SYZYGIES, MULTIGRADED REGULARITY AND TORIC VARIETIES

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Abstract. Using multigraded Castelnuovo-Mumford regularity, we study the equations defining a projective embedding of a variety X. Given globally generated line bundles $B_1, \dots, B_\ell$ on X and $m_1, \dots, m_\ell \in \mathbb{N}$, consider the line bundle $L := B_1^{m_1} \otimes \cdots \otimes B_\ell^{m_\ell}$. We give conditions on the m_i which guarantee that the ideal of X in $\mathbb{P}(H^0(X, L)^*)$ is generated by quadrics and the first p syzygies are linear. This yields new results on the syzygies of toric varieties and the normality of polytopes.

1. Introduction

Understanding the equations defining a projective variety X and the relations among them is a central problem in algebraic geometry. Green [Gre84a] shows that a sufficiently positive line bundle L on X gives an embedding $X \subseteq \mathbb{P}\big(H^0(X, L)^*\big)$ such that the first few syzygies are as simple as possible. Explicit conditions certifying that an ample line bundle is suitably positive are given by Green [Gre84a] for curves, Ein and Lazarsfeld [EL93] for smooth varieties, Gallego and Purnaprajna [GP99] for normal surfaces, and Pareschi and Popa [Par00, PP04, PP03] for abelian varieties; see Lazarsfeld [Laz04, §1.8.D] for a survey. The primary goal of this paper is to produce similar conditions for toric varieties.

Let X be a projective variety over a field of characteristic zero and let L be a globally generated line bundle on X. The associated morphism is $\phi_L \colon X \to \mathbb{P}\big(H^0(X, L)^*\big)$ and $S := \operatorname{Sym}^\bullet H^0(X, L)$ denotes the homogeneous coordinate ring of $\mathbb{P}\big(H^0(X, L)^*\big)$. Consider the graded S-module $R := \bigoplus_{j \geq 0} H^0(X, L^j)$ and a minimal free graded resolution $E_\bullet$ of R. Following Green and Lazarsfeld [GL85], we say that L satisfies *property* (N_p) for $p \in \mathbb{N}$ provided that $E_0 \cong S$ and $E_i = \bigoplus S(-i-1)$ for all $1 \leq i \leq p$. Hence, $\phi_L(X)$ is projectively normal if and only if L

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satisfies (N_0) and $\phi_L(X)$ is normal. If L satisfies (N_1), then the homogeneous ideal of $\phi_L(X)$ is generated by quadrics and (N_2) implies that the relations among the generators are linear. The concepts of normal generation and normal presentation introduced by Mumford [Mum70] correspond to (N_0) and (N_1) respectively. Typically, the property (N_p) is studied under the additional hypothesis that L is ample; our theorems do not require this assumption.

To examine this property, we use multigraded Castelnuovo-Mumford regularity. Fix a collection $B_1, \dots, B_\ell$ of globally generated line bundles on X. For $\boldsymbol{u} := (u_1, \dots, u_\ell) \in \mathbb{Z}^\ell$, set $B^{\boldsymbol{u}} := B_1^{u_1} \otimes \cdots \otimes B_\ell^{u_\ell}$ and let $\boldsymbol{\mathcal{B}}$ be the semigroup $\{B^{\boldsymbol{u}} : \boldsymbol{u} \in \mathbb{N}^\ell\} \subset \operatorname{Pic}(X)$. We say that a line bundle L is $\mathscr{O}_X$-regular (with respect to $B_1, \dots, B_\ell$) if $H^i(X, L \otimes B^{-\boldsymbol{u}}) = 0$ for all $i > 0$ and all $\boldsymbol{u} \in \mathbb{N}^\ell$ with $|\boldsymbol{u}| := u_1 + \cdots + u_\ell = i$. Our main result is the following.

Theorem 1.1. *Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4, \dots$ be a sequence in $\mathbb{N}^\ell$ such that $B^{\boldsymbol{w}_i} \in \bigcap_{j=1}^{\ell} (B_j \otimes \boldsymbol{\mathcal{B}})$ and set $\boldsymbol{m}_i := \boldsymbol{w}_1 + \cdots + \boldsymbol{w}_i$ for $i \geq 1$. If $B^{\boldsymbol{m}_1}$ is $\mathscr{O}_X$-regular then the line bundle $B^{\boldsymbol{m}_p}$ satisfies (N_p) for $p \geq 1$.*

The case $\ell = 1$ is in Gallego and Purnaprajna [GP99, Theorem 1.3]. Our proof is a multigraded variant of their arguments.

Applying Theorem 1.1 with $\ell = 1$ to line bundles on toric varieties yields the following.

Corollary 1.2. *Let L be an ample line bundle on an n-dimensional toric variety. If $d \geq n - 1 + p$ then the line bundle L^d satisfies (N_p).*

The case $p = 0$, an ingredient in our proof, was established by Ewald and Wessels [EW91]; other proofs appear in Liu, Trotter and Ziegler [LTZ93], Bruns, Gubeladze and Trung [BGT97], and Ogata and Nakagawa [ON02]. On a toric surface, Koelman [Koe93] proves that a line bundle L satisfies (N_1) if the associated lattice polytope contains more than three lattice points in its boundary. More generally, Bruns, Gubeladze and Trung [BGT97] show that R is Koszul when $d \geq n$ and this implies that L^d satisfies (N_1) when $d \geq n$. Assuming $n \geq 3$, Ogata [Oga03] establishes that L^{n-1} satisfies (N_1) and, building on this, Ogata [Oga04] proves that L^{n-2+p} satisfies (N_p) when $n \geq 3$ and $p \geq 1$.

With additional invariants, we can strengthen Corollary 1.2. Let $h_L(d) := \chi(L^d) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, L^d)$ be the Hilbert polynomial of L and let $r(L)$ be the number of integer roots of h_L.

Corollary 1.3. *Let L be a globally generated line bundle on a toric variety and let $r(L)$ be the number of integer roots of its Hilbert polynomial h_L. If $p \geq 1$ and $d \geq \max\{\deg(h_L) - r(L) + p - 1, p\}$ then the line bundle L^d satisfies (N_p).*

If $X = \mathbb{P}^n$ and $L = \mathscr{O}_X(1)$, then we have $h_L(d) = \binom{d+n}{n}$ and $r(L) = n$. In particular, we recover a result by Green [Gre84b, Theorem 2.2] that $\mathscr{O}_{\mathbb{P}^n}(d)$ satisfies (N_p) for $p \leq d$.

Using the dictionary between lattice polytopes and ample line bundles on toric varieties, Corollary 1.3 yields a normality criterion for lattice polytopes. A lattice polytope P is *normal* if every lattice point in mP is a sum of m lattice points in P. Let $r(P)$ be the largest integer such that $r(P)\,P$ does not contain any lattice points in its interior.

Corollary 1.4. *If P is a lattice polytope of dimension n, then the lattice polytope $\big(n - r(P)\big)\,P$ is normal.*

Theorem 1.1 also applies to syzygies of Segre-Veronese embeddings.

Corollary 1.5. *If $X = \prod_{i=1}^{\ell} \mathbb{P}^{n_i}$ then the line bundle $\mathscr{O}_X(d_1, \dots, d_\ell)$ satisfies (N_p) for $p \leq \min\{d_i : d_i \neq 0\}$.*

The Segre embedding $\mathscr{O}_X(1, \dots, 1)$ satisfies (N_p) if and only if $p \leq 3$; see Lascoux [Las78] or Pragacz and Weyman [PW85] for $\ell = 2$, and Rubei [Rub02, Rub04] for $\ell > 2$. Eisenbud *et al.* [EGHP04, §3] provides an overview of results and conjectures about the syzygies of Segre-Veronese embeddings.

Inspired by Ein and Lazarsfeld [EL93], we also examine the syzygies of adjoint bundles. Recall that a line bundle on a toric variety is numerically effective (nef) if and only if it is globally generated, and the dualizing sheaf K_X is a line bundle if and only if X is Gorenstein.

Corollary 1.6. *Let X be a projective n-dimensional Gorenstein toric variety and let $B_1, \dots, B_\ell$ be the minimal generators of* $\operatorname{Nef}(X)$*. If $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots$ is a sequence in $\mathbb{N}^\ell$ such that $B^{\boldsymbol{w}_i} \in \bigcap_{j=1}^{\ell}(B_j \otimes \boldsymbol{\mathcal{B}})$ and $\boldsymbol{m}_i := \boldsymbol{w}_1 + \cdots + \boldsymbol{w}_i$ for $i \geq 1$, then for $p \geq 1$*

$$K_X \otimes B^{\boldsymbol{m}_{n+1+p}} \text{ satisfies } \begin{cases} (N_{p+1}) & \text{if } X \neq \mathbb{P}^n, \\ (N_p) & \text{if } X = \mathbb{P}^n. \end{cases}$$

Ein and Lazarsfeld [EL93] prove that for a very ample line bundle L and a globally generated line bundle N on a smooth n-dimensional algebraic variety $X \neq \mathbb{P}^n$, $K_X \otimes L^{n+p} \otimes N$ satisfies (N_p). Corollary 1.6 gives a similar result for ample line bundles on possibly singular Gorenstein toric varieties. Specifically, if L is an ample line bundle such that $L \in \bigcap_{j=1}^{\ell}(B_j \otimes \boldsymbol{\mathcal{B}})$ and N is a nef line bundle on $X \neq \mathbb{P}^n$ then $K_X \otimes L^{n+p} \otimes N$ satisfies (N_p). The proof of Corollary 1.6 combines Theorem 1.1 with Fujita's Freeness conjecture for toric varieties, see Fujino [Fuj03].

Conventions. The nonnegative integers are denoted by $\mathbb{N}$. We work over a field of characteristic zero.

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2. Multigraded Castelnuovo-Mumford Regularity

In this section we review multigraded regularity as introduced by Maclagan and Smith [MS04]. Fix a collection $B_1, \dots, B_\ell$ of globally generated line bundles on X. For any element $\boldsymbol{u} := (u_1, \dots, u_\ell) \in \mathbb{Z}^\ell$, set $B^{\boldsymbol{u}} := B_1^{u_1} \otimes \cdots \otimes B_\ell^{u_\ell}$ and let $\boldsymbol{\mathcal{B}}$ be the semigroup $\{B^{\boldsymbol{u}} : \boldsymbol{u} \in \mathbb{N}^\ell\} \subset \operatorname{Pic}(X)$. If $\boldsymbol{e}_1, \dots, \boldsymbol{e}_\ell$ is the standard basis for $\mathbb{Z}^\ell$, then $B^{\boldsymbol{e}_j} = B_j$.

Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module and let L be a line bundle on X. We say that $\mathscr{F}$ is L-regular (with respect to $B_1, \dots, B_\ell$) provided $H^i(X, \mathscr{F} \otimes L \otimes B^{-\boldsymbol{u}}) = 0$ for all $i > 0$ and all $\boldsymbol{u} \in \mathbb{N}^\ell$ satisfying $|\boldsymbol{u}| := u_1 + \cdots + u_\ell = i$. As Mumford [Mum66] says, "this apparently silly definition reveals itself as follows."

Theorem 2.1. *If the coherent sheaf $\mathscr{F}$ is L-regular then for all $\boldsymbol{u} \in \mathbb{N}^\ell$:*

(1) *$\mathscr{F}$ is $(L \otimes B^{\boldsymbol{u}})$-regular;*

(2) *the natural map*

$$H^0(X, \mathscr{F} \otimes L \otimes B^{\boldsymbol{u}}) \otimes H^0(X, B^{\boldsymbol{v}}) \longrightarrow H^0(X, \mathscr{F} \otimes L \otimes B^{\boldsymbol{u}+\boldsymbol{v}})$$

is surjective for all $\boldsymbol{v} \in \mathbb{N}^\ell$;

(3) *$\mathscr{F} \otimes L \otimes B^{\boldsymbol{u}}$ is generated by its global sections, provided there exists $\boldsymbol{w} \in \mathbb{N}^\ell$ such that $B^{\boldsymbol{w}}$ is ample.*

When X is a toric variety, this follows from results in Maclagan and Smith [MS04, §6]. Our proof imitates Mumford [Mum70, Theorem 2] and Kleiman [Kle66, Proposition II.1.1].

Proof. By replacing $\mathscr{F}$ with $\mathscr{F} \otimes L$, we may assume that the coherent sheaf $\mathscr{F}$ is $\mathscr{O}_X$-regular. We proceed by induction on $\dim\bigl(\operatorname{Supp}(\mathscr{F})\bigr)$. The claim is trivial when $\dim\bigl(\operatorname{Supp}(\mathscr{F})\bigr) \leq 0$. As each B_j is basepoint-free, we may choose a section $s_j \in H^0(X, B_j)$ such that the induced map $\mathscr{F} \otimes B^{-\boldsymbol{e}_j} \longrightarrow \mathscr{F}$ is injective (see Mumford [Mum70, page 43]). If $\mathscr{G}_j$ is the cokernel, then we have $0 \longrightarrow \mathscr{F} \otimes B^{-\boldsymbol{e}_j} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}_j \longrightarrow 0$ and $\dim\bigl(\operatorname{Supp}(\mathscr{G}_j)\bigr) < \dim\bigl(\operatorname{Supp}(\mathscr{F})\bigr)$. From this short exact sequence, we obtain the long exact sequence

$$\cdots \longrightarrow H^i(X, \mathscr{F} \otimes B^{-\boldsymbol{u}-\boldsymbol{e}_j}) \longrightarrow H^i(X, \mathscr{F} \otimes B^{-\boldsymbol{u}}) \longrightarrow H^i(X, \mathscr{G}_j \otimes B^{-\boldsymbol{u}}) \longrightarrow H^{i+1}(X, \mathscr{F} \otimes B^{-\boldsymbol{u}-\boldsymbol{e}_j}) \longrightarrow \cdots .$$

By taking $|\boldsymbol{u}| = i$, we deduce that $\mathscr{G}_j$ is $\mathscr{O}_X$-regular. The induction hypothesis implies that $\mathscr{G}_j$ is (B_j)-regular. Setting $\boldsymbol{u} = -\boldsymbol{e}_j + \boldsymbol{u}'$ with $|\boldsymbol{u}'| = i$, we see that $\mathscr{F}$ is (B_j)-regular and (1) follows.

For (2), consider the commutative diagram:

$$\begin{array}{ccccccc} & & & & H^0(X,\mathscr{F}) \otimes H^0(X,B_j) & \longrightarrow & H^0(X,\mathscr{G}_j) \otimes H^0(X,B_j) \\ & & & \nearrow & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(X,\mathscr{F}) & \longrightarrow & H^0(X,\mathscr{F} \otimes B_j) & \longrightarrow & H^0(X,\mathscr{G}_j \otimes B_j)\,. \end{array}$$

Since $\mathscr{F}$ is $\mathscr{O}_X$-regular, the map in the top row is surjective. The induction hypothesis guarantees that the map in the right column is surjective. Thus, the Snake Lemma implies that the map in the middle column is also surjective. Therefore, (2) follows from the associativity of the tensor product and (1).

Lastly, consider the commutative diagram:

$$\begin{array}{ccc} H^0(X,\mathscr{F} \otimes B^{\boldsymbol{u}}) \otimes H^0(X,B^{\boldsymbol{v}}) \otimes \mathscr{O}_X & \longrightarrow & H^0(X,\mathscr{F} \otimes B^{\boldsymbol{u}+\boldsymbol{v}}) \otimes \mathscr{O}_X \\ \downarrow & & \downarrow{\scriptstyle \beta_{\boldsymbol{u}+\boldsymbol{v}}} \\ H^0(X,\mathscr{F} \otimes B^{\boldsymbol{u}}) \otimes B^{\boldsymbol{v}} & \xrightarrow{\ \beta_{\boldsymbol{u}} \otimes \mathrm{id}\ } & \mathscr{F} \otimes B^{\boldsymbol{u}+\boldsymbol{v}} \end{array}$$

Applying (2), we see that the map in the top row is surjective. By assumption, there is $\boldsymbol{w} \in \mathbb{N}^\ell$ such that $B^{\boldsymbol{w}}$ is ample. If $\boldsymbol{v} := k\boldsymbol{w}$, then Serre's Vanishing Theorem (Lazarsfeld [Laz04, Theorem 1.2.6]) implies that $\beta_{\boldsymbol{u}+\boldsymbol{v}}$ is surjective for k sufficiently large. Hence, $\beta_{\boldsymbol{u}}$ is also surjective which proves (3). $\square$

We end this section with an elementary observation.

Lemma 2.2. *Let* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *be a short exact sequence of coherent* $\mathscr{O}_X$*-modules. If* $\mathscr{F}$ *is* L*-regular,* $\mathscr{F}''$ *is* $(L \otimes B^{-\boldsymbol{e}_j})$*-regular for all* $1 \leq j \leq \ell$ *and* $H^0(X, \mathscr{F} \otimes L \otimes B^{-\boldsymbol{e}_j}) \to H^0(X, \mathscr{F}'' \otimes L \otimes B^{-\boldsymbol{e}_j})$ *is surjective for all* $1 \leq j \leq \ell$*, then* $\mathscr{F}'$ *is also* L*-regular.*

Sketch of Proof. This is similar to the proof of Theorem 2.1.1: tensor the exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ with $L \otimes B^{\boldsymbol{u}}$ and analyze the associated long exact sequence. $\square$

3. Proof of Main Theorem

The proof of Theorem 1.1 combines the multigraded Castelnuovo-Mumford regularity with a cohomological criterion for (N_p). Given a globally generated line bundle L on X, there is a natural surjective map $\mathrm{ev}_L \colon H^0(X,L) \otimes \mathscr{O}_X \longrightarrow L$ and we set $M_L := \mathrm{Ker}(\mathrm{ev}_L)$. Hence,

M_L is a vector bundle on X which sits in the short exact sequence

$$(\dagger) \qquad 0 \longrightarrow M_L \longrightarrow H^0(X,L) \otimes \mathscr{O}_X \longrightarrow L \to 0\,.$$

The following result shows that M_L governs the syzygies of $\phi_L(X)$ in $\mathbb{P}\big(H^0(X,L)^*\big)$.

Lemma 3.1. *Let L be a line bundle on X that is generated by its global sections and assume that the ground field has characteristic zero. If $H^1(X, M_L^{\otimes q} \otimes L^j) = 0$ for $q \leq p+1$ and $j \geq 1$, then L satisfies (N_p).*

Sketch of Proof. The arguments in Lazarsfeld [Laz89, §1.3] show that L satisfies (N_p) if and only if $H^1(X, \bigwedge^q M_L \otimes L^j) = 0$ for $q \leq p+1$ and $j \geq 1$. Since $\bigwedge^k M_L$ is a direct summand of tensor product M_L^k in characteristic zero, the claim follows. □

Proof of Theorem 1.1. Set $L := B^{\boldsymbol{m}_p}$ and let M_L be the vector bundle in (†). We first prove, by induction on q, that $M_L^{\otimes q}$ is $(B^{\boldsymbol{m}_q})$-regular for all $q \geq 1$. Since $B^{\boldsymbol{m}_1}$ is $\mathscr{O}_X$-regular, Theorem 2.1.2 implies that $H^0(X, B^{\boldsymbol{m}_1+\boldsymbol{u}}) \otimes H^0(X, B^{\boldsymbol{v}}) \longrightarrow H^0(X, B^{\boldsymbol{m}_1+\boldsymbol{u}+\boldsymbol{v}})$ is surjective for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^\ell$. In particular, the maps

$$H^0(X,L) \otimes H^0(X, B^{\boldsymbol{m}_1 - \boldsymbol{e}_j}) \longrightarrow H^0(X, L \otimes B^{\boldsymbol{m}_1 - \boldsymbol{e}_j}) \qquad 1 \leq j \leq \ell$$

are surjective because $B^{\boldsymbol{m}_1} \in \bigcap_{j=1}^{\ell} (B_j \otimes \boldsymbol{\mathcal{B}})$. Applying Theorem 2.1.1 and Lemma 2.2, we see that M_L is $(B^{\boldsymbol{m}_1})$-regular. For $q > 1$, tensor the sequence (†) with $M_L^{\otimes(q-1)}$ to obtain the exact sequence

$$0 \longrightarrow M_L^{\otimes q} \longrightarrow H^0(X,L) \otimes M_L^{\otimes(q-1)} \longrightarrow M_L^{\otimes(q-1)} \otimes L \longrightarrow 0\,.$$

Since the induction hypothesis states that $M_L^{\otimes(q-1)}$ is $(B^{\boldsymbol{m}_{q-1}})$-regular, Theorem 2.1.2 shows that the maps

$$H^0(X, M_L^{\otimes(q-1)} \otimes B^{\boldsymbol{m}_q - \boldsymbol{e}_j}) \otimes H^0(X,L) \longrightarrow H^0(X, M_L^{\otimes(q-1)} \otimes L \otimes B^{\boldsymbol{m}_q - \boldsymbol{e}_j})$$

are surjective for $1 \leq j \leq \ell$ because $B^{\boldsymbol{w}_q} \in \bigcap_{j=1}^{\ell} (B_j \otimes \boldsymbol{\mathcal{B}})$. Again by Theorem 2.1.1 and Lemma 2.2, M_L^q is $(B^{\boldsymbol{m}_q})$-regular.

By Lemma 3.1, it suffices to prove that $H^1(X, M_L^{\otimes q} \otimes L^j) = 0$ for $q \leq p+1$ and $j \geq 1$. Since $M_L^{\otimes q}$ is $(B^{\boldsymbol{m}_q})$-regular, Theorem 2.1.1 implies that $M_L^{\otimes q}$ is $(B^{\boldsymbol{m}_p})$-regular for $1 \leq q \leq p$; as $\mathscr{O}_X$ is $(B^{\boldsymbol{m}_1})$-regular, Theorem 2.1.1 also implies that $\mathscr{O}_X$ is $(B^{\boldsymbol{m}_p})$-regular. It follows that $H^1(X, M_L^{\otimes q} \otimes L^j) = 0$ for $q \leq p$ and $j \geq 1$. Moreover, Theorem 2.1.2 shows that $H^0(X,L) \otimes H^0(X, M_L^{\otimes p} \otimes L^j) \longrightarrow H^0(X, M_L^{\otimes p} \otimes L^{j+1})$ is surjective. Hence, the short exact sequence

$$0 \longrightarrow M_L^{\otimes(p+1)} \otimes L^j \longrightarrow H^0(X,L) \otimes M_L^{\otimes p} \otimes L^j \longrightarrow M_L^{\otimes p} \otimes L^{j+1} \longrightarrow 0$$

implies that $H^1(X, M_L^{\otimes(p+1)} \otimes L^j) = 0$ for $j \geq 1$. □

4. Applications to Toric Varieties

In this section, we apply our main theorem to line bundles on an n-dimensional projective toric variety X.

Consider a globally generated line bundle L on X and its associated lattice polytope P_L. Let $r(L)$ be the number of integer roots of the Hilbert polynomial $h_L(d) := \chi(L^d) = \sum_{i=1}^{n} (-1)^i \dim H^i(X, L^d)$. Since the higher cohomology of L^d vanishes and the lattice points in the polytope $dP_L = P_{L^d}$ form a basis for $H^0(X, L^d)$, it follows that $h_L(d)$ equals the Ehrhart polynomial of P_L; in other words, $h_L(d)$ is the number of lattice points in dP. If $r(P_L)$ is the largest integer such that $r(P_L)P_L$ does not contain any interior lattice points, then Ehrhart reciprocity (Stanley [Sta97, Corollary 4.6.28]) implies that the integer roots of $h_L(d)$ are $\{-1, \ldots, -r(P_L)\}$ and $r(P_L) = r(L)$.

Lemma 4.1. *If L is a globally generated line bundle on a toric variety X and $r(L)$ is the number of integer roots of its Hilbert polynomial h_L, then $L^{\deg(h_L)-r(L)}$ is $\mathscr{O}_X$-regular with respect to L.*

Proof. We must establish that $H^i(X, L^{\deg(h_L)-r(L)-i}) = 0$ for all $i > 0$. If $\deg(h_L) - r(L) - i \geq 0$, this follows from the vanishing of the higher cohomology of globally generated line bundles on a complete toric variety; see Fulton [Ful93, §3.5].

When $\deg(h_L) - r(L) - i < 0$, our argument follows Batyrev and Borisov [BB96, Theorem 2.5]. Let X' be the toric variety corresponding to the normal fan to P_L. There is a canonical toric map $\psi \colon X \longrightarrow X'$ and an ample line bundle A on X' such that $H^i(X, L) \cong H^i(X', A)$. A toric version of the Kodaira Vanishing Theorem establishes that $H^j(X, L^{-u}) = 0$ for $u > 0$ and $j \neq \deg(h_L) = \dim P_L = \dim X'$ (combine Serre duality in Fulton [Ful93, §4.4] with Mustaţă [Mus02, Theorem 3.4]). In particular, we have $H^i(X, L^{\deg(h_L)-r(L)-i}) = 0$ for $i \neq \deg(h_L)$. When $i = \deg(h_L)$, we also have

$$0 = h_L\big(-r(L)\big) = \chi(L^{-r(L)}) = (-1)^i \dim H^i(X, L^{\deg(h_L)-r(L)-i}) . \qquad \square$$

Proof of Corollary 1.3. By Lemma 4.1, we can apply Theorem 1.1 with $\ell = 1$, $B_1 = L$, $\boldsymbol{w}_1 = \max\{\deg(h_L) - r(L), 1\}$ and $\boldsymbol{w}_i = 1$ for $i > 1$. $\square$

Proof of Corollary 1.2. The case $p = 0$ is established in Ewald and Wessels [EW91]. When $p \geq 1$, the assertion follows from Corollary 1.3. $\square$

Proof of Corollary 1.4. Given a lattice polytope P, let X be the corresponding toric variety and L the associated ample line bundle on X. Since P is normal if and only if L satisfies (N_0), the result follows from Corollary 1.3 and the fact that $r(P) = r(L)$. $\square$

Proof of Corollary 1.5. Let $\pi_i \colon X \longrightarrow \mathbb{P}^{n_i}$ be the projection onto the i-th factor and let $B_i := \pi_i^*\big(\mathscr{O}_{\mathbb{P}^{n_i}}(1)\big)$. If $I := \big\{i \in \{1,\dots,\ell\} : d_i \neq 0\big\}$, then we have $\mathscr{O}_X(d_1,\dots,d_\ell) \cong \bigotimes_{i\in I} B_i^{d_i}$. Let $d := \min\{d_i - 1 : i \in I\}$ and let $\boldsymbol{\mathcal{B}}$ be the semigroup generated by $\{B_i : i \in I\}$. Maclagan and Smith [MS04, Proposition 6.10] prove that $\mathscr{O}_X$ is $\mathscr{O}_X$-regular with respect to $B_1,\dots,B_\ell$. Thus, Theorem 2.1 shows that $\bigotimes_{i\in I} B_i^{d_i-d}$ is $\mathscr{O}_X$-regular with respect to $\{B_i : i \in I\}$ and lies in $\bigcap_{i\in I}(B_i \otimes \boldsymbol{\mathcal{B}})$. Since we have $\bigotimes_{i\in I} B_i \in \bigcap_{j\in I}(B_j \otimes \boldsymbol{\mathcal{B}})$, Theorem 1.1 applies with $\boldsymbol{w}_1 = (d_1 - d,\dots,d_\ell - d)$ and $\boldsymbol{w}_j = (1,\dots,1)$ for $j \geq 2$. □

Let $\mathcal{B}$ be the semigroup generated by the nef line bundles. To apply our techniques to adjoint bundles, we need to find $\boldsymbol{u}$ with $K_X \otimes B^{\boldsymbol{u}} \in \boldsymbol{\mathcal{B}}$. Inspired by Fujita's conjectures, Fujino [Fuj03, Corollary 0.2] provides the necessary criterion.

Theorem 4.2 (Fujino)**.** *Let X be a projective toric variety (not isomorphic to $\mathbb{P}^n$) such that the canonical divisor K_X is $\mathbb{Q}$-Cartier. If D is a $\mathbb{Q}$-Cartier divisor such that $D \cdot C \geq n$ for all torus invariant curves C, then $K_X + D$ is nef.*

Proof of Corollary 1.6. If $X = \mathbb{P}^n$, then $K_X = \mathscr{O}_X(-n-1)$. Either Corollary 1.3 or Corollary 1.5 show that $K_X \otimes B^{\boldsymbol{m}_{n+1+p}}$ satisfies (N_p). Mustaţă [Mus02, Theorem 3.4] establishes that $K_X \otimes B^{\boldsymbol{m}_{n+1}}$ is $\mathscr{O}_X$-regular with respect to $B_1,\dots,B_\ell$. For any torus invariant curve C, there is a B_i such that $B_i \cdot C > 0$. Since $B^{\boldsymbol{m}_n} = B_i^n \otimes B'$ where B' is globally generated, Theorem 4.2 implies that $K_X \otimes B^{\boldsymbol{m}_n} \in \boldsymbol{\mathcal{B}}$. It follows that $K_X \otimes B^{\boldsymbol{m}_{n+1}} \in \bigcap_{j=1}^{\ell}(B_j \otimes \boldsymbol{\mathcal{B}})$ and Theorem 1.1 proves the claim. □

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