Dissections: How to cut things up

Matt Booth

PG Colloquium, University of Edinburgh

September 2017
What is a dissection?
What is a dissection?
Scissors-congruence

Definition

Two polygons $P, Q \subseteq \mathbb{R}^2$ are **congruent**, written $P \cong Q$, if $Q$ can be obtained from $P$ by translations and reflections.

Definition

Two polygons $P, Q$ are **scissors-congruent**, written $P \sim Q$, if they decompose as disjoint\(^a\) unions of polygons $P = \bigcup_{i=0}^n P_i$ and $Q = \bigcup_{i=0}^n Q_i$ with each $P_i \sim Q_i$.

\(^a\)up to boundary, i.e. $\text{area}(P_i \cap P_j) = 0$ for $i \neq j$
Scissors-congruence

- Note that we allow polygonal cuts: the definition is the same if we allow only straight-line cuts, but the minimal number of cuts needed to dissect one shape into another may change.

- Scissors-congruence is an equivalence relation! To see transitivity, superimpose cutting patterns.
The WBG theorem

Clearly if $P \sim Q$ then they have the same area.
The WBG theorem

- Clearly if $P \sim Q$ then they have the same area.

- The **Wallace-Bolyai-Gerwien theorem** (1807, 1833, 1835) says that the converse is also true: if two polygons have the same area, then they’re scissors-congruent.
To prove the WBG theorem, we’ll prove that:

Any polygon $P$ is scissors-congruent to a square of the same area.
The WBG theorem

To prove the WBG theorem, we’ll prove that:

Any polygon $P$ is scissors-congruent to a square of the same area.

First cut $P$ up into triangles. Then we’ll show that:

- A triangle is scissors-congruent to a parallelogram with the same base (and half the height)
- Two parallelograms of the same base and height are scissors-congruent
- Two squares are scissors-congruent to one big square

It’s possible to give ‘Euclid-style’ proofs of the above, but we’ll give a more modern proof using group actions on $\mathbb{R}^2$. 
Pak’s proof

Definition

Let $G$ be a group acting on $\mathbb{R}^2$. A **fundamental domain** for $G$ is a set $X \subseteq \mathbb{R}^2$ containing exactly one element from every orbit of $G$.

Lemma

Let $G$ be a discrete group of isometries of $\mathbb{R}^2$, and suppose that $P, Q$ are polygons that are (the closures of) fundamental domains for $G$. Then $P$ and $Q$ are scissors-congruent.
Proof of the Lemma

Since $Q$ is a fundamental domain, the translates $\{gQ : g \in G\}$ tile the plane. Since $G$ is discrete, only finitely many of the $gQ$ intersect $P$ nonemptily; write $P_i = P \cap g_i Q$ for these, so that $P = \bigcup_i P_i$. Set $Q_i := g_i^{-1} P_i$. Clearly $P_i \cong Q_i$, and since $P$ is a fundamental domain, $Q = \bigcup_i Q_i$ and the $Q_i$ are pairwise disjoint. Hence $P \sim Q$. 

Pak’s proof
$G = \mathbb{Z}^2$, acting by translations, extended by a copy of $\mathbb{Z}/2\mathbb{Z}$ acting by reflection in the origin; both the white triangles and the white parallelograms are fundamental domains.
Parallelograms into parallelograms

\[ G = \mathbb{Z}^2, \text{ acting by translations.} \]
$G = \mathbb{Z}^2$, acting by translations along the red axes.
More general decompositions

Definition

Two sets $X, Y \subseteq \mathbb{R}^n$ are **equidecomposable** if they decompose as disjoint unions $X = \bigcup_{i=0}^{k} X_i$ and $Y = \bigcup_{i=0}^{k} Y_i$, and there exist isometries $f_1, \ldots, f_k$ of $\mathbb{R}^n$ such that $f_i(X_i) = Y_i$.

(Tarski, 1924) For $n = 2$, any two polygons of equal area are equidecomposable.

(Banach-Tarski, 1924) If $n \geq 3$, any two bounded sets with nonempty interior are equidecomposable (not true if $n \leq 2$).
More general decompositions

**Definition**

Two sets \( X, Y \subseteq \mathbb{R}^n \) are **equidecomposable** if they decompose as disjoint unions \( X = \bigcup_{i=0}^{k} X_i \) and \( Y = \bigcup_{i=0}^{k} Y_i \), and there exist isometries \( f_1, \ldots, f_k \) of \( \mathbb{R}^n \) such that \( f_i(X_i) = Y_i \).

- (Tarski, 1924) For \( n = 2 \), any two polygons of equal area are equidecomposable.
More general decompositions

Definition

Two sets $X, Y \subseteq \mathbb{R}^n$ are **equidecomposable** if they decompose as disjoint unions $X = \bigcup_{i=0}^k X_i$ and $Y = \bigcup_{i=0}^k Y_i$, and there exist isometries $f_1, \ldots, f_k$ of $\mathbb{R}^n$ such that $f_i(X_i) = Y_i$.

- (Tarksi, 1924) For $n = 2$, any two polygons of equal area are equidecomposable.
- (Banach-Tarski, 1924) If $n \geq 3$, any two bounded sets with nonempty interior are equidecomposable (not true if $n \leq 2$).
Tarski’s circle-squaring problem

- **Tarski’s circle-squaring problem, 1925**: are a circle and a square of unit area equidecomposable?

- (Dubins, Hirsch, Karush, 1963) A unit circle and a unit square are not scissors-congruent. In fact, they’re not equidecomposable if the pieces have Jordan curve boundary.

- (Laczkovich 1990) The answer is yes! The proof uses about $10^{50}$ pieces, but they may not be (Lebesgue) measurable.

- (Grabowski, Máthé, Pikhurko, 2017) One can carry out the decomposition with measurable pieces.
Tarski’s circle-squaring problem

- **Tarski’s circle-squaring problem, 1925**: are a circle and a square of unit area equidecomposable?

- (Dubins, Hirsch, Karush, 1963) A unit circle and a unit square are not scissors-congruent. In fact, they’re not equidecomposable if the pieces have Jordan curve boundary.

- (Laczkovich 1990) The answer is yes! The proof uses about $10^{50}$ pieces, but they may not be (Lebesgue) measurable.

- (Grabowski, Máthé, Pikhurko, 2017) One can carry out the decomposition with measurable pieces.
Tarski’s circle-squaring problem

If one is allowed to use homotheties, there are much nicer solutions:
Hinged dissections

One can also consider dissections with hinges, such as Dudeney’s famous 1902 dissection of a square into a triangle:
Hinged dissections

In fact any scissors-congruence dissection can be ‘hingified’ by adding chains of triangles to move pieces around (AACDDDK 2012).

AACDDDK
Euclid knew that the volume of a tetrahedron is $\frac{1}{3}(\text{base}) \times (\text{height})$, but all known proofs use (some form of) calculus. Is there a scissors-congruence between a tetrahedron of unit volume and a cube of unit volume?
Euclid knew that the volume of a tetrahedron is \( \frac{1}{3} \times \text{(base)} \times \text{(height)} \), but all known proofs use (some form of) calculus. Is there a scissors-congruence between a tetrahedron of unit volume and a cube of unit volume?

Unlike in the 2d case, the answer is no. This was proved by Max Dehn in 1900, and was the first of Hilbert’s 23 problems to be solved. The idea of the proof is to define a new invariant of polyhedra.
Valuations

Definition
Let $\phi$ be a function from the set of convex polyhedra in $\mathbb{R}^3$ to some abelian group. Then $\phi$ is a \textbf{valuation} if it satisfies
\[ \phi(P_1 \cup P_2) = \phi(P_1) + \phi(P_2) \]
for disjoint (up to boundaries) $P_1, P_2$.

Definition
A valuation is \textbf{symmetric} if it’s invariant under rigid motions (rotations and translations).

Volume is an example of a symmetric valuation (convex polyhedra) $\rightarrow \mathbb{R}$. 
Proposition

Let $P_1, P_2$ be two scissors-congruent convex polyhedra. Let $\phi$ be any symmetric valuation. Then $\phi(P_1) = \phi(P_2)$.

Proof

Decompose $P_1 = \bigcup_{i=0}^{m} \Delta_i^1$ and $P_2 = \bigcup_{i=0}^{m} \Delta_i^2$ into tetrahedra with $\Delta_i^1 \simeq \Delta_i^2$. Then $\phi(P_1) = \sum_{i=0}^{m} \phi(\Delta_i^1) = \sum_{i=0}^{m} \phi(\Delta_i^2) = \phi(P_2)$.

So we want to find a symmetric valuation $\phi$ such that $\phi(\text{unit cube}) \neq \phi(\text{unit tetrahedron})$. 
Definition
Let $P$ be a convex polyhedron. If $e$ is an edge of $P$, let $\ell_e$ be the length of $e$, and let $\theta_e$ be the dihedral angle at $e$. The total mean curvature of $P$ is $H(P) := \frac{1}{2} \sum_e \ell_e \theta_e \in \mathbb{R}$. Dehn's idea is to take $\theta_e \pmod{\pi}$ to make $H$ into a symmetric valuation.
Total mean curvature

Definition

Let $P$ be a convex polyhedron. If $e$ is an edge of $P$, let $\ell_e$ be the length of $e$, and let $\theta_e$ be the dihedral angle at $e$. The total mean curvature of $P$ is $H(P) := \frac{1}{2} \sum_e \ell_e \theta_e \in \mathbb{R}$.

- $H$ is almost a symmetric valuation: we have $H(P \cup Q) = H(P) + H(Q) - H(P \cap Q)$. So if $P \cap Q$ is a polygon, we have $H(P \cap Q) = \sum_e \ell_e \pi$.
- Dehn’s idea is to take $\theta_e \mod \pi$ to make $H$ into a symmetric valuation.
The Dehn invariant is going to take values in the infinite-dimensional real vector space

\[ \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi) \]

If you don’t like tensor products: take a \(\mathbb{Q}\)-basis \(\{\pi\} \cup B\) of \(\mathbb{R}\). Then

\[ \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi) \cong \mathbb{R}^B \]

via the map \(\mathbb{R}^B \to \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi)\) that sends a basis vector \(b\) to \(1 \otimes b\).
The Dehn invariant

- Given a convex polygon $P$, we set $D(P) := \sum_e l_e \otimes \theta_e$. Then $D$ is indeed a symmetric valuation.
The Dehn invariant

- Given a convex polygon $P$, we set $D(P) := \sum e \ell_e \otimes \theta_e$. Then $D$ is indeed a symmetric valuation.
- It’s easy to compute that $D(\text{cube}) = 0$. 

A theorem of Sydler says that volume and Dehn invariant are enough to characterise scissors-congruence.
The Dehn invariant

- Given a convex polygon $P$, we set $D(P) := \sum_e \ell_e \otimes \theta_e$. Then $D$ is indeed a symmetric valuation.
- It’s easy to compute that $D(\text{cube}) = 0$.
- But the dihedral angles of a regular tetrahedron are all $\alpha = \arccos\left(\frac{1}{3}\right)$, which is not a rational multiple of $\pi$ [one can induct on $n$ to show that $\cos(n\alpha) \notin \mathbb{Z}$ for all $n$].
The Dehn invariant

- Given a convex polygon $P$, we set $D(P) := \sum_e \ell_e \otimes \theta_e$. Then $D$ is indeed a symmetric valuation.
- It’s easy to compute that $D(\text{cube}) = 0$.
- But the dihedral angles of a regular tetrahedron are all $\alpha = \arccos\left(\frac{1}{3}\right)$, which is not a rational multiple of $\pi$ [one can induct on $n$ to show that $\cos(n\alpha) \notin \mathbb{Z}$ for all $n$].
- Hence, a cube and a regular tetrahedron don’t have the same Dehn invariant, and are not scissors-congruent.
The Dehn invariant

- Given a convex polygon $P$, we set $D(P) := \sum e \, \ell_e \otimes \theta_e$. Then $D$ is indeed a symmetric valuation.
- It’s easy to compute that $D(\text{cube}) = 0$.
- But the dihedral angles of a regular tetrahedron are all $\alpha = \arccos\left(\frac{1}{3}\right)$, which is not a rational multiple of $\pi$ [one can induct on $n$ to show that $\cos(n\alpha) \notin \mathbb{Z}$ for all $n$].
- Hence, a cube and a regular tetrahedron don’t have the same Dehn invariant, and are not scissors-congruent.
- A theorem of Sydler says that volume and Dehn invariant are enough to characterise scissors-congruence.
References

- Ivan Izmestiev, *Intuition behind the Dehn Invariant*, https://mathoverflow.net/q/264880