The Hall algebra of curves and quivers: cuspidal functions, perverse sheaves and Kac polynomials

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The goal of this PhD thesis is the algebro-geometric study of Lie algebras arising from geometric constructions involving the categories of coherent sheaves on curves and their linearised versions, the categories of representations of quivers. These Lie algebras encode through their character enumerative invariants of the categories under consideration and through their algebraic structure their symmetries. The entailed categories enjoy favourable homological properties: both are of homological dimension one, resulting in the smoothness of the geometric spaces parametrizing their objects. The category of coherent sheaves on a smooth projective curve is a central object in the Langlands program, which leans on long-standing conjectures aiming at bridging the mathematical fields of harmonic analysis, number theory and algebraic geometry. On the other side, representations of quivers can be seen as a linearised version of coherent sheaves on curves. Their algebraic study was initiated by Pierre Gabriel in the 1960's. They are nowadays ubiquitous objects in mathematics, arising in representation theory of finite dimensional algebras, in enumerative geometry on 3-Calabi–Yau manifolds, in the description of physical systems (quiver gauge theories), and more generally to understand the structure of algebraic and geometric objects (cluster algebras describing the algebra structure of functions on unipotent groups, preprojective algebras of quivers giving local description of the moduli stack of objects in 2-Calabi–Yau categories, quiver with potential describing objects in 3-Calabi–Yau categories).

This thesis consists of four parts. The first part studies the cuspidal functions of the Hall algebra of a quiver. The cuspidal functions provide a canonical generating subspace of the Hall algebra and they satisfy quantum Serre relations, giving the Hall algebra the structure of a Borcherds–Kac–Moody quantum group. It is published in *International Mathematics Research Notices*[1]. The second part is a work involving computations with *SageMath* aiming at understanding the behaviour of Kac polynomials of a quiver when the quiver varies, in relation with the compatibilities of the Hall algebra with morphisms of quivers. An astonishing fact emerges: in a fixed dimension vector, the distribution of the coefficients of Kac polynomials converges to an Airy distribution

[1]**Isotropic cuspidal functions in the Hall algebra of a quiver**, *Int. Math. Res. Not. IMRN*, (15), 11514 – 11564, 51 pages, 2019 (online version), preprint : arXiv:1903.04378

when taking a limit as the number of arrows goes to infinity. This work appeared in *Experimental Mathematics*[2]. The third part involves the geometric analogue of the notion of cuspidality, given by microlocal conditions for constructible complexes on the stack of representations of the quiver, which makes a connection between representation theory and symplectic algebraic geometry. It contains a proof of a conjecture of Lusztig, giving a characterization of quiver character sheaves, and is published in *Representation Theory*[3]. The last work, accepted by *Transformation Groups*[4], is a glimpse into the link between the elliptic Hall algebra and the cohomological Hall algebra of an elliptic curve, and studies an analogue of Lusztig's conjecture for curves.

The Hall algebra of a quiver has been introduced by Ringel in the 1990's. It is associated to a quiver and a finite field. It admits a basis given by the set of isoclasses of representations of the quiver over the finite field and is endowed with the structure of a (twisted) Hopf algebra: multiplication, comultiplication, antipode, unit, counit. It can be seen as the algebra of symmetries of the extension structure of the category of quiver representations: the structure constants of the operations are built by using the enumeration of extensions between objects. For some particular quivers, we retrieve well-studied algebras: the punctual quiver with one vertex and no arrows gives the polynomial algebra in one variable; the Jordan quiver with one vertex one loop produces the algebra of symmetric functions of Macdonald. The Hall algebra of a general quiver contains as a subalgebra the positive part of Drinfeld's quantum group, as shown by Ringel. In general, the Hall algebra is much bigger than the quantum group, and the difference is measured by the space of *cuspidal functions*, which gives a minimal generating subspace of the Hall algebra. The dimensions of the graded components of this space are given by polynomials with integer coefficients, a theorem of Bozec and Schiffmann of 2017. Our goal was to provide a parametrisation of a natural basis of this space. Using Auslander-Reiten theory of finite dimensional algebras, we obtained such a parametrisation for affine quivers (the quivers giving affine Lie algebras and appearing in McKay correspondence), generalizing the known parametrisation for finite type quivers (which correspond to simple simply-laced Lie algebras), leaving open for now the case of wild quivers. We give a conjecture concerning possible cohomological interpretations of the coefficients of these polynomials, having as a corollary the positivity of these coefficients. These conjecture is verified for a class of dimension vectors called *isotropic*.

The second part concerns the study of Kac polynomials of a given quiver. These are the polynomials counting absolutely indecomposable representations over a finite field of a quiver in a given dimension vector. By a theorem of Hausel–Letellier–Rodriguez-Villegas of 2013, their coefficients are non-negative, and have a cohomological interpretation related to Nakajima quiver varieties. As consequence of a work of Ben Davison of 2013, they can also be interpreted as *BPS invariants* of a certain quiver with potential, the tripled quiver. These polynomials give the character of the BPS Lie algebra of the quiver and also, after taking the plethystic exponential (a combinatorial operation relating the character of a vector space with the character of the symmetric algebra over this vector space), the character of the Hall algebra of the quiver. The construction of the Hall algebra is functorial with respect to inclusions of quivers, and the existence of a limiting object is expected but not obvious to define due to the divergence of the dimensions of graded components. This motivates the study of the evolution of Kac polynomials when the dimension

[2] **Asymptotic behaviour of Kac polynomials**, *Experimental Mathematics*, 19 pages, 2021 (online version), preprint: arXiv:2003.06929

[3] **Microlocal characterization of Lusztig sheaves for affine and g-loops quivers**, *Representation Theory* 26, pp. 17 – 67, 51 pages, 2022, (online version), preprint: arXiv:2006.12780

[4] **Perverse sheaves with nilpotent singular support on the stack of coherent sheaves on an elliptic curve**, 31 pages, 2021, preprint: arXiv:2101.03813, submitted to *Transformation Groups*, accepted for publication

vector is fixed but the number of arrows tends to infinity. An amazing property appears: the distribution of the coefficients converges to an Airy distribution. Such phenomena appear also in other contexts as e.g. the cohomology of semiprojective hyperkähler varieties (including the moduli space of Higgs bundles), but a conceptual explanation is still missing.

The quantum group associated to a quiver defined by Drinfeld and its canonical basis defined by Kashiwara have been categorified by Lusztig by geometrising the construction of the Hall algebra of Ringel, providing at the same time a geometric notion of cuspidality. Lusztig considered perverse sheaves on the algebraic stack parametrising representations of the quivers, and defined a category of *character sheaves* for quivers with induction and restriction operations. The Grothendieck group of the category of character sheaves is isomorphic to the positive part of the quantum group and the simple character sheaves provide its canonical basis. There is an analogous construction in terms of a Lagrangian substack of the cotangent stack of the stack of representations of the quiver. This is the *nilpotent stack*. It can be described as the stack of nilpotent representations of the preprojective algebra of the quiver and this symplectic construction provides the *semicanonical basis* of the unipotent enveloping algebra. The link between the two constructions is made using the characteristic cycle map of Kashiwara–Schapira. Lusztig conjectured (and proved for finite type quivers) that character sheaves for quivers are characterized by the nilpotency of their singular support, a tool of *microlocal geometry* introduced by Kashiwara–Schapira. We extended this characterization to affine quivers and to totally negative quivers, which are the quivers with a totally negative Euler form. The strategy is to understand the geometry of the stack of representations of affine or totally negative quivers and their cotangent stack by relying on Auslander–Reiten theory and explicit techniques. We also give generalisations of Lusztig's conjecture for arbitrary quivers, containing possible loops. A fruitful approach to these conjectures from the viewpoint of quantisation of Nakajima quiver varieties is expected.

The last part is motivated by understanding the global situation of curves and by the study of the *Hall algebra* of a curve (inside which the space of cuspidal forms for GL_n sits) together with the *cohomological Hall algebra* of Higgs bundles over the same curve (which contains the cohomology of the space of semistable Higgs bundles). We prove that the characteristic cycle map from the Grothendieck group of the category of Eisenstein spherical sheaves of an elliptic curve to the abelian group of Lagrangian cycles of the global nilpotent cone is an isomorphism. This answers a question of Bezrukavnikov and Losev. This is a result analogous to the local situation of quivers investigated in third part. We rely on Atiyah's classification of vector bundles on a smooth projective curve. We expect this result to hold for arbitrary smooth projective curves, and approaches involving cohomological Hall algebras seem promising.

In this thesis, we made progress towards a fine understanding of the constructible Hall algebra of a quiver by describing cuspidal functions for affine quivers and its relation to the cohomological Hall algebra of the preprojective algebra, via the categorification of the quantum group provided by Lusztig perverse sheaves and microlocal considerations. We obtained a proof of a conjecture of Lusztig and opened the path of the study of wild quivers by formulating some precise conjectures. We highlighted surprising combinatorial properties of Kac polynomials, related to stable cohomology of semi-projective hyperkähler varieties, giving more weight to the putative existence of a limiting Hall algebra when letting the set of arrows of the quiver vary. Last, we answered a question of Bezrukavnikov–Losev on the characteristic cycle map for the category of spherical Eisenstein sheaves on an elliptic curve, which can be seen as an analogue of Lusztig's conjecture for curves.