STATISTICAL MECHANICS OF THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION - EXERCISES

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1. Exercise session 1

Exercise 1. Let $A : D \subseteq L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ be a densely defined, nonnegative, self-adjoint operator which is translation invariant. Recall that this implies that for every $n \in \mathbb{Z}^d$, there exists $\lambda_n \in \mathbb{R}$ such that for every $f \in D$,

$$\widehat{Af}(n) = \lambda_n^2 \widehat{f}(n).$$

Moreover,

$$D = \{f : \sum_{n\in\mathbb{Z}^d} \lambda_n^4 |\widehat{f}(n)|^2 < \infty\}.$$

Suppose moreover that for some $s_0 > 0$, $c > 0$, we have

$$\langle Au, u\rangle \geq c\|u\|^2_{H^{s_0}}.$$

We want to build a measure μ, formally given by

$$\mu \sim \exp\Big(-\frac{1}{2}\langle Au, u\rangle\Big) du d\overline{u},$$

which has the following properties

i. There exists $s \in \mathbb{R}$ such that the measure μ is a Gaussian measure on $H^s(\mathbb{T}^d)$.
ii. For every $f, g \in C^\infty(\mathbb{T}^d)$, we have that

$$\int \langle f, u\rangle d\mu(u) = 0,$$
$$\int \langle f, u\rangle \overline{\langle g, u\rangle} d\mu(u) = \langle A^{-1} f, g\rangle.$$

Show the following.

(1) Let $\{g_n\}_{n\in\mathbb{T}^d}$ be a family of i.i.d. complex valued standard random variables.[1] Define

$$X = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n\in\mathbb{Z}} \frac{g_n}{\lambda_n} e^{in\cdot x}.$$

Firstly, show that

$$\mathbb{E}\|X\|^2_{H^s} < \infty$$

for every $s < s_0 - \frac{d}{2}$. Then, show that $\mu = \mathrm{Law}(X)$ satisfies ii.

[1] We say that g is a complex valued standard random variable if $\mathrm{Re}\, g \sim N(0, \frac{1}{2})$, $\mathrm{Im}\, g \sim N(0, \frac{1}{2})$, and they are independent.

(2) Now suppose that μ is a measure that satisfies both i. and ii. Show that if $\mathrm{Law}(u) = \mu$, then the random variables

$$G_n := \frac{\lambda_n}{(2\pi)^{\frac{d}{2}}} \langle u, e^{in\cdot x} \rangle$$

are i.i.d. complex valued standard random variables.

(3) Let $s < s_0 - \frac{d}{2}$ be such that μ is concentrated on H^s. For G_n as in (2), show that

- $\int \|u\|_{H^s}^2 d\mu(u) < \infty$,
- $\lim_{N\to\infty} \int \|u - \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{|n|\le N} \frac{G_n}{\lambda_n} e^{in\cdot x}\|_{H^s}^2 d\mu(u) = 0.$

Deduce that i. and ii. uniquely determine the measure μ, and that every such measure admits a random series representation

$$\mu = \mathrm{Law}\Big(\frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n\in\mathbb{Z}} \frac{g_n}{\lambda_n} e^{in\cdot x} \Big).$$

Exercise 2. Let

$$X = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x}.$$

We want to show that the property $X \in H^{\alpha-\frac{d}{2}-\varepsilon}$ is optimal, or in other words, that

$$\|X\|_{H^{\alpha-\frac{d}{2}}}^2 = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2\alpha-d} |\widehat{X}(n)|^2 = \infty \quad \text{a.s.}$$

For $N \in 2^{\mathbb{N}}$, define

$$X_N := \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\frac{N}{2}<|n|\le N} \widehat{X}(n) e^{in\cdot x}.$$

(1) Show that

$$\mathbb{E}\|X_N\|_{L^2}^2 \sim N^{d-2\alpha}.$$

(2) Show that

$$\mathbb{E}\big|\|X_N\|_{L^2}^2 - \mathbb{E}\|X_N\|_{L^2}^2\big|^2 \lesssim N^{d-4\alpha}.$$

(3) Deduce that

$$\sum_N N^{2\alpha-d} \big|\|X_N\|_{L^2}^2 - \mathbb{E}\|X_N\|_{L^2}^2\big| < \infty \quad \text{a.s.}$$

(4) From $\|X\|_{H^{\alpha-\frac{d}{2}}}^2 \gtrsim \sum_N N^{2\alpha-d} \|X_N\|_{L^2}^2$, deduce that

$$\|X\|_{H^{\alpha-\frac{d}{2}}}^2 = \infty \quad \text{a.s.}$$

Exercise 3 (Wick renormalisation)**.** Let $X \sim N(0, \sigma^2)$. Define recursively the *Wick powers* of x in the following way.

$$
\begin{aligned}
{:}x^0{:} &= 1,\\
{:}x{:} &= x,\\
{:}x^{n+1}{:} &= x \cdot {:}x^n{:} - \sigma^2 \partial_x({:}x^n{:}).
\end{aligned}
$$

i. Show that for every $n \ge 1$, $\partial_x({:}x^n{:}) = n{:}x^{n-1}{:}$.

ii. Proceeding inductively, show that

$$\mathbb{E}[:x^n::x^m:] = n!\delta_{nm}\sigma^{2n}.$$

It might be useful to recall that

$$(x + \sigma^2\partial_x)e^{-\frac{x^2}{2\sigma}} = 0.$$

iii. For $z \in \mathbb{R}$, seeing $:x^k:$ as a polynomial in x, show that

$$:(x+z)^k: = \sum_{h=0}^{k} \binom{n}{k} :x^{n-k}:z^k.$$

iv. Let (x, y) be jointly Gaussian with zero average, and let $:x^n:, :y^m:$ be their Wick powers. Show that

$$\mathbb{E}[:x^n::y^m:] = n!\delta_{nm}\mathbb{E}[xy]^n.$$

It might be useful to write $y = \lambda x + w$, with $\lambda \in \mathbb{R}$ and w independent from x.

Exercise 4 (Wick renormalisation of the Gaussian free field)**.** Consider the random series on $\mathbb{T}^2$,

$$X = \operatorname{Re} \frac{1}{2\pi} \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{in\cdot x}.$$

Our goal is to show that for any $k \in \mathbb{N}$, the Wick powers $:X^k:$ are well-defined distributions, and moreover

$$(1-\Delta)^{-\frac{s}{2}}:X^k: \in L^\infty(\mathbb{T}^2)$$

for every $s > 0$.

For a distribution u, define

$$u_N := \frac{1}{2\pi} \sum_{|n|\leq N} \widehat{u}(n)e^{in\cdot x}.$$

Recall that for any $f \in C^\infty(\mathbb{T}^2)$, and for every $0 < s < 2$, we have that

$$(1-\Delta)^{-\frac{s}{2}} f(x) = \int K_s(x-y)f(y)dy,$$

where K_s satisfies

$$|K_s|(x) \lesssim \frac{1}{|x|^{2-s}}.$$

i. Let $N \leq M$. show that

$$\mathbb{E}[X_N(x)X_M(y)] = K_N(x-y),$$

where

$$\widehat{K_N}(n) = \frac{\mathbf{1}_{\{|n|\leq N\}}}{\langle n\rangle^2}.$$

Deduce that for every $1 \leq p < \infty$,

$$\sup_N \|K_N\|_{L^p(\mathbb{T}^2)} < \infty.$$

ii. Exploiting point iv. of Exercise 3, show that for every $k \in \mathbb{N}$,

$$\sup_N \mathbb{E}|\big((1-\Delta)^{-\frac{s}{2}}:X_N^k:\big)(x)|^2 < \infty.$$

iii. Recall the (simplified) variational formula

$$\mathbb{E}[\exp(F(X))] \leq \mathbb{E}[\sup_{V \in H^1} F(X+V) - \frac{1}{2}\|V\|_{H^1}^2]$$

for any measurable functional F bounded below.

Show that for every $k \in \mathbb{N}$, there exists $\varepsilon_0 = \varepsilon_0(k,s) \ll 1$ such that

$$\sup_N \mathbb{E}\Big[\exp\Big(\varepsilon_0 |\big((1-\Delta)^{-\frac{s}{2}} {:}X_N^k{:}\big)(x)|^{\frac{2}{k}}\Big)\Big] < \infty.$$

Deduce that for every $1 \leq q < \infty$,

$$\sup_N \mathbb{E}\|(1-\Delta)^{-\frac{s}{2}} {:}X_N^k{:}\|_{L^p}^q < \infty.$$

iv. (optional) Show that for $N \leq M$,

$$\mathbb{E}\big[\big({:}X_M^k{:} - {:}X_N^k{:}\big)(x)\big({:}X_M^k{:} - {:}X_N^k{:}\big)(y)\big] = k!(K_M(x-y) - K_N(x,y)).$$

Arguing as in ii.-iii., deduce that for some $\varepsilon_0 = \varepsilon_0(k,s) > 0$ and some $\delta = \delta(k,s) > 0$, we have that

$$\sup_{N \leq M} \mathbb{E}\Big[\exp\Big(\varepsilon_0 N^\delta |\big((1-\Delta)^{-\frac{s}{2}}({:}X_M^k{:} - {:}X_N^k{:})\big)(x)|^{\frac{2}{k}}\Big)\Big] < \infty,$$

and hence show that for every $1 \leq p < \infty$, the sequence

$$(1-\Delta)^{-\frac{s}{2}} {:}X_N^k{:}$$

is a Cauchy sequence in $L^p(\mathbb{T}^2)$.

v. Denote by ${:}X^k{:}$ the a.s. limit of ${:}X_N^k{:}$ as $N \to \infty$. Show that for every $s > 0$, there exists $\varepsilon_0 = \varepsilon_0(k,s)$ such that

$$\mathbb{E}\Big[\exp\Big(\varepsilon_0 \|(1-\Delta)^{-\frac{s}{2}} {:}X^k{:}\|_{L^\infty(\mathbb{T}^2)}^{\frac{2}{k}}\Big)\Big] < \infty.$$

2. Exercise session 2

Exercise 5. Recall the Boué-Dupuis formula: for every real-valued functional F on distributions, Borel and bounded, we have

$$\log \int \exp(F(u)) d\mu(u) = \sup_{V \in \mathbb{H}_a^1} \mathbb{E}\Big[F(X(1) + V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|_{H^1}^2\Big],$$

where

$$X(t) = \operatorname{Re} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{W_n(t)}{\langle n \rangle} e^{in \cdot x},$$

and

$$\mathbb{H}_a^1 := \{V \in L^2(\mathbf{P}, H^1) : V(0) = 0, \|\dot{V}\| \in L^2([0,1], H^1) \text{ a.s.}, \\ V \text{ is adapted with respect to the filtration induced by } X\}.$$

Our goal is to show that the similar formula

$$\log \int \exp(F(u)) d\mu(u) = \sup_{V \in \mathbb{H}_{a,+}^1} \mathbb{E}\Big[F(X(1) + V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|_{H^1}^2\Big]$$

holds under more general hypotheses on F. Here $\mathbb{H}^1_{a,+}$ denotes

$$\mathbb{H}^1_{a,+} = \mathbb{H}^1_a \cap \{\mathbb{E}[F_-(X(1)+V(1))] < \infty\}.$$

We suppose the following on F.

(1) For every $V \in H^1$,

$$F_-(X(1)+V) \le G_1(X(1)) + G_2(V),$$

where

$$\mathbb{E}|G_1(X(1))| < \infty,$$

and $G_2 : H^1 \to \mathbb{R}$ is bounded on every ball of H^1.

(2)

$$\sup_{V\in\mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big] < \infty.$$

We start by considering the case of F bounded above by F^*, and satisfying (1). Notice that in this case $\mathbb{H}^1_{a,+} = \mathbb{H}^1_a$. Define $F_M = F \vee (-M)$.

Step 1. Notice that

$$\int \exp(F_M(u))d\mu(u) = \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F_M(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big]$$
$$\ge \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big].$$

Deduce that

$$\int \exp(F(u))d\mu(u) \ge \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big] > -\infty.$$

Step 2. Let $L > 0$. Using a stopping time argument, show that

$$\sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big]$$
$$\ge \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[\Big(F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big)\mathbf{1}_{\{\int_0^1 \|\dot V\|^2_{H^1} ds \le L\}}\Big]$$
$$\ge \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1))\mathbf{1}_{\{\int_0^1 \|\dot V\|^2_{H^1} ds \le L\}} - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big].$$

Step 3. Let V^ε_M be an almost optimiser for the following sup, in the sense that

$$\mathbb{E}\Big[F_M(X(1)+V^\varepsilon_M(1)) - \frac{1}{2}\int_0^1 \|\dot V^\varepsilon_M\|^2_{H^1}\Big] \ge \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F_M(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot V\|^2_{H^1}\Big] - \varepsilon.$$

Exploiting the fact that $F \le F^* < \infty$, show that

$$\mathbf{P}\Big(\int_0^1 \|\dot V^\varepsilon_M\|^2_{H^1} ds \ge L\Big) \lesssim L^{-1},$$

where the implicit constant is independent of M and of $\varepsilon \le 1$. Using (1), exploit this to show that for every $L > 0$,

$$\lim_{M\to\infty} \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F_M(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big]$$
$$\le \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1))\mathbf{1}_{\{\int_0^1 \|\dot{V}^\varepsilon\|^2_{H^1} ds \le L\}} - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big] + O(L^{-1}).$$

With steps 1,2, deduce that

$$\int \exp(F(u))d\mu(u) \le \sup_{V\in\mathbb{H}^1_a} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big].$$

We now move to the general case of F satisfying both (1) and (2). For $M > 0$, define $F^M := F \wedge M$. Note that F^M satisfies (1) and it is bounded above, so by the previous steps, the Boué-Dupuis fomula applies to F^M.

Step 4. Show that

$$\int \exp(F(u))d\mu(u) = \lim_{M\to\infty}\int \exp(F^M(u))d\mu(u)$$
$$\le \sup_{V\in\mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big].$$

Step 5. Let $V^\varepsilon \in \mathbb{H}^1_{a,+}$ be an almost optimiser for the sup, in the sense that

$$\mathbb{E}\Big[F(X(1)+V^\varepsilon(1)) - \frac{1}{2}\int_0^1 \|\dot{V}^\varepsilon\|^2_{H^1}\Big] \ge \sup_{V\in\mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big] - \varepsilon.$$

Show that

$$\mathbb{E}\Big[F(X(1)+V^\varepsilon(1)) - \frac{1}{2}\int_0^1 \|\dot{V}^\varepsilon\|^2_{H^1}\Big] \le \lim_{M\to\infty} \mathbb{E}\Big[F^M(X(1)+V^\varepsilon(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big].$$

Deduce that

$$\log\int \exp(F(u))d\mu(u) = \sup_{V\in\mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big].$$

Exercise 6. Let $d = 2, 3$. Given $M \gg 1$, define X_M by its Fourier coefficients as follows. For $|n| \le M$, $\widehat{X}_M(n,t)$ is a solution of the following differential equation:

$$\begin{cases} d\widehat{X}_M(n,t) = \frac{M}{\langle n\rangle}(\widehat{X}(n,t) - \widehat{X}_M(n,t))dt \\ \widehat{Z}_M|_{t=0} = 0, \end{cases}$$

and we set $\widehat{X}_M(n,t) \equiv 0$ for $|n| > M$. Show that $X_M(t)$ is a centered Gaussian process in $L^2(\mathbb{T}^d)$, which is frequency localized on $\{|n| \le M\}$, satisfying for every fixed function

$f \in H^{-1}(\mathbb{T}^d)$,

$$\begin{aligned}
&\mathbb{E}\big[X_M^2(x)\big] \sim \begin{cases} \log M & \text{if } d=2, \\ M & \text{if } d=3, \end{cases} \\
&\mathbb{E}\Big[2\int_{\mathbb{T}^d} X X_M dx - \int_{\mathbb{T}^d} X_M^2 dx\Big] = (2\pi)^d \mathbb{E}\big[X_M^2(x)\big](1+o(1)), \\
&\mathbb{E}\Big[\Big|\int_{\mathbb{T}^d} X_M f dx\Big|^2\Big] \lesssim \|f\|_{H^{-1}}^2, \\
&\mathbb{E}\Big[\Big|\int_{\mathbb{T}^d} :(X-X_M)^2: dx\Big|^2\Big] \lesssim \begin{cases} M^{-2}\log M & \text{if } d=2, \\ M^{-1} & \text{if } d=3, \end{cases} \\
&\mathbb{E}\Big[\int_0^1 \Big\|\frac{d}{ds}X_M(s)\Big\|_{H^1}^2 ds\Big] \lesssim M^d,
\end{aligned}$$

where $X = X(1), X_M = X_M(1)$, and

$$:(X-X_M)^2: = (X-X_M)^2 - \mathbb{E}\big[(X-X_M)^2\big].$$

Hint: Write and solve the SDE satisfied by $Y_n(s,n) := \widehat{X}(s,n) - \widehat{X_M}(s,n)$.

Exercise 7. Let $d = 2$, $p > 4$. Show that there exists $\gamma > 0$ such that for every $\sigma > 0$,

$$\int \exp\Big(\frac{\sigma}{4}\int_{\mathbb{T}^2} :u^4: dx - \|u\|^{\gamma}_{W^{-\frac{1}{2},p}(\mathbb{T}^2)}\Big) d\mu(u) < \infty.$$

Here

$$\|u\|_{W^{s,p}} := \|(1-\Delta)^{\frac{s}{2}} u\|_{L^p}.$$

The simplified variational formula is enough to show the above. It might be convenient to use the following interpolation inequalities.

- (Gagliardo-Nirenberg-Sobolev). For every $0 \le \theta \le 1$, $s_j \in \mathbb{R}$, $1 < p_j < \infty$, $\frac{1}{r} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, we have that

$$\|u\|_{W^{\theta s_1 + (1-\theta)s_2, r}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}.$$

- (Sobolev embeddings). Suppose that $1 < q < 4$, $s > 0$, and

$$\frac{s}{d} \ge \frac{1}{q} - \frac{1}{4}.$$

  Then

$$\|u\|_{L^4(\mathbb{T}^d)} \lesssim \|u\|_{W^{s,q}(\mathbb{T}^d)}.$$

3. Exercise session 3

Let F be a real-valued functional on distributions that satisfies (1) and (2) in Exercise 5. Define the probability measure

$$d\rho_F = \frac{\exp(-F(u))d\mu(u)}{\int F(u)d\mu(u)}$$

Let $V^n \in \mathbb{H}^1_{a,+}$ be a sequence of almost optimisers for the variational formula associated to $\exp(F(u))$, or more precisely,

$$\sup_{V \in \mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big] = \lim_{n\to\infty} \mathbb{E}\Big[F(X(1)+V^n(1)) - \frac{1}{2}\int_0^1 \|\dot{V}^n\|^2_{H^1}\Big].$$

Let now g be a real-valued, bounded, Borel functional on distributions. We want to show that

$$\int g(u) d\rho_F(u) = \lim_{n\to\infty} \mathbb{E}\Big[g(X(1)+V^n(1))\Big].$$

Step 1. Show that

$$\int g(u) d\rho_F(u) = \lim_{\lambda\to 0} \frac{1}{\lambda} \log\Big(\int \exp(\lambda g(u)) d\rho_F(u)\Big).$$

Step 2. Show that for every $\lambda \in \mathbb{R}$, the functional $\lambda g + F$ satisfies (1) and (2) in Exercise 5, and deduce that

$$\begin{aligned} &\log\Big(\int \exp(\lambda g(u)) d\rho_F(u)\Big) \\ &= \sup_{V \in \mathbb{H}^1_{a,+}} \mathbb{E}\Big[\lambda g(X(1)+V(1)) + F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big] \\ &\quad - \sup_{V \in \mathbb{H}^1_{a,+}} \mathbb{E}\Big[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}\Big]. \end{aligned}$$

Step 3. Deduce that, for every $\lambda \in \mathbb{R}$,

$$\log\Big(\int \exp(\lambda g(u)) d\rho_F(u)\Big) \le \liminf_{n\to\infty} \lambda \mathbb{E}[g(X(1)+V^n(1))].$$

Step 4. By considering separately the cases $\lambda \uparrow 0$ and $\lambda \downarrow 0$, show that

$$\limsup_{n\to\infty} \mathbb{E}[g(X(1)+V^n(1))] \le \log\Big(\int \exp(\lambda g(u)) d\rho_F(u)\Big) \le \liminf_{n\to\infty} \mathbb{E}[g(X(1)+V^n(1))].$$

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