Dynamic threshold modelling and the US business cycle (Supporting Information)

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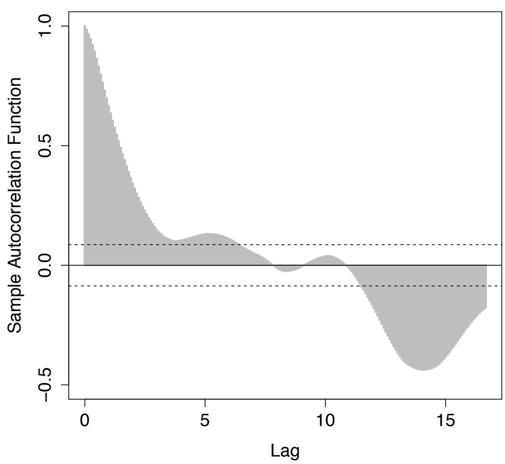
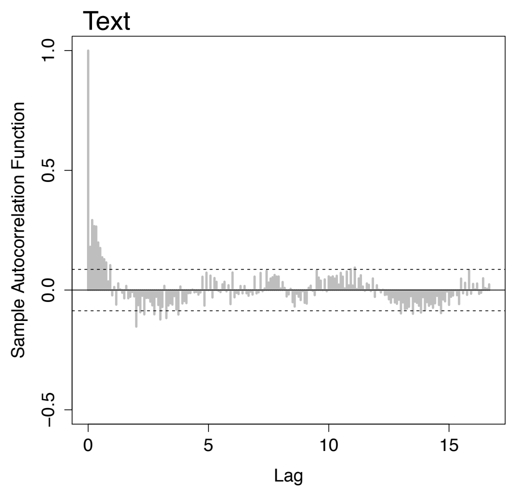
Here we report further data analyses, numerical experiments, comments on the large sample characterization of the dynamic threshold, and extra diagnostics.

1. Further exploratory analyses



(a)



(b)

Figure 1. Correlagram for US monthly unemployment rate for the original data (a) and data in first differences (b).

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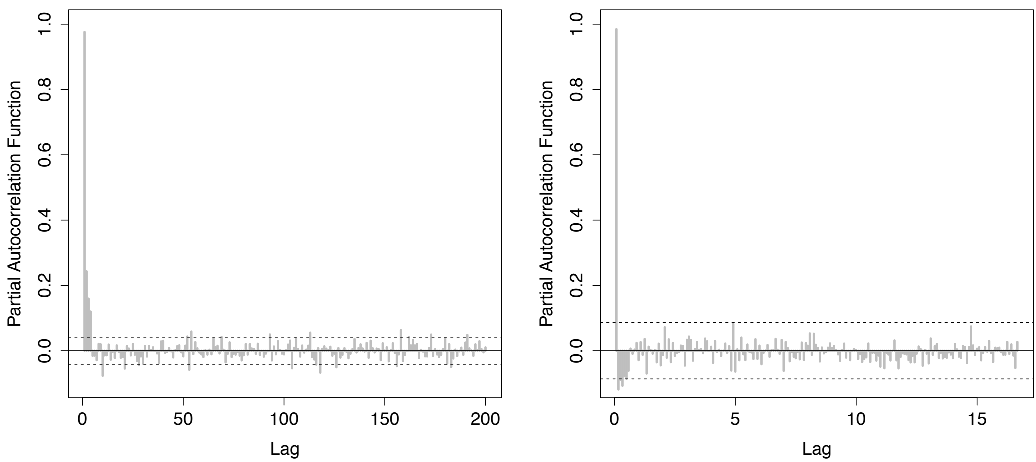
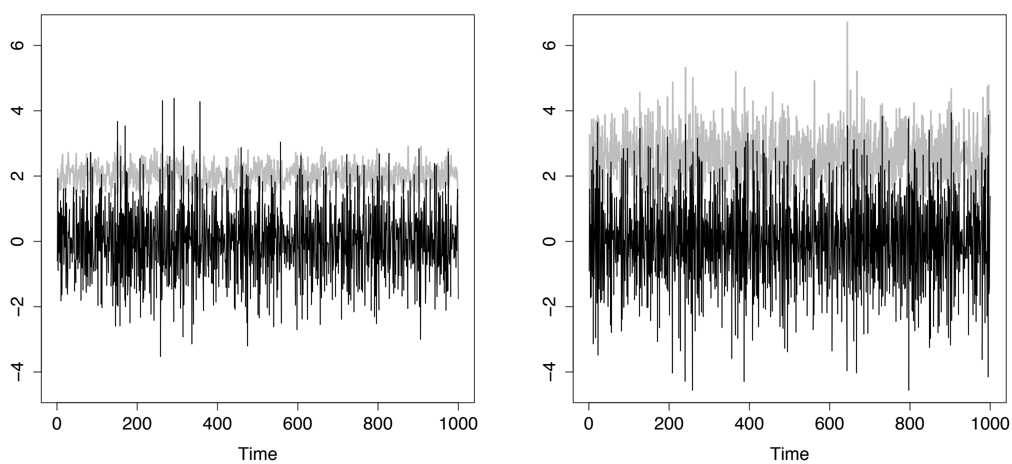


(a) (b)

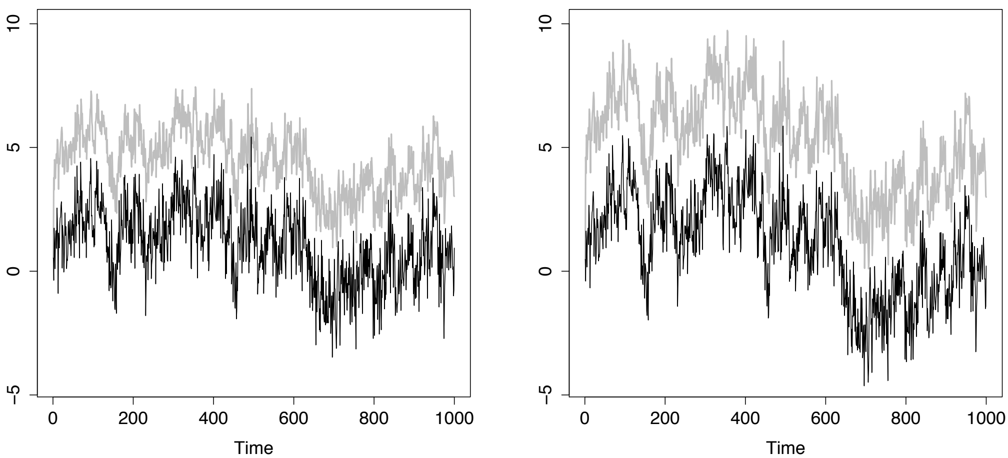
Figure 2. Partial correlograms for: (a) weekly number of unemployment insurance claims in the US; (b) US monthly unemployment rate.

2. Examples of numerical experiments with simulated data



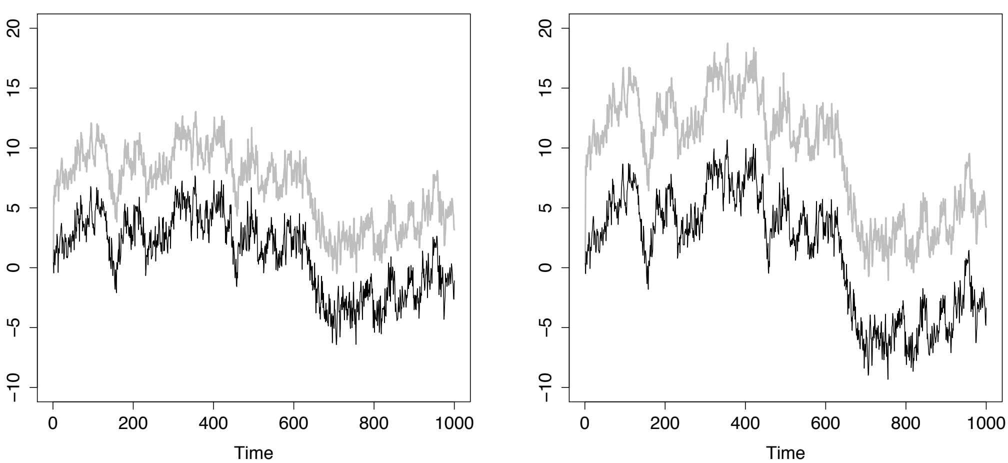
(a) (b)

Figure 3. Solid gray lines represent the dynamic thresholds for $n = 1000$ observations simulated from: (a) FARIMA(0.2,-0.43,0.4); (b) FARIMA(0.2,-0.89,0.4).



(a) (b)

Figure 4. Solid gray lines represent the dynamic thresholds for $n = 1000$ observations simulated from: (a) FARIMA(0.2,0.6,0.4); (b) FARIMA(0.2,-0.7,0.4).



(a) (b)

Figure 5. Solid gray lines represent the dynamic thresholds for $n = 1000$ observations simulated from: (a) FARIMA(0.2,0.7,0.4); (b) FARIMA(0.2,-0.8,0.4).

3. Large sample characterization of the dynamic threshold

The following theorem follows directly by combining the delta method, with the Theorem 2 in Hurvich *et al.* (1998); we recall that the derivative of the Pochhammer symbol is given by

$$\frac{\mathrm{d}\langle\alpha\rangle_i}{\mathrm{d}\alpha} = \langle\alpha\rangle_i\{\Psi(\alpha+i) - \Psi(\alpha)\}, \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}_0^-,$$

with Ψ denoting the digamma function, i.e., $\Psi(x) = (\mathrm{d}/\mathrm{d}x)\log\Gamma(x)$, for $x \in \mathbb{R}\backslash\mathbb{Z}_0^-$.

THEOREM 1. *Let $\{Y_t\}$ denote a series with Gaussian increments, and let $\mathcal{F}_t = \{Y_{\tau-1}\}_{\tau\leq t}$. Consider the dynamic threshold*

$$\widehat{u}_t = u + \sum_{i=1}^{\infty} \frac{\langle\widehat{\alpha}\rangle_i}{\Gamma(i+1)} \{Y_{t-i}I(i \text{ odd}) - Y_{t-i}I(i \text{ even})\}, \tag{1}$$

where $\langle\alpha\rangle_i = \alpha(\alpha-1)\cdots(\alpha-i+1)$, and $\alpha \in (-1/2, 1/2)\backslash\{0\}$. If $m = o(n^{4/5})$ and $\log^2 n = o(m)$, then it holds that $m^{-1/2}(\widehat{u}_t - u_t) \mid \mathcal{F}_t$ converges weakly to

$$\mathrm{N}\left(0, \pi^2/24\left[\sum_{i=1}^{\infty} \frac{\langle\alpha\rangle_i\{\Psi(\alpha+i)-\Psi(\alpha)\}}{\Gamma(i+1)}\{Y_{t-i}I(i \text{ odd}) - Y_{t-i}I(i \text{ even})\}\right]^2\right), \tag{2}$$

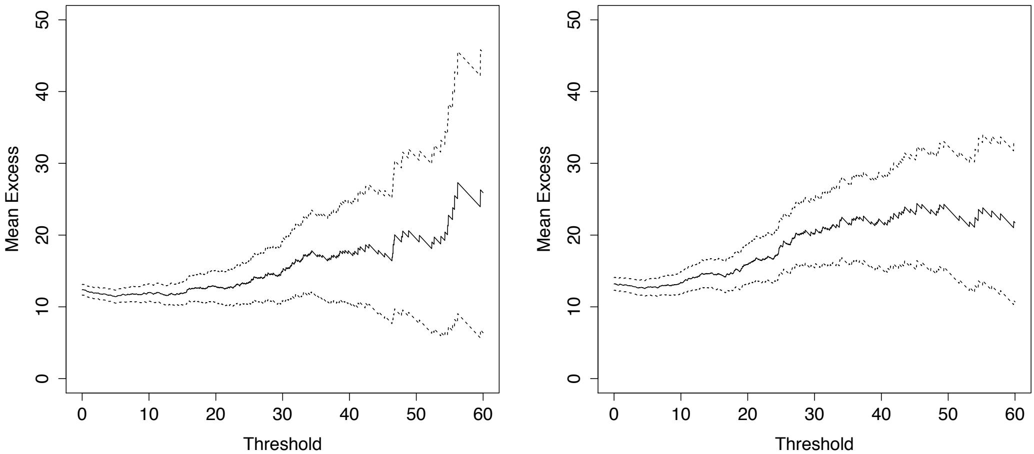
as $n \to \infty$.

It is also possible to combine the delta method with other central limit theorems for α which exist in the literature; for example:

- by Theorem 2 in Deo and Hurvich (2001), we can remove the assumption of Gaussian increments, at the cost of imposing more strict conditions on the rate of increase of m, but the limiting distribution remains unchanged;
- by Theorem 3 in Velasco (1999), we can obtain a similar result for $\alpha \in [1/2, 3/4)$ for a modified GPH estimator which trims low frequency ordinates, but the expression for the variance will differ from the one presented in (2); the assumptions on the rate of increase of m need also to be changed.

Results about consistency of the GPH estimator (Hurvich *et al.*, 1998; Velasco , 1999; Robinson and Marinucci, 2001; Phillips, 2007), can be readily adapted to establish consistency of the dynamic threshold in (1), at every $t \in \mathbb{Z}$. Since for each $t \in \mathbb{Z}$, $u_t \mid \mathcal{F}_t$ is a continuous function of α, except at a countable number of points ($\mathbb{Z}_0^-$), the generalized continuous mapping theorem of Billingsley (1999, Thm 2.7), can be used to establish consistency of $\widehat{u}_t \mid \mathcal{F}_t$, at every $t \in \mathbb{Z}$, for the same settings where the GPH estimator has been shown to be consistent.

4. Extra diagnostics



(a) (b)

Figure 6. Mean residual life plot; left and right tails are in (a) and (b), respectively.

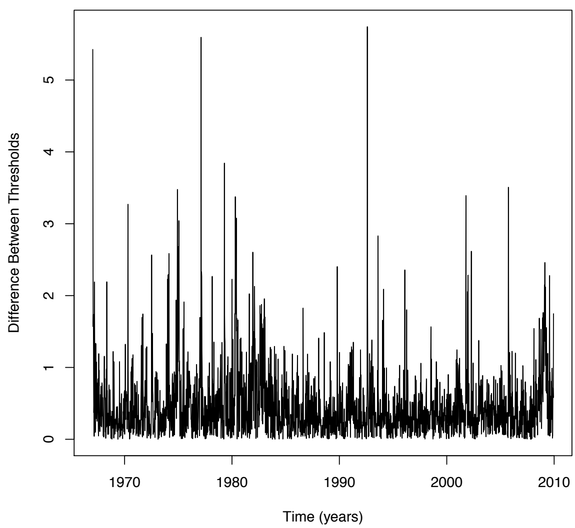


Figure 7. Difference in absolute value between the threshold obtained with a long memory parameter of 0.96 and 1.

Table 1. Left tail exceedances which occurred until two months after each of the nine mirror filtered exceedanc
falling outside a contraction period.

	Weeks After							
Date	1	2	3	4	5	6	7	8
July 17, 1971	—	—	—	—	—	—	—	-45.26
June 6, 1975	—	—	-38.04	—	—	—	—	—
January 22, 1977	—	—	-82.41	-78.14	-39.33	—	—	—
April 7, 1979	—	-72.28	—	—	—	—	—	—
August 2, 1986	—	—	—	—	—	—	—	—
January 1, 1994	—	—	—	—	-42.91	—	—	—
January 20, 1996	—	—	—	—	—	—	—	—
March 30, 2002	—	—	—	—	—	—	—	—
September 10, 2005	—	-61.18	—	—	—	—	—	—

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