Undular Bores and the Morning Glory

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The Morning Glory is an internal atmospheric wave which occurs in northern Australia and which takes the form of a series of roll clouds. It is generated by the interaction of nocturnal sea-breezes over Cape York Peninsula and propagates in a south-westerly direction over the Gulf of Carpentaria. It is shown that the morning glory can be modelled by the resonant flow of a two layer fluid over topography, the topography being the mountains of Cape York Peninsula. In the limit of a deep upper layer, the equations of motion reduce to a forced Benjamin-Ono equation. In this context, resonant means that the underlying flow velocity of the sea-breezes is near a linear longwave velocity for one of the longwave modes. The morning glory is then given by the undular bore solution of the modulation equations for the Benjamin-Ono equation. The predictions of the modulation solution are compared with observational data on the Morning Glory and good agreement is found for the pressure jump due to the lead wave of the morning glory, but not for the speed and half-width of this lead wave. The reasons for this are discussed.

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I. INTRODUCTION

The Morning Glory is the name given to a roll cloud, or a series of roll clouds, which propagate in a south-westerly direction from Cape York Peninsula in north Queensland, Australia, over the Gulf of Carpentaria towards land again in the Gulf Country of north Queensland. Figure 1 shows the relevant geography of Australia. These roll clouds have a typical width of about 4 km and can reach heights of more than 2 km from a base height of a few hundred metres [2]. The Morning Glories can retain their identities for several hours and can extend in length from several kilometres to hundreds of kilometres [3]. The origin of the Morning Glory is the collision of opposing sea breezes from the Coral Sea and the Gulf of Carpentaria over Cape York Peninsula in the early evening [6]. Shortly after these breezes collide large disturbances in the form of solitary waves with amplitudes between 700 m and 1000 m develop, followed by a decelerating gravity current [6]. The roll clouds are formed at the leading edges of the solitary waves as they lift the moist ambient air to the condensation level. The clouds tend to arrive in the Gulf Country of the southern Gulf of Carpentaria in the early morning, hence the name Morning Glory. After the clouds dissolve, the Morning Glory can propagate for a large distance inland, and have been observed up to 300 km inland [3]. Furthermore Morning Glories excite waves on an upper level waveguide and these waves have been observed on following nights at Warramunga near Tennant Creek in the Northern Territory, approximately 1000 km from the origin of the Morning Glory. The Gulf Country is particularly suited for the observation of the Morning Glory due to the moist air.

A series of measurements by Clarke et al [6] showed that the Morning Glory is an undular bore propagating on the shallow nocturnal boundary layer. Furthermore Christie [2, 3] found that the atmosphere in which the Morning Glory propagates can be approximated by a surface based, stable lower layer, in which the Morning Glory propagates, lying below a deep upper layer. The appropriate equation governing the Morning Glory is then the forced Benjamin-Ono equation [2, 3], the forcing being due to the mountains of Cape York Peninsula. There has been extensive research, both theoretical and experimental, on fluid flow forced by topography. An excellent review and summary of this work is given in the book by Baines [1]. Of relevance to the current work is the modulation theory solution for resonant flow over topography, governed by the forced Korteweg-de Vries (KdV) equation, given in Grimshaw and Smyth [8] and Smyth [13]. When the upstream flow speed of a fluid flowing over some topography is close to the speed of one of the linear, longwave modes, energy cannot escape from the forcing. The flow then becomes nonlinear, with undular bores propagating upstream and downstream of the forcing. Smyth [13] used the simple wave (undular bore) solution of the modulation equations for the KdV equation, derived by Whitham [14, 15], to derive solutions for these undular bores produced in a band around exact linear resonance. In the present work a similar modulation theory analysis will be used to derive the undular bore solution of the modulation equations for the Benjamin-Ono equation, appropriate for the Morning Glory.
II. MODULATION SOLUTION

Let us consider the two dimensional flow over topography of a density stratified, inviscid fluid in a waveguide of depth $h$ which lies underneath a deep upper layer for the case in which the fluid in the waveguide has velocity $U$ at upstream infinity. Let us further assume that the topographic forcing is localised. We take the coordinate $Z$ to be orientated vertically upwards, with $Z = 0$ being the flat ground away from the topography, and the $X$ coordinate to be orientated in the direction of the upstream flow, with $X = 0$ being some point in the topography. The displacement $A$ of the mode is governed by the forced Benjamin-Ono equation

$$\frac{\partial A}{\partial T} - \delta \frac{\partial A}{\partial X} + \alpha A \frac{\partial A}{\partial X} + \beta P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A Y}{Y - X} dY + \frac{c_n}{2} \frac{dG(X)}{dX} = 0$$

for the case in which the underlying velocity $U$ is close to the modal velocity $c_n$ [8]. In non-dimensional form this equation is

$$\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial x} + P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u_y y}{y - x} dy + \frac{d g(x)}{dx} = 0,$n

where P.V. represents the principal value, $u$ is the non-dimensional amplitude of the disturbance and $g$ is a function representing the forcing, for instance the mountains of Cape York Peninsula [8]. In the absence of forcing, the Benjamin-Ono equation has the periodic wave solution

$$u = c + a + b + 2(b-a) \frac{b - c - \sqrt{(b-c)(a-c)} \cos \phi}{a + b - 2c - 2\sqrt{(b-c)(a-c)} \cos \phi},$$

with the phase $\phi$ being

$$\phi(x, t) = \Delta - (a - b)x - \frac{(a^2 - b^2)}{2} t$$

[7]. The mean level $m$ of this periodic solution is

$$m = \frac{1}{2\pi} \int_0^{2\pi} u(x, t) \, d\phi = c + b - a$$

and its amplitude $S$ is

$$S = 4\sqrt{(b-c)(a-c)}.$$

The limit $a \to b$ of this travelling wave solution gives the soliton solution and the limit $a \to c$ gives the linear periodic wave solution of the Benjamin-Ono equation.

Modulation theory for the Benjamin-Ono equation is now constructed by allowing the parameters $a$, $b$ and $c$ in the periodic wave solution (3) to slowly vary in $x$ and $t$. Dobrokhotov and Krichever [7] showed that the modulation equations for these periodic wave parameters are

$$a = \text{constant on } \frac{dx}{dt} = \Delta - 2a$$

$$b = \text{constant on } \frac{dx}{dt} = \Delta - 2b$$

$$c = \text{constant on } \frac{dx}{dt} = \Delta - 2c$$

in characteristic form. These modulation equations have the simple wave solution

$$a(x, t) = \begin{cases} B, & x < (\Delta - 2B)t, \\ \frac{1}{2} (\Delta - \frac{2}{3})t, & (\Delta - 2B)t \leq x \leq \Delta t, \\ 0, & x > \Delta t \end{cases}$$

with $b = B$ and $c = 0$. This simple wave solution represents the dispersive resolution of a jump of height $B$, with solitons at its leading edge $x = (\Delta - 2B)t$ and linear waves at its trailing edge $x = \Delta t$. It is then the undular
bore solution of the Benjamin-Ono equation. This undular bore solution will be used as the basis of the solution of the forced Benjamin-Ono equation (2).

Figure 2 shows the evolution of the numerical solution of the forced Benjamin-Ono equation (2). It can be seen that there are distinct upstream and downstream wavetrains linked by a steady profile over the topography and that both the upstream and downstream wavetrains have the form of undular bores. The upstream wavetrain forced by the topography corresponds to the Morning Glory.

A. Upstream Solution

Let us first derive the approximate solution for the upstream wavetrain. As the upstream bore has solitons at its leading edge, \( a \rightarrow b \) at the leading edge. It is therefore apparent from the mean level expression (5) that for the upstream bore to rise from the undisturbed level we require \( c = 0 \). The upstream solution is then the simple wave solution \( b = B_- \) and

\[
a = \begin{cases} 
B_-, & x < (\Delta - 2B_-)t \\
\frac{1}{2}(\Delta - \frac{x}{t}), & (\Delta - 2B_-)t \leq x \leq \Delta t \\
0, & \Delta t < x \leq 0
\end{cases}
\]

of the modulation equations (7), where \( B_- > 0 \). This bore solution gives a smooth jump from the mean level \( 0 \) to the mean level \( B_- \). It should be noted that the trailing edge of the wavetrain is \( x = \Delta t \). Since the upstream wavetrain must propagate into \( x < 0 \) only, the solution (9) is valid for \( \Delta \leq 0 \). For \( \Delta > 0 \), the upstream simple wave solution must be terminated at \( x = 0 \) so that it does not partly propagate downstream, as was done by Smyth [13] for the equivalent upstream solution of the forced Korteweg-de Vries equation. The upstream simple wave solution for \( \Delta > 0 \) is therefore

\[
a = \begin{cases} 
\frac{1}{2}(\Delta - \frac{x}{t}), & x < (\Delta - 2B_+)t \\
(\Delta - 2B_+)t, & (\Delta - 2B_+)t \leq x \leq 0
\end{cases}
\]

(10)

where \( B_+ > 0 \).

B. Downstream Solution

The solution downstream of the forcing can be similarly found as a simple wave solution of the modulation equations (7). As can be seen from Figure 2, the main difference between the upstream and downstream solutions is that the downstream solution jumps from a negative mean level to the undisturbed level. Now the leading edge of the downstream bore consists of linear waves. As the linear wave limit of the periodic solution has \( a \rightarrow c \), we see from the mean level expression (5) that we require \( b = 0 \) for the leading edge to approach the undisturbed level. The downstream wavetrain is therefore the simple wave solution \( b = 0, c = C_+ \) and

\[
a = \begin{cases} 
0, & 0 \leq x < \Delta t \\
\frac{1}{2}(\Delta - \frac{x}{t}), & \Delta t \leq x \leq (\Delta - 2C_+)t \\
C_+, & x > (\Delta - 2C_+)t
\end{cases}
\]

(11)

of the modulation equations (7), where \( C_+ < 0 \). This downstream bore solution gives a jump from the negative mean level \( C_+ \) to \( 0 \). As for the upstream simple wave solution for \( \Delta > 0 \), the simple wave solution (11) will partly propagate upstream when \( \Delta < 0 \). Hence, as for the upstream solution, this simple wave solution is truncated at \( x = 0 \) when \( \Delta < 0 \), as was done by Smyth [13] in the corresponding case for the downstream solution of
the forced Korteweg-de Vries equation. The downstream simple wave solution for \( \Delta < 0 \) is then \( b = 0, c = C_- \) and
\[
a = \left\{ \frac{1}{2} \left( \frac{\Delta - \frac{1}{4}}{C_-} \right), \quad 0 \leq x \leq (\Delta - 2C_-)t \right. \quad \text{and} \\
\left. \frac{1}{2} \left( \frac{\Delta - \frac{1}{4}}{C_-} \right), \quad x > (\Delta - 2C_-)t, \right. \tag{12}
\]
where \( C_- < 0 \).

The upstream and downstream solutions are now complete, except for the constants \( B_-, B_+, C_- \) and \( C_+ \), which give the size of the jumps across the simple waves. These constants will be found as for the forced KdV equation [13].

It can be seen from Figure 2 that the flow over the forcing is steady, even though the upstream and downstream flows are not. In the limit of a broad forcing, so that the dispersive term can be neglected, the solution of the steady, forced Benjamin-Ono equation which is continuous at \( x = 0 \) and has the correct sign upstream and downstream is
\[
u_s = \left\{ \begin{array}{ll}
\frac{\Delta}{2} + \sqrt{g_0 - g(x)}, & x \leq 0 \\
\frac{\Delta}{2} - \sqrt{g_0 - g(x)}, & x \geq 0,
\end{array} \right. \tag{13}
\]
where \( g_0 = g(0) \). The simple wave solutions of the previous section will now be matched in the mean to this steady solution over the forcing as \( x \to \infty \) and \( x \to -\infty \). In this manner, using the mean level expression (5) gives
\[
B_- = \frac{\Delta}{2} + \sqrt{g_0} \tag{14}
\]
for \( \Delta \leq 0 \) and
\[
B_+ = \Delta + \sqrt{g_0} \tag{15}
\]
for \( \Delta > 0 \) for the upstream wavetrain. For the downstream wavetrain, a similar matching in the mean gives
\[
C_+ = \frac{\Delta}{2} - \sqrt{g_0} \tag{16}
\]
for \( \Delta \geq 0 \) and
\[
C_- = \Delta - \sqrt{g_0} \tag{17}
\]
for \( \Delta < 0 \).

<table>
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<th>( G_0 ) (m)</th>
<th>( \Delta )</th>
<th>( \Delta P ) (hPa)</th>
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Table III: Pressure jump \( \Delta P \) of lead wave of upstream wavetrain (Morning Glory) as given by modulation solution for a range of mountain heights \( G_0 \). \( \Delta \) calculated from results of Smith and Noonan [12]. Waveguide depth is \( h = 500 \)m.

\[
C_0 (\text{m}) \quad \Delta \quad |V| (\text{m/s}) \quad |W| (\text{m}) \quad X_s / h
\begin{array}{cccc}
100 & -0.799 & 4.7 & 657 & 0.62
100 & -1.131 & 4.7 & 1165 & 0.35
150 & -0.799 & 5.7 & 450 & 0.90
150 & -1.131 & 5.7 & 641 & 0.63
200 & -0.799 & 6.6 & 355 & 1.14
200 & -1.131 & 6.6 & 465 & 0.87
\end{array}
\]

Table IV: Speed \(|V|\), half-width \(W\) and amplitude to waveguide depth ratio \(A_s/h\) for lead wave of upstream wavetrain (Morning Glory) as given by modulation solution for a range of mountain heights \( G_0 \). \( \Delta \) calculated from results of Smith and Noonan [12]. Waveguide depth is \( h = 500 \)m.

The constants \( B_- \) and \( B_+ \) must be positive and the constants \( C_- \) and \( C_+ \) must be negative, by construction. It can therefore be seen from (14){(17) that the modulation solution is valid for
\[
-2\sqrt{g_0} < \Delta < 2\sqrt{g_0}. \tag{18}
\]
Outside of this range, the flow is qualitatively similar to the linear solution in that there is no large, permanent upstream disturbance, see Grimshaw and Smyth [8] and Smyth [13]. This band of \( \Delta \) is the resonant or transcritical range for flow over the forcing.

### III. COMPARISON WITH OBSERVATIONAL DATA

In this section the modulation solution will be compared with the observational data of Christie [2, 3] and Menhofer et al [9] for the Morning Glory. These authors reported observational data on the Morning Glory at various points along the Queensland coast of the Gulf of Carpentaria. To compare with these data, some assumptions will have to be made about the stratification state of the atmosphere into which the Morning Glory propagates. As a model of the atmosphere, Christie [2] assumed that the lower waveguide layer in which the Morning Glory propagates has depth \( h \) and a constant Brunt-Väisälä frequency \( N \) and that the upper layer is neutrally stable \( (N = 0) \). It was then shown that in the limit of a small density jump across the interface between the upper and lower layers, the coefficients in the dimensional Benjamin-Ono equation (1) are
\[
\alpha = \frac{4N}{\pi}, \quad \beta = \frac{8Nh^2}{\pi^3} \quad \text{and} \quad c_n = c_1 = \frac{2Nh}{\pi} \tag{19}
\]
for a first mode wave. Using these parameter values the non-dimensional forcing height \( g_0 \) is related to the dimensional height \( G_0 \) by
\[
g_0 = \frac{\pi^4}{32h} G_0. \tag{20}
\]
Christie [2] gives that for the assumed stratification, the pressure change \( \Delta P \) due to a wave of dimensional height
A is
\[ \Delta P = \frac{2\rho h N^2}{\pi} A, \]
for air of density \( \rho \). Finally the pressure change is given in terms of the non-dimensional amplitude \( u \) by
\[ \Delta P = \frac{8\rho h^2 N^2}{\pi^3} u. \]

The observational data of Christie [2, 3] and Menhofer et al [9] for the pressure jump of the lead wave of the Morning Glory are presented in Table I. Christie [3] gives that the depth \( h \) of the waveguide layer varies between 400\( \text{m} \) and 1200\( \text{m} \) and that a typical value of the Brunt-Väisälä frequency is \( N = 0.0233 \text{s}^{-1} \). Christie [3] and Clarke [4] found that the Morning Glory is generated in a region between 40\( \text{km} \) and 70\( \text{km} \) inland from the Gulf coast of Cape York Peninsula, in which the mountains range between 100\( \text{m} \) and 200\( \text{m} \) in height, with most of the elevation being towards the lower end of this range. Numerical studies by Clarke [5] found that atmospheric bores can be generated by colliding sea breezes, provided they have speeds less than 7\( m/s \). The final piece of data needed is the density of the air in the waveguide, which can be taken to be 1.16\( \text{kg/m}^3 \) for a typical summer daytime temperature of 30\( ^\circ \text{C} \). Using these values, the pressure jump due to the lead wave of the upstream wavetrain as predicted by the modulation solution can be calculated and the results are presented in Table II.

It can be seen from Tables I and II that the modulation solution predictions are in good agreement with the observations for waveguide depths of 400\( m \) and 500\( m \). In this regard, numerical calculations of Smith and Noonan [12] of the atmospheric flow over the Gulf of Carpentaria-Cape York Peninsula-Coral Sea region show that over land a shallow stable layer of depth 500\( m \) underlying a deep upper layer with weak vertical gradients forms during the night. Furthermore Christie [2] states that a typical depth of the lower waveguide is 500\( m \).

Table III lists the pressure jump (22) for the lead wave of the upstream modulated wavetrain, but for values of \( \Delta \) corresponding to the bounds 4\( m/s \) and 5\( m/s \) for the onshore sea breeze as predicted from the numerical calculations of Smith and Noonan [12]. Comparing these values with the observational values listed in Table I, we see that the modulation solution is in accord with the observational data, except for the lower bound on the mountain height \( G_0 = 100 \text{m} \) for \( \Delta = -1.131 \) (for 4\( m/s \)), for which the pressure jump is too low.

In Table IV the lead wave half-widths \( W \) and speeds \( |V| \) are listed for the same range of mountain heights as in the previous tables and for the upper and lower values of \( \Delta \) from the numerical calculations of Smith and Noonan [12], based on a waveguide depth of \( h = 500 \text{m} \). Noonan and Smith [10] reported observed Morning Glory speeds of between 9.6\( m/s \) and 11.1\( m/s \) and lead wave half-widths \( W \) of between 1500\( m \) and 2000\( m \) and Menhofer et al [9] reported Morning Glory speeds of between 5.2\( m/s \) and 15.3\( m/s \). It can be seen that the modulation theory predicts lead wave speeds towards the lower end of the observed speeds for \( G_0 = 150 \text{m} \) and \( G_0 = 200 \text{m} \). However for these mountain heights, the half-widths of the lead wave are smaller than the observed half-widths by a factor of between 2 and 5. One reason for this discrepancy is the questionable applicability of the Benjamin-Ono equation to the Morning Glory due to the amplitude to waveguide depth ratio \( A/h \) not being small, as was noted by Noonan and Smith [10]. Also, for weakly nonlinear theory to be within its range of validity, we need to assume that \( G_0/h \ll 1 \), which is questionable for a waveguide of depth \( h = 500 \text{m} \) and a mountain height of \( G_0 = 200 \text{m} \).