Perverse sheaves & hyperbolic localization

X algebraic variety $/\mathbb{C}$

R coefficient ring $\quad R = \mathbb{Q}, \mathbb{C},$

* $D^b_c(X, R) = D^b(Sh_c(X, R))$

$X = \bigsqcup_{\alpha \in A} X_\alpha$ finite stratification

locally closed, algebraic subvarieties

$\mathcal{F} \in Sh_c(X, R)$. $\mathcal{F}|_{X_\alpha}$ is locally constant)

$\mathcal{F}$ sheaf of R-vector spaces on X . the fiber is finite dimensional.

,

* $\mathcal{D}$ = full subcat $\subset D^b(X, R)$ of complexes w/ constructible cohomology.

* If $\mathcal{S}$ is a strat of X, $D^b_{\mathcal{S}}(X, R)$.

$\cdot \ \mathcal{N} = \bigsqcup_{G \text{ nilp orbit}} G$;

6-functors formalism

$f : X \longrightarrow Y$

(f^*, f_*) $\quad f_* = R f_*$

$(f_!, f^!)$

$\left(- \overset{L}{\otimes} -, R\mathcal{H}om(-,-)\right)$

- $f^* \rightsquigarrow$
- f_*
- $f_! = Lf_!$
- $f^!$ is not defined at the level of $Sh_c(-, R)$.
right adjoint to $f_!$

X, Y

this is if f is a locally closed immersion.

$- \overset{L}{\otimes} -$

$R\mathcal{H}om(-,-)$

Verdier duality

$\mathbb{D} : D^b_c(X, R)^{op} \longrightarrow D^b_c(X, R)$

$\mathbb{D}^2 \cong \mathrm{Id}$

$$\cdot \quad X \xrightarrow{p} pt \;; \qquad \mathbb{D} = \underset{D^b_c(X,R)}{R\mathcal{H}om}\Big(- \, , \, \underset{\overset{\|}{p^! R_X}}{\omega_X}\Big).$$

(duality, $\mathbb{D}^2 = \mathrm{Id}$, $\mathbb{D} p_* \mathbb{D} = p_!$)

<u>Remark:</u> the existence of $\mathbb{D}$ is equivalent to $\forall p : X \to Y$

the existence of $f^!$ $(\forall f)$.

right adjoint $f_!$

$$f^! = \mathbb{D}_X f^* \mathbb{D}_Y \qquad f : X \to Y$$

$R = \mathbb{Q}$

- If X is smooth of dim n, $\mathbb{D}\mathbb{Q}_X = \mathbb{Q}_X[2n]$.
- $\forall f : X \to Y$, $\quad f_! \to f_*$; isomorphism if f is proper.

  and $j^! = j^*$ for open immersion j.
- If $X = U \sqcup Z$ $\quad U$ open, $Z = X \setminus U$,

$U \xrightarrow{j} X$

$Z \xrightarrow{i} X$,

$i_! i^! \longrightarrow id \longrightarrow j_* j^* \longrightarrow$

$j_! j^! \longrightarrow id \longrightarrow i_* i^* \longrightarrow$.

* Proper base change:

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "g"] \arrow[dl, bend right=80] \\
Y' \arrow[r, "f"'] & Y \arrow[l, bend left=80]
\end{tikzcd}$$

- g proper

$g_! = g_*$.

$$f^* \circ g_! \simeq g'_! \circ (f')^* .$$

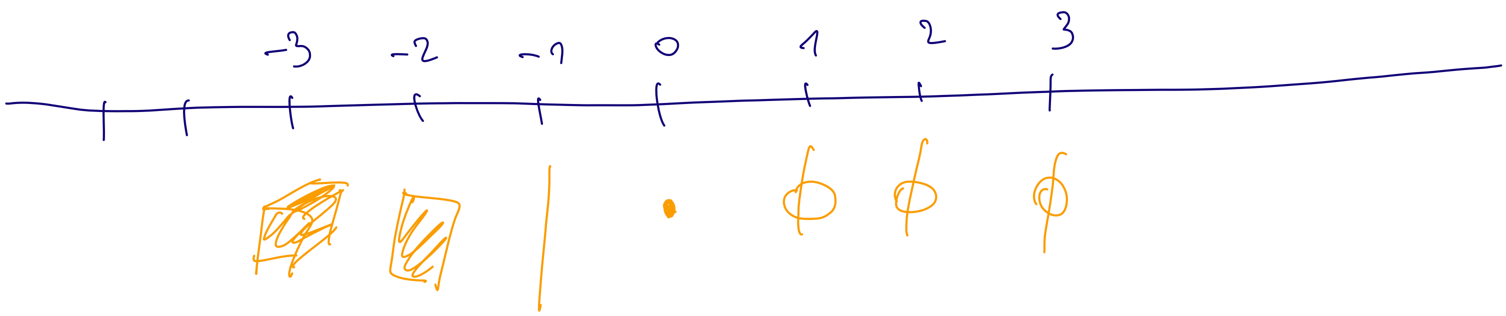
Perverse sheaves

abelian subcategory of $D^b_c(X,R)$.

- $\mathrm{Perv}(X,R)$.
- $\mathcal{F}^\bullet \in D^b_c(X,R)$, $\quad \mathcal{H}^i(\mathcal{F}^\bullet) \in Sh_c(X,R)$.

t-structure

- $D_c^{b,\le 0}(X,R) = \{\mathcal{F}^\bullet \in D^b_c(X,R),\ \forall i,\ \dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}^\bullet) \le -i\}$



- $D_c^{b,\ge 0}(X,R) = \{\mathcal{F}^\bullet \mid \mathbb{D}\mathcal{F}^\bullet \in D^{b,\le 0}(X,R)\}$.

- $\mathrm{Perv}(X,R) = D_c^{b,\le 0}(X,R) \cap D_c^{b,\ge 0}(X,R)$

$\circlearrowleft \mathbb{D}$

[middle perversity perverse sheaves]

- exact sequences of $Perv(X, R)$ are

$$0 \to \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \to \mathcal{H}^\bullet \longrightarrow 0$$

where $\mathcal{F}^\bullet, \mathcal{G}^\bullet, \mathcal{H}^\bullet \in Perv(X, R)$ and

$\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \longrightarrow$ is a distinguished triangle of $D^b_c(X, R)$.

- $Perv(X, R)$ is noetherian artinian.
- simple objects: $Y \overset{j}{\hookrightarrow} X$ smooth loc. closed irreducible, $\mathcal{L}$ simple loc system on Y,

$$IC(\overline{Y}, \mathcal{L}) = j_{!*} \mathcal{L}[\dim Y]$$

simple perverse sheaves,

$$IC(\overline{Y}, \mathcal{L})\big|_Y \simeq \mathcal{L}$$

Example: $\{0\} \xhookrightarrow{i} \mathbb{C} \xhookleftarrow{j} \mathbb{C}^*$

$$j_! j^! \to id \to i_* i^* \to \qquad \mathbb{Q}_{\mathbb{C}}[1]$$

$$\mathbb{Q}_{\{0\}} \to j_! \mathbb{Q}_{\mathbb{C}^*}[1] \to \mathbb{Q}_{\mathbb{C}}[1] \to \mathbb{Q}_{\{0\}}[1] \to$$

simple p.s.   perverse sheaf on $\mathbb{C}$.   simple p.s

$$0 \to \mathbb{Q}_{\mathbb{C}}[1] \to \boxed{j_* \mathbb{Q}_{\mathbb{C}^*}[1]} \to \mathbb{Q}_{\{0\}} \to 0$$

complex on $\mathbb{C}$ is a p.s.

Perverse cohomology

$D_c^{b,\le 0}(X) \hookrightarrow D_c^b(X)$ has a right adjoint $\tau^{\le 0}$.

$D_c^{b,\ge 0}(X) \hookrightarrow D_c^b(X)$ has a left adjoint $\tau^{\ge 0}$.

$${}^{p}\mathcal{H}^0 : D^b_c(X, R) \longrightarrow \mathrm{Perv}(X, R)$$
$$\mathcal{F}^\bullet \longmapsto \tau^{\leq 0}\tau^{\geq 0}\mathcal{F}^\bullet$$

$${}^{p}\mathcal{H}^i(\mathcal{F}^\bullet) = {}^{p}\mathcal{H}^0(\mathcal{F}^\bullet[i]).$$

Decomposition theorem

$\boxed{R = \mathbb{C}}$

- Semisimple complex.

$\mathcal{F}^\bullet \in D^b_c(X, R)$ s.t.

$$\mathcal{F}^\bullet = \bigoplus_{i \in \mathbb{Z}} \boxed{{}^{p}\mathcal{H}^i(\mathcal{F}^\bullet)}[-i]$$

and each ${}^{p}\mathcal{H}^i(\mathcal{F}^\bullet)$ is a semisimple p. sheaf.

Thm (BBDG) If $f : X \to Y$ proper morphism

$\mathcal{F}$ is a semisimple complex on X.

Then $f_*\mathcal{F}$ is a semisimple complex on Y.

Smallness :

$f : X \to Y$ proper, surjective.

Semismallness is the condition on the dimension of the fibers of f ensuring that $f_* \underline{\mathbb{Q}}_X[\dim X]$ is a perverse sheaf on Y.

f semismall if $\forall k \geq 0$,

$$\underbrace{\{ y \in Y \mid \dim f^{-1}(y) \geq k \} + 2k}\ \overset{(*)}{\leq} \dim X.$$

f small if $\forall k \geq 1$, $(*)$ is strict.

$\Rightarrow$ the simples p.s. appearing in $f_* \underline{\mathbb{Q}}_X[\dim X]$ are supported on the whole of Y.

- Kaledin : symplectic resolutions are semismall.

Hyperbolic localization

- Braden 2003
- some natural morphism of functors is an isomorphism.

- X $\mathbb{C}$-variety, normal.
  $\circlearrowleft$
  $\mathbb{G}_m$.

$$X^{\mathbb{G}_m} := \{ x \in X \mid t \cdot x = x \quad \forall t \in \mathbb{G}_m \}$$

$$X^+ = \bigsqcup_{F \in \pi_0(X^{\mathbb{G}_m})} X_F^+ \quad F$$

$$X^- = \bigsqcup_{F \in \pi_0(X^{\mathbb{G}_m})} X_F^-$$

$\mathbb{G}_m \hookrightarrow \mathbb{P}^1$
"$\mathbb{P}^1 \setminus \{0, \infty\}$".

$$X_F^{\pm} = \{ x \in X \mid \lim_{t \to 0,\ \infty} t \cdot x \in F \}$$

ex. $X = \mathbb{P}^1 \circlearrowleft \mathbb{G}_m$, $X^{\mathbb{G}_m} = \{0, \infty\}$

$$\mathbb{P}^1 \neq X^+ = (\mathbb{P}^1 \setminus \{\infty\}) \sqcup \{\infty\}$$

$\sqcup \bullet$

$\mathbb{C}$

$$X^- = \{0\} \sqcup (\mathbb{P}^1 \setminus \{0\})$$

$$\mathbb{P}^1 = X^+ \longrightarrow \{0, \infty\}$$

$$y \longmapsto \begin{cases} 0 & \text{if } y \neq \infty \\ \infty & \text{if } y = \infty \end{cases}$$

set theoretically.

$$x \longmapsto \lim_{t \to 0} t \cdot x$$

$$\begin{tikzcd}
 & X^+ \arrow[dl, "g^+"'] \arrow[dr, shift left, "\pi^+"] & \\
X & & X^{\mathbb{G}_m} \arrow[ul, shift left, "f^+"] \arrow[dl, shift left, "f^-"] \\
 & X^- \arrow[ul, "g^-"] \arrow[ur, shift left, "\pi^-"] &
\end{tikzcd}$$

hyperbolic restriction

$$(f^+)^! \circ (g^+)^* : D^b_c(X) \longrightarrow D^b_c(X^{\mathbb{G}_m})$$

$(\pi_+)_!$

$$(f^-)^* \circ (g^-)^!$$

$(\pi_-)_*$

We have a natural morphism of functors

becomes an isomorphism when evaluated on $\mathcal{F}^\bullet$ weakly equivariant complex.

$\mathcal{F}^\bullet \in D^b_c(X)$ is weakly equivariant if

$$\mu : \mathbb{G}_m \times X \longrightarrow X$$

$$\mu^* \mathcal{F}^\bullet \simeq \mathcal{L} \boxtimes \mathcal{F}^\bullet \quad , \quad \mathcal{L} \text{ locally constant on } \mathbb{G}_m .$$

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example.

$$\begin{tikzcd}
& (\mathbb{P}^1 \setminus \{0\}) \sqcup \{0\} \arrow[dl, "g^+"'] \arrow[dr, bend left, "\pi^+"] & \\
\mathbb{P}^1 & & \{0, \infty\} \arrow[ul, "f^+"] \arrow[dl, "f^-"] \\
& (\mathbb{P}^1 \setminus \{\infty\}) \sqcup \{\infty\} \arrow[ul, "g^-"] \arrow[ur, bend right, "\pi^-"'] &
\end{tikzcd}$$

$\mathbb{P}^1$ $\mathbb{P}^1$

$\mathcal{F} = \mathbb{Q}_{\mathbb{P}^1}[1]$ (perverse sheaf)

$$(f^+)^! (g^+)^* \mathcal{F} = (f^+)^! \left(\mathbb{Q}_{\mathbb{P}^1 \setminus \{0\}}[1] \oplus \mathbb{Q}_{\{0\}}[1] \right)$$

$$= \mathbb{Q}_{\{\infty\}}[-1] \oplus \mathbb{Q}_{\{0\}}[1]$$

$$(f^{-})^{*}(g^{-})^{!} = \ldots \text{ "}$$

$$(\{0,\infty\} \longrightarrow \mathbb{P}^1)^{*} \underline{\mathbb{Q}}_{\mathbb{P}^1}[1] = (\{0,\infty\} \longrightarrow \mathbb{P}^1)^{!} \mathbb{Q}_{\mathbb{P}^1}[1],$$

- purity is preserved by HL. $\rightsquigarrow$ ℓ-adic sheaves $\mathbb{F}_q$ $\to$ Fr.
  $\rightsquigarrow$ MHM

$\mathbb{G}_m$ fibers

$$\pi : Y \underset{s}{\overset{\longrightarrow}{\longleftarrow}} X \quad \text{v-bundle.}$$

zero section

$\pi_! \mathcal{F} \cong s^! \mathcal{F}$ if $\mathcal{F}$ is weakly $\mathbb{G}_m$-equivariant.

- If $\mathcal{F}$ s.s on X then $HR(\mathcal{F})$ is s.s on $X^{\mathbb{G}_m}$
  weakly $\mathbb{G}_m$-equivariant.