(p - r)}{p - q} \left[ \left( m - \frac{p - q}{p} \right) u + \frac{p - q}{p} E(u, m) \right], \]

\[ u = \text{sn}^{-1} \left( \frac{p - f}{p - q} \right), \quad m = \frac{p - q}{p - r}. \]

Here \( E(u, m) \) is the incomplete elliptic integral of the second kind and \( \text{sn}(u, m) \) is the Jacobian elliptic sine function. Both of these have modulus squared \( m \). From the steady periodic wave solution (2.5), we see that the wave oscillates between \( p \) and \( q \). The zero \( p \) is the crest height and \( q \) is the trough height. Hence, the amplitude of the wave is \( a = (p - q)/2 \). The dispersion relation is determined by the requirement that \( f \) be \( 2\pi \) periodic in \( \theta \). From the periodic solution (2.5), the dispersion relation is then

\[ \pi \left( \frac{2}{U k^2} \right)^{\frac{1}{2}} = \frac{4p(p - r)^{\frac{1}{2}}}{p - q} \left[ \left( m - \frac{p - q}{p} \right) K(m) + \frac{p - q}{p} E(m) \right], \]

where \( K(m) \) and \( E(m) \) are the complete elliptic integrals of the first and second kinds, respectively.

In the limit \( m \to 1 \), the solution is a solitary wave solution, called a magmon. Taking this limit \( (q \to r) \) in the periodic wave (2.5) gives the magmon solution implicitly as

\[ \left( \frac{2}{U} \right)^{\frac{1}{2}} \theta = \frac{2q}{(p - q)^{\frac{1}{2}}} \text{tanh}^{-1} \left( \frac{p - f}{p - q} \right)^{\frac{1}{2}} + 2(p - f)^{\frac{1}{2}}, \]

where we have set \( k = 1 \) without loss of generality. This solitary wave solution is equivalent to that obtained by Scott & Stevenson (1984).
Modulation theory as originally developed by Whitham (1974) was based on averaging a Lagrangian for the governing equation. However, for the magma equation (1.2) it is easier to average conservation equations to obtain the relevant modulation equations. By Nöther’s Theorem, averaging the Lagrangian and averaging the conservation equations are equivalent.

The steady periodic wave train solution (2.5) forms the basis of modulation theory in that we now consider a slowly varying wave train and wish to find equations for the amplitude, wave number and mean level of this slowly varying wave train. Let us then consider a slowly varying wave train of amplitude $a$, wave number $k$, frequency $\omega$ and mean level $\beta$, these parameters being functions of the slow space coordinate $X = \epsilon x$ and the slow time $T = \epsilon t$. Here, $0 < \epsilon \ll 1$ is a measure of the slow variation.

For a slowly varying wave train, the phase function is given by

$$\theta = \epsilon^{-1} \Theta(X, T), \quad \omega = -\Theta_T, \quad k = \Theta_X,$$

where $\omega$ is the local frequency and $k$ is the local wave number. Let us denote the mean value of $u(\theta, X, T)$ over a wave period by $\bar{u}$,

$$\bar{u} = \frac{1}{2\pi} \int_0^{2\pi} u \, d\theta.$$

Averaging (2.2) for the periodic wave over a wave period, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} k^2 f'^2 \, d\theta = \frac{1}{2\pi} \oint k^2 f' \, df = kW,$$

where

$$W(A, B, U) = \frac{1}{2\pi} \oint f^{-1}[-2U^{-1}f^3 + 2Af^2 - 2f + 2B]^{1/2} \, df.$$

Here, the integral $\oint$ denotes integration in $f$ over a wave period. The integral $W$ plays a central role in determining the modulation equations and, although it could now be evaluated in terms of elliptic integrals, it is best left in this form while manipulations are done and only expressed in terms of elliptic integrals when the modulation equations are in final form.

The magma equation (1.2) is in conservation form. Noting that $f = f(\theta, X, T)$, we deduce

$$-\omega f + \epsilon f_T + k(f^3 + \omega k f^3 f_{\theta\theta} + \epsilon \omega f^3 f_{\theta} + \epsilon \omega_X f^3 f_\theta - \epsilon k f^3 f_{\theta T})_\theta + \epsilon (f^3 + k \omega f^3 f_{\theta\theta})_X + O(\epsilon^2) = 0.$$  

Averaging this equation over a wave period then gives

$$\beta_T + (\bar{f}^3 + k \omega \bar{f}^3 f_{\theta\theta})_X = 0,$$

since $\beta$ denotes the mean of $f$. Now from (2.2) for the steady periodic wave,

$$k^2 \bar{f}^3 f_{\theta\theta} = -U^{-1} \bar{f}^3 + \beta - 2B.$$  

Substituting this expression into the average (2.13), we then obtain the mean flow equation

$$\beta_T + (U \beta - 2UB)_X = 0.$$

Since this modulation equation governs the mean wave level, it can be considered to take the role of a mass conservation equation.
A second conservation equation for the magma flow equation can be obtained by dividing (1.2) by $f^2$. This second conservation equation is

$$(-f^{-1} - f_x^2)_t + (3f - ff_x)_x = 0. \quad (2.16)$$

The second conservation equation (2.16) for the magma flow equation does not correspond to energy or momentum conservation and is an unusual conservation equation in that it contains the term $f^{-1}$. We note that this term is acceptable since $f$ does not become zero as the magma flow occurs on a constant background state $f = f_b > 0$. Barcilon & Richter (1986) also obtained the conservation law (2.16) and commented that its physical meaning was unclear.

On noting that $f = f(\theta, X, T)$, we obtain from the second conservation equation (2.16)

$$\omega(f^{-1} + k^2 f_\theta^2 + 2\epsilon f_\theta f_X)_\theta + k(3f + \omega f_\theta f_\theta + \epsilon f_\theta f_{\theta X} - \epsilon k f_f f_\theta - \epsilon k f_f f_T)_\theta$$

$$+ \epsilon(-f^{-1} - k^2 f_\theta^2)_T + \epsilon(3f + \omega f f_\theta)_X + O(\epsilon^2) = 0. \quad (2.17)$$

Averaging over a wave period, we deduce the equation

$$(-f^{-1} - k^2 f_\theta^2)_T + (3\beta + \omega k f f_\theta^2)_X = 0. \quad (2.18)$$

To determine the averaged quantities in this equation, we again use (2.2) for the periodic wave solution. From these equations, we have

$$\omega k f f_\theta = -\beta + UF^{-1} - 2BU^{-2}, \quad k^2 f_\theta^2 = 2A - 2U^{-1} \beta - 2f^{-1} + 2Bf^{-2}. \quad (2.19)$$

Hence, the second modulation equation is

$$\left( f^{-1} + 2U^{-1} \beta - 2Bf^{-2} - 2A \right)_T + \left( 2\beta + Uf^{-1} - 2BU^{-2} \right)_X = 0. \quad (2.20)$$

It now remains to determine the average values of $f^{-1}$ and $f^{-2}$. These averages can be expressed in terms of the integral $W$ defined in (2.11). Averaging (2.2) for the periodic wave and using the relation (2.10) between $W$ and $f$, we have

$$\overline{f^{-1}} = -U^{-1} \beta - \frac{1}{2}kW + A + Bf^{-2}. \quad (2.21)$$

To find the average of $f^{-2}$, we differentiate the expression (2.11) for $W$ once with respect to each of $A$, $B$ and $U$ and use the periodic wave equation (2.2) to obtain

$$k = W_A^{-1}, \quad \overline{f^{-2}} = kW_B \quad \text{and} \quad \beta = kU^{-2}W_U. \quad (2.22)$$

Substituting the averages (2.21) and (2.22) into (2.19), we finally obtain the second modulation equation as

$$\left( UW_A^{-1}W_U - \frac{1}{2}W_A^{-1}W - A - BW_A^{-1}W_B \right)_T$$

$$+ \left( U^2W_A^{-1}W_U - \frac{1}{2}UW_A^{-1}W - UBW_A^{-1}W_B + AU \right)_X = 0. \quad (2.23)$$
The first of the modulation equations, (2.15), can also be re-expressed in terms of $W$ on using (2.22) to give
\[(U^2 W_A^{-1} W_U)_T + (U^3 W_A^{-1} W_U - 2UB)_X = 0. \tag{2.24}\]
The third modulation equation needed to determine the three quantities $\beta$, $k$ and $\alpha$ is obtained from the consistency relation $k_T + \omega_X = 0$, which, combined with $\omega = kU$, gives
\[(W_A^{-1})_T + (U W_A^{-1})_X = 0. \tag{2.25}\]
The full set of modulation equations for the magma flow equation (1.2) are then (2.23)–(2.25). These equations are a set of three pdes for $A$, $B$ and $U$, with the physical variables $a$, $k$ and $\beta$ given by (2.4) and (2.22) in terms of $A$, $B$ and $U$. Whitham (1974) wrote the system of modulation equations for the KdV equation in terms of the zeros of the cubic polynomial of (2.3), $p$, $q$ and $r$, instead of the variables $U$, $A$ and $B$. This allowed him to set the modulation equations in Riemann invariant form. We will also write the modulation equations for magma flow in terms of the zeros of the cubic polynomial, instead of the variables $U$, $A$ and $B$. To simplify the algebra associated with this, we perform the change of variable $f_1 = U^{-\frac{1}{2}} f$ in the definition (2.11) for $W$, giving
\[
W = \frac{1}{2\pi} \int f_1^{-1} \left[ -2f_1^3 + 2AU^\frac{1}{2} f_1^2 - 2U^\frac{1}{3} f_1 + 2B \right]^{\frac{1}{2}} df_1
\]
where the zeros of the new cubic polynomial corresponding to (2.3) are determined by
\[
p_1 + q_1 + r_1 = AU^\frac{1}{2}, \quad p_1 q_1 + p_1 r_1 + q_1 r_1 = U^\frac{1}{3} \quad \text{and} \quad p_1 q_1 r_1 = B. \tag{2.27}\]
Identities relating the second derivatives of $W$ to its first derivatives are now needed. These identities, which are found by integration by parts, are
\[
W_{p_1 q_1} = \frac{W_{q_1} - W_{p_1}}{2(p_1 - q_1)}, \quad W_{p_1 p_1} = \frac{W_{p_1} - W_{r_1}}{2(p_1 - r_1)} = \frac{W_{p_1} - W_{q_1} - W_{r_1}}{2p_1} - W_{p_1 q_1}, \tag{2.28}\]
plus similar expressions for the other second derivatives obtained by interchanging $p_1$, $q_1$ and $r_1$ (by symmetry).

The modulation equations can be written as the matrix equation
\[A v_t + B v_x = 0, \tag{2.29}\]
where $v = (p_1, q_1, r_1)^T$ and $A$ and $B$ are $3 \times 3$ matrices with their coefficients presented in Appendix A (see (A.1)). We then solve for the eigenvalues $\lambda_i$ of the system $(B - \lambda A)$ and the left eigenvectors $I_i$, which satisfy $I_i (B - \lambda_i A) = 0$. Then the system of pdes (2.29) is reduced to
\[I_i A \frac{dv}{dt} = 0, \quad \text{on} \quad \frac{dx}{dt} = \lambda_i, \quad i = 1, 2, 3, \tag{2.30}\]
which is a set of ordinary differential equations (odes) along the characteristic directions. Whitham (1974) was able to set the system of KdV modulation equations in Riemann invariant form by integrating odes of the form (2.30) analytically. However, it is not possible to set the magma flow modulation
equations in Riemann invariant form, so the eigenvalue equation and the characteristic equations (2.30) must be solved numerically. In fact, from the numerical evidence presented in Section 5, the modulation equations can be either hyperbolic or of mixed type, depending on the exact values of the zeros.

Flaschka et al. (1980) showed that the KdV modulation equations can be derived from the inverse scattering solution for the KdV equation and that their Riemann invariant form was a consequence of an invariant representation of the modulation equations evaluated near certain branch points (related to the zeros of a cubic polynomial). Integrability and a periodic solution given explicitly in terms of hyperelliptic functions are thus linked to the ability to set the modulation equations into Riemann invariant form. As the magma equation (1.2) is not integrable, it is not surprising that a Riemann invariant form for (2.29) cannot be found.

3. The IBV problem and the simple wave solution

In this section, approximate solutions of the IBV problem (1.5) for the magma equation will be found. We will focus on the physically realistic case for which \( f_r \) and \( f_b \) are both positive, and it is assumed that both \( f_r \) and \( f_b \) are constant. We also note that a simple scaling of (1.5) indicates that the solution depends on the single parameter \( f_r / f_b \).

In the IBV problem for the KdV equation, three different types of solution occur in the parameter space when the initial value \( u_r \) and the boundary value \( u_b \) are both positive (see Marchant & Smyth, 1991). They are a mean height variation (which occurs for \( 0 \leq u_r < u_b \)), a full undular bore (\( 0 \leq u_r / 2 < u_b < u_r \)) and a partial undular bore (\( 0 \leq u_b < u_r / 2 \)). The KdV undular bore solution is described by a simple wave solution of the KdV modulation equations which is found analytically in terms of complete elliptic integrals of the first and second kinds. The bore solution consists of cnoidal waves, with solitons (modulus \( m = 1 \)) at its leading edge and sinusoidal waves (modulus \( m = 0 \)) at its trailing edge. The IBV problem for the KdV equation was considered by Marchant & Smyth (1991), with approximate solutions constructed using the KdV undular bore solution.

These three different types of solution also occur in the magma flow IBV problem (1.5). The mean height variation solution can be derived by neglecting the dispersive effects in (1.2). This gives \( f_t + 3 f^2 f_x = 0 \) which has the solution

\[

d = \begin{cases} 
  f_r, & 0 \leq \frac{x}{t} \leq 3 f_r^2, \\
  \sqrt{\frac{x}{3 t}}, & 3 f_r^2 \leq \frac{x}{t} \leq 3 f_b^2, \\
  f_b, & \frac{x}{t} \geq 3 f_b^2.
\end{cases}
\]

(3.1)

This solution is valid in the region \( 0 \leq f_r \leq f_b \) and hence represents the case for which the liquid fraction in the reservoir is lower than the background level. Magma then drains back into the reservoir with no waves generated.

The magma undular bore solution will now be found from a simple wave solution of the modulation equations. The simple wave solution only depends on \( \frac{x}{t} \) and is an expansive fan in the modulus squared, \( m \), along the middle characteristic. The limits of this expansion fan are magmons at the leading edge (where \( m = 1 \)) and sinusoidal waves at the trailing edge (where \( m = 0 \)). If the eigenvalues are ordered such that \( \lambda_1 < \lambda_2 < \lambda_3 \), then the fan must occur along the middle characteristic, hence \( \frac{x}{t} = \lambda_2 \). Information from both ahead of and behind the bore can then propagate into the bore along the \( \lambda_1 \) and \( \lambda_3 \) characteristics. An expansive fan cannot be found along \( \lambda_1 \) or \( \lambda_3 \); e.g. if \( \frac{x}{t} = \lambda_3 \) was assumed, then the information from behind the bore cannot propagate through the bore as it is not travelling fast enough.
As the waves generated by the reservoir must occur in \( x > 0 \), the slope of the middle characteristic cannot be negative. Therefore, as the reservoir is at the fixed position \( x = 0 \), we require \( \lambda_2 = 0 \) at \( x = 0 \). A partial undular bore will then result if \( \lambda_2 = 0 \) for some \( m = m_0 > 0 \). The bore is referred to as partial since it does not extend fully back to \( m = 0 \) but stops at \( x = 0 \). In this case, cnoidal waves of modulus squared \( m_0 \) are created at the trailing edge of the partial undular bore at \( x = 0 \). On the other hand, since \( m = 0 \) is the minimum possible value of the modulus squared, if \( \lambda_2 > 0 \) for \( m = 0 \), a full undular bore will result. This full undular bore will propagate away from \( x = 0 \) as the velocity of its trailing edge is positive, so that at the boundary \( x = 0 \), \( \beta = f_r \) with no superimposed waves.

We need to solve for \( p_1, q_1 \) and \( r_1 \) over the range of the undular bore (between \( m = m_0 \) and \( m = 1 \)), subject to the appropriate boundary conditions, where the odes (2.30) describe how \( p_1, q_1 \) and \( r_1 \) vary through the bore. For the undular bore solution of the KdV equation, the Riemann invariants corresponding to \( \lambda_1 \) and \( \lambda_3 \) are constant throughout the bore; hence, they provide explicit relations describing the variation of \( p_1, q_1 \) and \( r_1 \) throughout the bore. In the case of the magma modulation equations, the undular bore solution is described by the two odes corresponding to \( \lambda_1 \) and \( \lambda_3 \), supplemented by an ode obtained by differentiating the relation \( m = (p_1 - q_1)/(p_1 - r_1) \). This system of odes can be written as

\[
A_1 \begin{bmatrix} dp_1 \\ dq_1 \\ dr_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dm \end{bmatrix}, \quad \text{where } A_1 = \begin{bmatrix} l_{1p_1} & l_{1q_1} & l_{1r_1} \\ l_{3p_1} & l_{3q_1} & l_{3r_1} \\ \frac{(q_1-r_1)}{(p_1-r_1)^2} & \frac{-1}{(p_1-r_1)} & \frac{(p_1-q_1)}{(p_1-r_1)^2} \end{bmatrix}.
\]

Equation (3.2) provides the basis for numerically integrating through the undular bore. By making a small step, \( \Delta m \), in the modulus squared \( m \) and then solving (3.2), the appropriate changes in \( p_1, q_1 \) and \( r_1 \) can be found. Hence, we can integrate from the leading to trailing edge of the undular bore.

By using (2.22) and (2.27), it can be shown that a cnoidal wave is described by the expressions

\[
U = (p_1 q_1 + q_1 r_1 + p_1 r_1)^3, \quad a = U^\frac{1}{2} \frac{(p_1 - q_1)}{2},
\]

\[
\beta = \frac{U^{\frac{1}{3}}}{3N} \left[ p_1 r_1 + q_1 r_1 + 2r_1^2 - p_1 q_1 + 2P(m)(p_1 - r_1)(p_1 + q_1 + r_1) \right],
\]

\[
k = \frac{\pi (p_1 - r_1)^{\frac{1}{2}}}{2\sqrt{2U^{2/3}K(m)N}},
\]

where

\[
N = r_1 + (p_1 - r_1)P(m), \quad P(m) = \frac{E(m)}{K(m)}, \quad m = \frac{p_1 - q_1}{p_1 - r_1}.
\]

These equations relate the zeros to the physical variables of phase speed \( U \), wave number \( k \) and mean liquid fraction of magma \( \beta \). \( K(m) \) and \( E(m) \) are complete elliptic integrals of the first and second kinds of modulus squared \( m \), respectively.

The expressions (3.3) are now used to obtain boundary conditions for the odes (3.2) in terms of the zeros of the cubic. The boundary condition at the leading edge of the bore is that the mean liquid fraction \( \beta = f_b \). At the leading edge, \( m = 1 \) or \( q_1 = r_1 \). Letting \( q_1 = r_1 \) in the expression for \( \beta \) in (3.3) gives

\[
f_b = q_1^2 (q_1 + 2p_1), \quad \text{at } m = 1.
\]
At the trailing edge of the bore, the mean liquid fraction $\beta = f_r$ and $m = m_0$. For a partial bore, $\lambda_2(m_0) = 0$, while for a full bore, $m_0 = 0$ and $\lambda_2(0) \geq 0$. Hence,

$$f_r = \frac{U}{3N} \left[ p_1 r_1 + q_1 r_1 + 2 r_1^2 - p_1 q_1 + 2 P(m)(p_1 - r_1)(p_1 + q_1 + r_1) \right],$$

(3.5)

at $m = m_0$, where $N$ and $P(m)$ are defined in (3.3). The boundary conditions (3.4) and (3.5) represent two equations in the two unknowns $p_1$ and $q_1$, but with the difficulty that the conditions are applied at the leading and trailing edges of the undular bore. However, the values of the zeros at the leading and trailing edges of the bore can be related by solving the ode system (3.2). The non-linear equations (3.2) were solved using a commercially available root-finding package. The solution of (3.2)–(3.5) shows that the transition from a full to a partial undular bore occurs when $f_r = 1.60 f_b$. For the KdV equation, the equivalent result can be obtained analytically, with the transition occurring along the line $u_r = 2 u_b$. Hence, for the magma equations a full bore occurs when $0 \leq 0.625 u_r < u_b < u_r$ and a partial undular bore occurs for $0 \leq u_b < 0.625 u_r$.

Figure 1 shows the liquid magma fraction $f$ versus $x$ at $t = 60$ for the parameter values $f_r = 0.4$ and $f_b = 0.2$. The approximate solution consists of a partial undular bore which lies in the region $0 < x < 16.68$ and takes the solution from a mean level of $f = f_r = 0.4$ at $x = 0$ to $f = f_b = 0.2$ at $x = 16.68$. At the leading edge of the bore magmons occur (with $m = 1$), while at $x = 0$ cnoidal waves of modulus squared $m_0 = 0.329$ are generated. As modulation theory gives averaged information about the undular bore, the mean level, $\beta$, and the wave peak and trough envelopes, $p$ and $q$, are plotted from the modulation solution. An excellent comparison between the modulation and numerical solutions is obtained.

Figure 2 shows the liquid magma fraction $f$ versus $x$ at $t = 100$ for the parameter values $f_r = 0.4$ and $f_b = 0.3$, whilst Fig. 3 shows a perspective plot of this solution up to $t = 100$. The solution of Fig. 2 is similar to that of Fig. 1, except that the modulation solution now consists of a full undular bore which lies in $11.14 < x < 38.83$ and takes the solution from a mean level of $f = f_r = 0.4$ at $x = 11.14$ to $f = f_b = 0.3$ at $x = 38.83$. At the leading edge of the bore magmons occur, while at the trailing edge linear (small amplitude) waves occur. As for Fig. 1, an excellent comparison between the modulation and numerical solutions occurs. For $0 < x < 11.14$, the numerical solution consists of small amplitude waves on a mean level of $f = 0.4$, these waves not being captured by the modulation equations.

As mentioned in Section 1, magmon solutions of (1.2) are unstable to transverse perturbations. Wiggins & Spiegelman (1995) perform numerical simulations which illustrate the break up of 1D magmons into sets of spherical 3D magmons. From their Fig. 2, the time to instability is $t \approx 25$ for a magmon amplitude of four on a background melt fraction of unity. Scaling the background melt fraction and using the expression (3.34) from Barcilon & Lovera (1989) indicates that the leading magmons of Figs 1 and 2 are stable to transverse perturbations up to $t = 150$, well beyond the time at which the results are presented.

4. Approximations for the magmon amplitude

A key point of interest in the IBV problem (1.5) is the amplitude of the magmon generated at the leading edge of the undular bore. In this section, we compare the predictions from modulation theory, an alternative approximation, which assumes that a uniform train of magmons is generated, and numerical solutions of (1.5).
4.1 A simplified estimate from modulation theory

The prediction of the magmon amplitude from modulation theory is obtained by solving the ode system (3.2) and boundary conditions (3.4) and (3.5). Solutions of these equations show that through the undular bore the quantities \( p + q \) and \( q + r \) vary only by a small amount \( (p, q \) and \( r \) are the original, unscaled zeros of the cubic); e.g. in Fig. 1 these quantities vary by no more than 2% through the undular bore. Hence, we assume that the quantities \( p + q \) and \( q + r \), which are the Riemann invariants associated with KdV modulation theory, are constant through the magma bore also.

The magma equation can be approximated by the KdV equation when long time-scales and space scales and small disturbances from the background melt fraction are considered (see Takahashi et al., 1990). Moreover, the periodic solutions of the magma equation (described by the ode (2.3)) and the KdV equation are closely related (see p. 418 of Takahashi & Satsuma, 1988). As modulation theory considers the slow evolution of the periodic wave solution, it is unsurprising that there is a close connection between the magma and KdV modulation theory.

For a full undular bore, \( \beta = f_b = q = r \) at the leading edge and \( \beta = f_r = p = q \) at the trailing edge of the bore. Hence, we assume that \( q + r = 2f_{b} \) and \( p + q = 2f_{r} \) are invariant throughout the bore. Using the definition of \( m \) in (2.5), we obtain an explicit expression for the magmon amplitude of \( a = p - q = 2(f_r - f_b) \). This is the same result as obtained from KdV modulation theory, namely that the soliton amplitude is twice the difference in the mean height at the leading and trailing edges of the bore. For a partial undular bore, the amplitude is found as the solution of the transcendental equations (3.4) and (3.5), with the zeros at the leading and trailing edges of the bore linked by assuming \( p + q \) and \( q + r \) are invariant, rather than by the system of odes (3.2).
4.2 A uniform train of magmons

An alternative approximate solution of the IBV problem (1.5) for magma flow can be found by assuming that the wave train generated is a uniform train of magmons. In a study of the forced KdV equation, Grimshaw & Smyth (1986) assumed that the upstream flow resulting from the flow of a fluid over a topographic forcing at a Froude number near one was a uniform train of KdV solitons. This solution was found to be in good agreement with numerical solutions and the appropriate solution of KdV modulation theory (see Smyth, 1987).

From the magmon solution (2.7), we see that $q = f_b$. Hence, in the magmon limit $q \to r$, we find the magmon speed from (2.4) as $U = 2a_s f_b + 3 f_b^2$, where the magmon amplitude is $a_s = p - q$. To find an approximate solution of (1.5), let us assume that at time $t$, there are $N$ magmons of uniform amplitude $a_s$ and spacing $h$ in the region $x > 0$. Integrating the conservation equations (1.2) and (2.16) on the semi-infinite line gives

$$\frac{d}{dt} \int_0^\infty (f - f_b) \, dx = f_r^3 [1 - f_{tx}(0,t)] - f_b^3,$$

$$\frac{1}{2} \frac{d}{dt} \int_0^\infty (f_{b}^{-1} - f^{-1} - f_x^2) \, dx = f_r - \frac{3}{2} f_b + \frac{1}{2} f_r [1 - f_{tx}(0,t)].$$

(4.1)

In deriving these expressions, we have used the fact that $f \to f_b$ as $x \to \infty$ since there are no magmons ahead of the line $x = Ut$. Since we are assuming that there are $N$ magmons in $x > 0$ at time $t$, the integrals on the left hand side of (4.1) are $N$ times the integral for one magmon. It can be found from the periodic wave solution (2.3) with $q = r = f_b$ that for one magmon

$$\int_{-\infty}^{\infty} (f - f_b) \, dx = 2(2U a_s)^{\frac{1}{2}} \left( f_b + \frac{2}{3} a_s \right),$$

$$\frac{1}{2} \int_{-\infty}^{\infty} (f_{b}^{-1} - f^{-1} - f_x^2) \, dx = (2U a_s)^{\frac{1}{2}} H,$$

(4.2)

where

$$H = f_b^{-1} - \frac{2}{3} U^{-1} a_s + 2a^{-1} f_b (1 + f_b a_s^{-1}) \frac{1}{2} \cosh^{-1} (1 + a_s f_b^{-1}) \frac{1}{2} - 2U^{-1} f_b.$$

The flux terms on the right hand sides of (4.1) involve the value of $f_{tx}(0,t)$, which is unknown unless the exact solution is to be determined. We shall assume here that $f_{tx}(0,t) = 0$. In doing so, we are assuming that the derivatives of $f$ are small compared with the values of $f$ at $x = 0$. It is expected that this will be so as long as the magmon amplitude is not too large. Camassa & Wu (1989) similarly ignored $u_{xx}(0,t)$ when solving the corresponding IBV problem (1.4) for the KdV equation using inverse scattering and obtained results in good agreement with numerical solutions. Another way of justifying neglecting $f_{tx}(0,t)$ is that the average value of $f_{tx}(0,t)$ will be zero over the time of generation of a magmon (by symmetry of the magmon solution about its peak).

Using (4.2) in (4.1), neglecting $f_{tx}(0,t)$ and manipulating the expressions, gives

$$(f_r^2 + f_r f_b + f_b^2) H = 3 f_b + 2a_s,$$

(4.3)

which implicitly determines the amplitude $a_s$ of the magmon in terms of $f_r$ and $f_b$. 
4.3 Comparison of the approximate solutions

Figure 4 shows the magmon amplitude \( a \) versus liquid magma fraction \( f_b \) for \( f_r = 0.4 \). Compared in the figure are the magmon amplitude at the leading edge of the undular bore as given by the uniform magmon approximation (4.3), modulation theory and the numerical solution of (1.5). Note that the modulation theory predictions are calculated from (3.4) and (3.5) and on assuming that \( p + q \) and \( q + r \) are invariant across the bore. The resulting amplitude in Fig. 4 is the same, to graphical accuracy, as the solution of (3.4) and (3.5) and the ode system (3.2).

A summary of the different solution types which can occur is as follows. As \( f_b \to 0.4 \), \( f_r - f_b \to 0 \) and the magmon amplitude becomes small, whilst in the limit as \( f_b \to 0 \), the amplitude becomes large. For \( f_b > 0.25 \) a full undular bore occurs, whilst a partial bore occurs for \( f_b < 0.25 \). For \( f_b < 0.1 \) a train of near magmons occurs as the modulus squared of the waves generated at the boundary is \( m_0 > 0.7 \). From Fig. 4, it can be seen that for \( f_b > 0.15 \) modulation theory is extremely accurate but loses accuracy as \( f_b \to 0 \) since it underestimates the magmon amplitude significantly in this limit. On the other hand, the uniform magmon theory maintains good accuracy for the range of \( f_b \) displayed in this figure. It performs better for small \( f_b \) since the magmon approximation is valid for a train of uniform magmons (where \( m_0 \to 1 \)).

Both approximate theories break down as \( f_b \to 0 \), however, as the conservation law (2.16) contains the quantity \( f^{-1} \). Modulation theory does much worse than uniform magmon theory in this limit in that it significantly underpredicts the magmon amplitude. This is due to the fact that for small \( f_b \) the solution is very nearly a train of magmons. Now solitary waves of crest height \( p \) and trough height \( q \) have mean height \( q \) (because the period is infinite), while periodic waves with \( m < 1 \) have mean height greater than their trough height. The boundary condition at \( x = 0 \) is satisfied in the mean for the modulation theory solution (see (3.4) and (3.5)). As the solution becomes more like a train of magmons, the satisfaction of the boundary condition at \( x = 0 \) in the mean will become less valid.
It should also be pointed out that while uniform magmon theory gives a good prediction of the magmon amplitude for small $f_b$, a comparison of amplitudes at the trailing edge of the undular bore shows the superiority of modulation theory for large values of $f_b$. This is because magmon theory predicts constant amplitude throughout the bore, while modulation theory shows that the amplitude decreases to a fixed value at the trailing edge of the bore (see Figs 1 and 2).

5. Stability of the magma flow equations

The longitudinal (1D) stability of the periodic magma waves will now be investigated. This stability investigation is unrelated to the transverse (2D) stability analysis of Barcilon & Lovera (1989). The modulation equations (2.29) can have three real characteristics, which correspond to a hyperbolic system, or can have one real and two complex characteristics, which correspond to a system of mixed hyperbolic/elliptic type. The wave train is stable if the system of modulation equations is hyperbolic and the wave train is unstable for systems of mixed or elliptic type (see Whitham, 1974). The characteristics of the modulation equations are given by $|B - \lambda A| = 0$, which can be written in the form

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$

where the coefficients of the cubic equation are functions of the coefficients of the $A$ and $B$ matrices. The number of real roots of (5.1) depends on

$$D = Q^2 - P^3, \quad P = \frac{1}{9}b^2 - \frac{1}{3}ac \quad \text{and} \quad Q = \frac{1}{6}abc - \frac{1}{6}a^2d - \frac{1}{27}b^3.$$  

(5.2)

There is one real root if $D$ is positive and three if $D$ is negative. Hence, the system of modulation equations is hyperbolic (stable) for negative $D$ and of mixed type (unstable) for positive $D$. Numerical solutions of (5.2) indicate that the modulation equations are hyperbolic (stable) when all the zeros $p_1,$

![Fig. 5. Lines along which $D = 0$ in the $(a, k)$ plane. The three lines are for $\beta = 0.375$ (bottom curve), 0.25 (middle curve) and 0.175 (top curve).](image)
$q_1$ and $r_1$ are positive and of mixed type (unstable) when all the zeros are negative. As the liquid fraction must be positive, the case of all positive zeros is physically realistic, while the case of all negative zeros is not. Physically realistic periodic waves do also occur with $p_1$ and $q_1$ positive and $r_1$ negative, however, a case which can be either stable or unstable.

The expression $D = 0$ is a function of the zeros $p_1$, $q_1$ and $r_1$, so it represents a surface in $(p_1, q_1, r_1)$ space which divides the parameter space into regions in which the modulation equations are hyperbolic (stable) or of mixed type (unstable). Equations (3.3) relate the physical variables to the zeros, so (3.3) and $D = 0$ represent a set of transcendental equations for the surface in physical parameter space.

Figure 5 shows lines in the $(a, k)$ plane for which $D = 0$. Note that the curves displayed in Fig. 5 are self-similar; $a\beta^{-1}$ and $ka^3$ are invariant quantities. Below each line (for that particular value of $\beta$) the modulation equations are hyperbolic (stable), while above the line the equations are of mixed type (unstable). Moving to the right along a given curve increases the modulus squared $m$ of the cnoidal wave until the magmon limit is approached. Hence, the periodic magma waves of (1.2) are not stable for all physically realistic parameter values. For the IBV problem (1.5), the zeros of the cubic are always positive, hence, the modulation equations are always hyperbolic (stable) for an undular bore, as they must be.

6. Conclusions

Modulation theory has been developed for the magma flow equation (1.2) and used to solve the associated IBV problem. Excellent comparisons between the modulation solution and numerical solutions of (1.5) are obtained. Approximate expressions for the magmon amplitude at the leading edge of an undular bore are also obtained from modulation theory and by assuming that a uniform train of magmons is generated. These simple expressions allow the amplitude to be found either explicitly (the modulation theory estimate for a full bore) or implicitly, as the solution of transcendental equations.

A strong connection to KdV modulation theory was also identified. Both the KdV and magma flow modulation equations are expressed in terms of an integral of a cubic polynomial. Moreover, the invariants of the KdV undular bore vary only slightly within the magma bore, a result which leads to simple expressions for the magmon amplitude. This connection warrants further investigation and may be useful in developing modulation theory for other equations, such as the BBM, modified KdV and Camassa–Holm equations.

REFERENCES


### Appendix A. Coefficients of the modulation equations

The coefficients of the matrices $A$ and $B$ in the modulation equation (2.29) are given in terms of the zeros of the cubic polynomial, $p_1$, $q_1$ and $r_1$, as

\[ a_{11} = \frac{1}{2} p_1 (q_1 - r_1) W_{p_1} + \frac{1}{2} q_1 r_1 (W_{q_1} - W_{r_1}) + 2 (q_1 + r_1) U^{-\frac{1}{2}} V, \]

\[ a_{21} = q_1 r_1 [(r_1 - q_1) W_{p_1} + (p_1 + q_1 - r_1) W_{q_1} + (q_1 - p_1 - r_1) W_{r_1}] \]

\[ + 3 (p_1 - q_1) (p_1 - r_1) (q_1 - r_1) W_{p_1}, \]

\[ a_{31} = p_1^2 (q_1 - r_1) W_{p_1} + \frac{1}{2} q_1 (p_1 r_1 + q_1 r_1 - p_1 q_1) W_{q_1} \]

\[ + \frac{1}{2} r_1 (p_1 r_1 - p_1 q_1 - q_1 r_1) W_{r_1} + (q_1 + r_1) U^{-\frac{1}{2}} \left[ p_1 (q_1 - r_1) (p_1 q_1 + p_1 r_1 - q_1 r_1 + 2 p_1^2) W_{p_1} \right. \]

\[ - q_1 (p_1 - r_1) (2 q_1^2 + p_1 q_1 + q_1 r_1 - p_1 r_1) W_{q_1} \]

\[ + r_1 (p_1 - q_1) (2 r_1^2 + q_1 r_1 + p_1 r_1 - p_1 q_1) W_{r_1} \], \tag{A.1} \]

\[ b_{11} = U a_{11} - 3 U^\frac{3}{2} (p_1 + r_1) V, \]

\[ b_{21} = U a_{21} - 2 U^{\frac{3}{2}} V [(q_1 + r_1) (6 p_1 q_1 r_1 U^{-\frac{1}{2}} - p_1 - q_1 - r_1) + 2 U^{\frac{3}{2}} + 2 q_1 r_1] \]

\[ b_{31} = U a_{31} - V [(6 p_1 q_1 r_1 q_1 + r_1) U^\frac{1}{2} + 2 q_1 r_1 U^\frac{3}{2}], \]
where
\[ V = p_1^2(q_1 - r_1)W_{p_1} - q_1^2(p_1 - r_1)W_{q_1} + r_1^2(p_1 - q_1)W_{r_1}. \]

Only the first columns of the matrices are presented, which represent the coefficients of \( p_{1T} \) and \( p_{1X} \). The coefficients in the second column, which represent the coefficients of \( q_{1T} \) and \( q_{1X} \), can be found by letting \( p_1 \to q_1, q_1 \to r_1 \) and \( r_1 \to p_1 \) in (A.1). The third column of the matrices can be found by a similar cyclic permutation of the zeros.

**Appendix B. The numerical scheme**

In order to make comparisons with the approximate solutions, numerical solutions of the magma IBV problem (1.5) were calculated using finite differences. As the magma equation is qualitatively similar to the BBM equation, a similar type of implicit scheme as that used by Eilbeck & McGuire (1975) for the BBM equation shall be derived. We take central differences in space and time, which results in an implicit three-level scheme with second-order accuracy. The solutions are then described by

\[ f^n_j = f(n\Delta t, j\Delta x), \quad j = 0, \ldots, J. \]  

(B.1)

The initial and boundary conditions for (1.5) are

\[ f^0_j = f_b, \quad j = 1, \ldots, J, \quad f^n_0 = f_r, \quad f^n_J = f_b, \quad n = 0, 1, \ldots \]  

(B.2)

The boundary and initial conditions represent an initial step (or shock) which is smoothed out by the undular bore over time. The numerical scheme is given by

\begin{align*}
  f^{n+1}_j & = f^n_j + \frac{3}{4} f^3_j (f^{n+1}_{j+1} - f^n_{j+1})(f^n_{j+1} - f^n_{j-1}) - f^3 f^{n+1}_{j+1} + f^{n+1}_j) \\
  & = f^n_j - 3 f^2 (f^n_{j+1} - f^n_{j-1})\Delta t \Delta x \\
  & - f^3 (f^{n-1}_{j+1} + f_{j-1}^{n-1}) - 2 f^n_{j-1}) - \frac{3}{4} f^2 (f^n_{j+1} - f^n_{j-1})(f^n_{j+1} - f^n_{j-1}), \\
  & \quad j = 1, \ldots, J - 1, \\
  & \quad (B.3)
\end{align*}

where \( f \) takes some local value such as \( f^n_j \) or an average about \((n\Delta t, j\Delta x)\). This gives a tridiagonal system of equations to be solved at each time step, which can be efficiently done using a standard algorithm. Since it is a three-level scheme, at the first time step a less accurate two-level scheme must be used.

We shall now consider the stability of the difference scheme (B.3). The Fourier series representation of \( f^n_j \) is

\[ f^n_j = \sum_{k=-\infty}^{\infty} C_k (g(k))^n e^{ikj\Delta x}. \]  

(B.4)

We need to determine the amplification factor \( g(k) \) to see if (B.3) is stable to small perturbations. Substituting the Fourier series representation (B.4) into (B.3), we find that the quadratic equation for \( g \) is

\begin{align*}
  g^2 \Delta x^2 + 2 f^3 \cos \psi + 3 f^2 \sin^2 \psi & + i6 g f^2 \Delta t \Delta x \sin \psi \\
  - f^3 \Delta x^2 + 2 f^3 \cos \psi - 1 & - 3 f^2 \sin^2 \psi = 0, \\
  & \quad (B.5)
\end{align*}

where \( \psi = k\Delta x \). The product of the roots of this quadratic is \(-1\). Hence, for stability we require the modulus of both the roots to be \(1\). We then require \(|3 f^2 \Delta t / \Delta x| \leq 1\) for stability. The scheme will be stable for most realistic choices of \( \Delta t \) and \( \Delta x \).