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On the distribution of linear combinations of independent Gumbel random variables (Supplementary Material)

Filipe J. Marques · Carlos A. Coelho · Miguel de Carvalho

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1 Mathematica code

The 'sample code' presented in Figures 1 and 2 was developed for `Mathematica` 7.0; further code is available from the corresponding author. In Figure 1 we provide code for computing the near-exact distribution function, and in Figure 2 we provide code for computing the corresponding near-exact probability density function.

For example, if we wish to plot the density and cumulative distribution functions of W for $\boldsymbol{\mu} = (-20, -1, -50, 12, 40)$, $\boldsymbol{\sigma} = (2, 1/2, 5/4, 10, 50)$ and $\boldsymbol{\alpha} = (2, 12, 24, 50, 10)$, with $\gamma = 6$, we should use

```
mu={-20,-1,-50,12,40};
sigma={2,1/2,5/4,10,50};
alpha={2,12,24,50,10};
gamma= 6;
Plot[LinearGumbelsPDF[alpha, mu, sigma, gamma, w],
     {w,-2000,4000}]
Plot[LinearGumbelsCDF[alpha, mu, sigma, gamma, w],
     {w,-2000,4000}]
```

and the result should be the first two plots in Figure 3; similar code was use to produce all the remainder examples of near-exact densities and near-exact distribution functions in Figures 3 and 4.

F. J. Marques · C. A. Coelho
DM–CMA–FCT, Universidade Nova de Lisboa, Portugal
email: fjm@fct.unl.pt

M. de Carvalho
Pontificia Universidad Católica de Chile, Santiago, Chile

2 Further numerical reports

2.1 Sums of independent Gumbel random variables

In Table 1 and 2 we report further numerical results for sums of independent Gumbel random variables, all with the same scale parameter, according to the following scenarios:

—Scenario 1: $\boldsymbol{\mu}_1 = (2, 3)$, $\boldsymbol{\sigma}_1 = 1/100 \times \mathbf{1}_2^{\mathrm{T}}$, and $\boldsymbol{\alpha}_1 = \mathbf{1}_2^{\mathrm{T}}$;
—Scenario 2: $\boldsymbol{\mu}_2 = (-4, -1, 2, 3)$, $\boldsymbol{\sigma}_2 = 5 \times \mathbf{1}_4^{\mathrm{T}}$, and $\boldsymbol{\alpha}_2 = \mathbf{1}_4^{\mathrm{T}}$;
—Scenario 3: $\boldsymbol{\mu}_3 = (-10, 10, 20, 30, 40)$, $\boldsymbol{\sigma}_3 = 50 \times \mathbf{1}_5^{\mathrm{T}}$, and $\boldsymbol{\alpha}_3 = \mathbf{1}_5^{\mathrm{T}}$.

Table 1 Values of $\varDelta$ for Scenarios 1, 2, and 3

γ	Scenario 1 $(\boldsymbol{\mu}_1, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1)$ $p = 2$	Scenario 2 $(\boldsymbol{\mu}_2, \boldsymbol{\sigma}_2, \boldsymbol{\alpha}_2)$ $p = 4$	Scenario 3 $(\boldsymbol{\mu}_3, \boldsymbol{\sigma}_3, \boldsymbol{\alpha}_3)$ $p = 5$
4	1.3×10^{-4}	4.5×10^{-5}	3.3×10^{-5}
10	7.5×10^{-6}	2.4×10^{-6}	1.8×10^{-6}
15	2.1×10^{-6}	6.9×10^{-7}	5.0×10^{-7}
20	8.8×10^{-7}	2.8×10^{-7}	2.1×10^{-7}
50	5.4×10^{-8}	1.7×10^{-8}	1.3×10^{-8}
100	6.7×10^{-9}	2.1×10^{-9}	1.6×10^{-9}
500	5.3×10^{-11}	1.7×10^{-11}	1.2×10^{-11}

In Table 2 we present the computation time, in seconds, for the calculation of the p-values 0.10, 0.05, 0.01, using the near-exact quantiles. These calculations were done using an Intel i7 2GHz processor; for values of γ larger than 50 the computation times start to increase quite a bit.

Table 2 Computation time (in seconds) for the near-exact cumulative distribution functions for Scenarios 1–3

	Scenario 1 $(\boldsymbol{\mu}_1, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1)$ $p = 2$			Scenario 2 $(\boldsymbol{\mu}_2, \boldsymbol{\sigma}_2, \boldsymbol{\alpha}_2)$ $p = 4$			Scenario 3 $(\boldsymbol{\mu}_3, \boldsymbol{\sigma}_3, \boldsymbol{\alpha}_3)$ $p = 5$		
γ	p-values			p-values			p-values		
	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
4	0.03	0.02	0.02	0.08	0.08	0.09	0.14	0.14	0.16
10	0.13	0.11	0.13	0.58	0.62	0.72	1.14	1.19	1.34
15	0.23	0.25	0.30	1.50	1.52	1.62	3.01	3.18	3.57
20	0.42	0.44	0.53	3.03	3.23	3.67	6.03	6.44	7.24
50	3.37	3.70	4.35	29.1	31.0	35.1	63.1	66.2	74.4

Fig. 1 Mathematica module for the cumulative distribution function of positive linear combinations of independent Gumbel random variables

```
LinearGumbelsCDF[alpha_,mu_,sigma_,gamma_,w_]:=Module[{l,rho,theta,mom,mom1,v,vs,n,isc,lambda,r,shift,c,g,P},
  mom=Table[SetPrecision[I^(-h)*D[Product[Gamma[gamma-I*t*sigma[[j]]*alpha[[j]]]/Gamma[gamma],
        {j,1,Length[mu]}],{t,h}]/.t->0,150],{h,1,3}];
  mom1=Table[SetPrecision[I^(-h)*D[l^rho*(l-I*t)^(-rho)*Exp[I*t*theta],{t,h}]/.t->0,150],{h,1,3}];
  {rho,l,theta}={rho,l,theta}/.Flatten[Solve[{mom[[1]]==mom1[[1]],mom[[2]]==mom1[[2]],
   mom[[3]]==mom1[[3]]},{rho,l,theta}]];
  v=Flatten[{Table[Table[((1+k)/(sigma[[j]]*alpha[[j]])),{k,0,gamma-2}],{j,1,Length[sigma]}]}];
  vs = Sort[v];n=Length[v];lambda={vs[[1]]};r={1};isc=1;
  Do[If[vs[[i]]==vs[[i-1]],{r[[isc]]=r[[isc]]+1},{isc=isc+1,lambda=Append[lambda,vs[[i]]],
    r=Append[r,1]}],{i, 2, n}];
  If[Count[r,_Integer]==Length[r] && And @@ Positive[r] && And @@ Positive[lambda], g = Length[r];
   shift = theta + Sum[alpha[[j]]*mu[[j]], {j, 1, Length[sigma]}];
   c=Table[Table[0,{j,1,Max[r]}],{i,1,g}];
   Table[c=ReplacePart[c,(Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,1,i-1}]*
         Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,i+1,g}])/(r[[i]]-1)!,{i,r[[i]]}],{i,1,g}];
   Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,
    {h,1,i-1}]+Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,{h,i+1,g}])*
    c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j, 1, k}]/k,{i,r[[i]]-k}],{k,1,r[[i]]-1}],{i,1,g}];
   l^rho*(w - shift)^rho/Gamma[rho + 1]*Hypergeometric1F1[rho,rho+1,-l(w-shift)]
     -Product[lambda[[j]]^r[[j]],{j,1,g}]*l^rho*Sum[Exp[-lambda[[j]]*(w-shift)]*
     Sum[c[[j]][[k]]/lambda[[j]]^k*Gamma[k]*Sum[(w-shift)^(rho+i)*lambda[[j]]^i/Gamma[rho+1+i]
     *Hypergeometric1F1[rho,rho+1+i,-(l-lambda[[j]])*(w-shift)],{i,0,k-1}],{k,1,r[[j]]}],{j,1,g}]]]
```

Fig. 2 Mathematica module for the density function of positive linear combinations of independent Gumbel random variables

```
LinearGumbelsPDF[alpha_,mu_,sigma_,gamma_,w_]:=Module[{l,rho,theta,mom,mom1,v,vs,n,isc,lambda,r,shift,c,g,P},
  mom=Table[SetPrecision[I^(-h)*D[Product[Gamma[gamma-I*t*sigma[[j]]*alpha[[j]]]/Gamma[gamma],
        {j,1,Length[mu]}],{t,h}]/.t->0,150],{h,1,3}];
  mom1=Table[SetPrecision[I^(-h)*D[l^rho*(l-I*t)^(-rho)*Exp[I*t*theta],{t,h}]/.t->0,150],{h,1,3}];
  {rho,l,theta}={rho,l,theta}/.Flatten[Solve[{mom[[1]]==mom1[[1]],mom[[2]]==mom1[[2]],
   mom[[3]]==mom1[[3]]},{rho, l, theta}]];
  v=Flatten[{Table[Table[((1+k)/(sigma[[j]]*alpha[[j]])),{k,0,gamma-2}],{j,1,Length[sigma]}]}];
  vs=Sort[v];n=Length[v];lambda={vs[[1]]};r={1};isc=1;
  Do[If[vs[[i]]==vs[[i-1]],{r[[isc]]=r[[isc]]+1},{isc=isc+1,lambda=Append[lambda,vs[[i]]],
    r=Append[r,1]}],{i, 2, n}];
  If[Count[r,_Integer]==Length[r] && And @@ Positive[r] && And @@ Positive[lambda], g = Length[r];
   shift=theta+Sum[alpha[[j]]*mu[[j]],{j,1,Length[sigma]}];
   c=Table[Table[0,{j,1,Max[r]}],{i,1,g}];
   Table[c=ReplacePart[c,(Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,1,i-1}]*
         Product[(lambda[[j]]-lambda[[i]])^(-r[[j]]),{j,i+1,g}])/(r[[i]]-1)!,{i,r[[i]]}],{i,1,g}];
   Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,
    {h,1,i-1}]+Sum[r[[h]]/(lambda[[i]]-lambda[[h]])^j,{h,i+1,g}])*
    c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j,1,k}]/k,{i,r[[i]]-k}],{k,1,r[[i]]-1}],{i,1,g}];
   Product[lambda[[j]]^r[[j]],{j,1,g}]*l^rho*Sum[Exp[-lambda[[j]]*(w-shift)]*Sum[c[[j]][[k]]*
   Gamma[k]/Gamma[k + rho]*(w-shift)^(k+rho-1)*Hypergeometric1F1[rho,k+rho,-(l-lambda[[j]])*
    (w-shift)],{k, 1, r[[j]]}], {j, 1, g}]]]
```

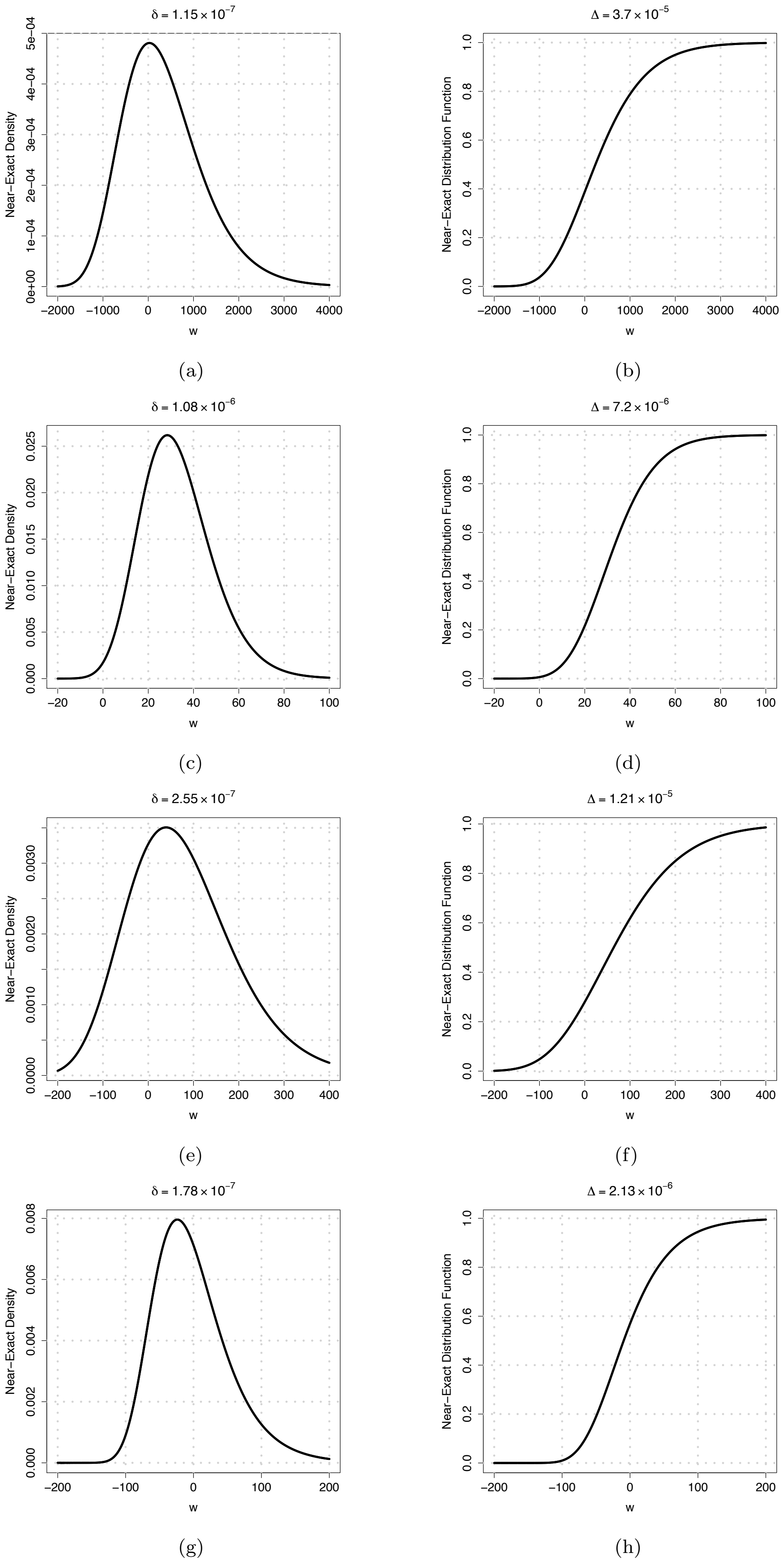

Fig. 3 Near-exact densities and near-exact distribution functions for positive linear combinations of independent Gumbel random variables: (a–b) $\boldsymbol{\mu} = (-20, -1, -50, 12, 40)$, $\boldsymbol{\sigma} = (2, 1/2, 5/4, 10, 50)$, $\boldsymbol{\alpha} = (2, 12, 24, 50, 10)$, and $\gamma = 6$; (c–d) $\boldsymbol{\mu} = (2, 3, 4, 5^{1/2}, \pi, -6, -7, -7)$, $\boldsymbol{\sigma} = (1/2, \pi, \exp(1), 2^{1/2}, 1.2, 3.1, 2, 1)$, $\boldsymbol{\alpha} = (1, 2, 3, 1/2, 5, 1, 1, 1)$, and $\gamma = 8$; (e–f) $\boldsymbol{\mu} = (1, 2, 3, -3, -2, -1)$, $\boldsymbol{\sigma} = (1, 2, 3, 4, 5, 6)$, $\boldsymbol{\alpha} = (2, 4, 6, 8, 10, 12)$, and $\gamma = 8$; (g–h) $\boldsymbol{\mu} = (-2, -4)$, $\boldsymbol{\sigma} = (5, 6)$, $\boldsymbol{\alpha} = (3, 7)$, and $\gamma = 20$.

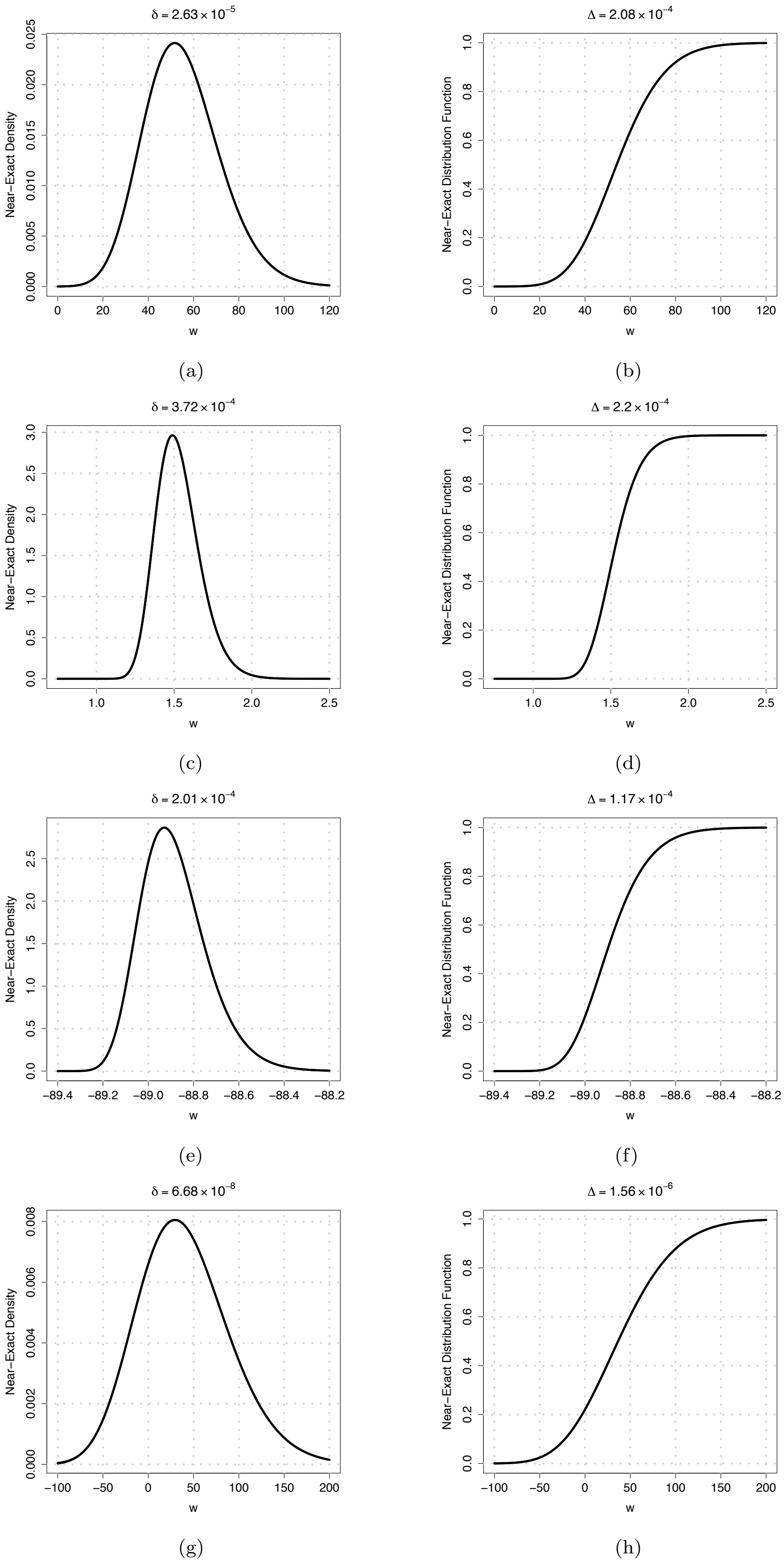

Fig. 4 Near-exact densities and near-exact distribution functions for sums of independent Gumbel random variables: (a–b) $\boldsymbol{\mu} = (2, 3, 4, 5, 6, 7, 8)$, $\boldsymbol{\sigma} = 5 \times \mathbf{1}_7^{\mathrm{T}}$, and $\gamma = 2$; (c–d) $\boldsymbol{\mu} = (2/10, 3/10, 4/10, 5/10)$, $\boldsymbol{\sigma} = 55/1000 \times \mathbf{1}_4^{\mathrm{T}}$, and $\gamma = 5$; (e–f) $\boldsymbol{\mu} = (-29, -25, -35)$, $\boldsymbol{\sigma} = 1/15 \times \mathbf{1}_3^{\mathrm{T}}$, and $\gamma = 7$; (g–h) $\boldsymbol{\mu} = (-9, -5, -5, -7, 2, 3, 1/2)$, $\boldsymbol{\sigma} = 15 \times \mathbf{1}_7^{\mathrm{T}}$, and $\gamma = 9$.

2.2 Further reports on measuring accuracy

In Figure 3 we plot the near-exact density and near-exact distribution function of four examples of positive linear combinations of independent Gumbel random variables. To assess the quality of our approximation we also report in Figure 3 the values of $\varDelta$, and of the measure

$$\delta = \frac{1}{2\pi} \int_{\mathbb{R}} |\varPhi_W(t) - \varPhi_{W^\star}(t)| \; \mathrm{d}t \, , \qquad (1)$$

which provides an upper bound analogous to equation (14) in the paper, but for the case of the near-exact density $f_{W^\star} = \mathrm{d}F_{W^\star}/\mathrm{d}w$, i.e.

$$\|f_W - f_{W^\star}\|_\infty \leq \delta \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\varPhi_{W_1}(t) - \varPhi_{W_1^\star}(t)|\mathrm{d}t,$$

with $\|f_W - f_{W^\star}\|_\infty = \sup_{w\in\mathbb{R}} |f_W(w) - f_{W^\star}(w)|$, and where $f_W = \mathrm{d}F_W/\mathrm{d}w$ is the exact density. Thus similarly to $\varDelta$, the measure δ also provides an upper bound—in the sup-norm—for the error of our approximation, but for $f_{W^\star}$ instead of $F_{W^\star}$; further details on the measure δ can be found in Marques and Coelho (2008, p. 732).

The quality of our approximation is visible in the extremely reduced values of $\varDelta$ and δ, which also show that if we plotted the exact density and the exact distribution function, obtained through the inversion formulas in Gil-Pelaez (1951), these would be virtually indistinguishable from our near-exact approximations. Similar conclusions can be drawn for Figure 4, where we plot near-exact densities and near-exact distribution functions, but now considering examples of sums of independent Gumbel random variables, all with the same scale parameter.

References

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