## LOCAL COMPLETE INTERSECTION MORPHISMS

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ABSTRACT. We discuss local complete intersection morphisms, with some examples.

# 1. LOCAL COMPLETE INTERSECTION MORPHISMS

Let  $i: Z \to X$  be a closed immersion. Then, i is *regular* if locally at the target, it can be written Spec $(A/I) \to$  Spec A and the ideal I is generated by a regular sequence, That is  $I = (x_1, \ldots, x_r)$  and for any  $1 \le i \le r$ ,  $x_i$  is not a zero divisor in  $A/(x_1, \ldots, x_{i-1})$ .

Let  $f: X \to Y$  be a morphism between algebraic varieties. Then, f is called *local complete intersection* (l.c.i.) if f can be written as the composition  $f = g \circ h$  of a regular closed immersion g and a smooth morphism h.

The basic examples are morphisms between smooth varieties  $f: X \to Y$ . Indeed, we can write  $f = g \circ h$ , where  $g: X \to X \times Y$  is the graph of f and  $h: X \times Y \to Y$  is the projection.

The *codimension* of an l.c.i. map  $f: X \to Y$  is by definition dim  $Y - \dim X$ . For example, if f is smooth, it is l.c.i. and its codimension is the opposite of the relative dimension.

## 2. Borel-Moore Homology

If  $f: X \to Y$  is a local complete intersection morphism of codimension d, then there is a morphism of sheaves

$$\mathbb{D}\mathbf{Q}_Y \to f_*(\mathbb{D}\mathbf{Q}_X)[2d]$$

which gives the virtual pullback in Borel–Moore homology

$$f^! \colon \operatorname{H}^{\operatorname{BM}}_*(Y) \to \operatorname{H}^{\operatorname{BM}}_{*-2d}(X)$$

by taking derived global sections. We recall that  $\mathrm{H}_{i}^{\mathrm{BM}}(X) = \mathrm{H}^{-i}(\mathbb{D}\mathbf{Q}_{X}).$ 

We do not give the details of the construction of this map. The theory for stacks is presented in [Ols15] but for schemes, it existed earlier.

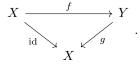
Here are some elements. Since for f smooth,  $f^! \cong f^*[2d]$  where d is the relative dimension of f, we have a morphism  $\mathbb{D} \mathbf{Q}_Y \to f_* \mathbb{D} \mathbf{Q}_X[-2d]$  coming from the isomorphism  $f^* \mathbb{D} \mathbf{Q}_Y \cong \mathbb{D} \mathbf{Q}_X[-2d]$ . Therefore, in virtue of the factorization of l.c.i. morphism as a regular closed immersion followed by a smooth map, it suffices to construct the virtual pullback for regular closed immersions. It is then a theorem that the map obtained is independent of the factorization.

#### 3. Examples

## 3.1. Section of a smooth map.

**Proposition 3.1.** Let  $g: Y \to X$  be a smooth map and  $f: X \to Y$  a section of g, that is  $g \circ f = id_X$ . Then f is a regular immersion.

*Proof.* After writing the more complicated proof below, I understood that one can just simply apply Lemma 37.60.10 of the stacks project to the diagram

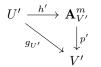


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*Proof.* Let  $x \in X$  and y = f(x). There exists a neighbourhood  $U \subset Y$  of Y and a neighbourhood  $V \subset X$  of x = g(y) such that  $g(U) \subset V$  and the restriction  $g_U$  factors as



where h is étale and p is the projection. We replace V by  $V' = f^{-1}(U) \cap V$  and U by  $U' = g^{-1}(V') \cap U$ . Then,  $g(U') \subset V'$  and  $f(V') \subset U$  and, since gf(V') = V',  $f(V') \subset g^{-1}(V')$ , proving that  $f(V') \subset U'$ . We therefore obtain a diagram



with p' the projection, U' étale, and  $f_{V'}: V' \to U'$  gives a section of  $g_{U'}$ . It follows that  $h' \circ f_{V'}$  gives a section of p'. The map  $h' \circ f_{V'}$  is clearly l.c.i., as it is even complete intersection. Now, we have  $h' \circ f_{V'}$  complete intersection and h' is smooth (since étale) and so, by Lemma 37.60.10 of the stacks project,  $f_{V'}$  is l.c.i.

3.2. Section of a smooth map and virtual pullback. We saw in §3.1 that a section  $f: X \to Y$  of a smooth map (of relative dimension d) is l.c.i. This is actually a very favourable situation among all l.c.i. situations, as the morphism of sheaves

$$\mathbb{D}\mathbf{Q}_Y \to f_*(\mathbb{D}\mathbf{Q}_X)[2d]$$

comes by adjunction from the isomorphism  $f^* \mathbb{D} \mathbf{Q}_Y \to (\mathbb{D} \mathbf{Q}_X)[2d]$ .

*Proof.* We prove that when  $f: X \to Y$  is a section of a smooth map  $g: Y \to X$ , then  $f^* \mathbb{D} \mathbf{Q}_Y \cong (\mathbb{D} \mathbf{Q}_X)[2d]$ . Indeed, by smoothness of g, we have  $g! = g^*[2d]$ . We therefore have

$$\mathbf{Q}_X \cong (\mathrm{id}_X)^! \, \mathbf{Q}_X \cong f^! g^! \, \mathbf{Q}_X \cong f^! \, \mathbf{Q}_Y[2d].$$

By taking Verdier duality, we obtain an isomorphism  $\mathbb{D}\mathbf{Q}_X \cong f^*(\mathbb{D}\mathbf{Q}_Y)[-2d]$ .

As shown in 3.3, this situation is not general. This is nevertheless the kind of situations appearing in geometric representation theory, when constructing cohomological Hall algebra products.

3.3. The nodal singularity. We let  $Y = \mathbf{C}^2$  and  $X = \{xy = 0\} \subset \mathbf{C}^2$ . Then,  $f: X \to Y$  is l.c.i. as X is complete intersection in Y. It is of codimension 1. But we do not have  $f^* \mathbb{D} \mathbf{Q}_Y \cong (\mathbb{D} \mathbf{Q}_X)[2]$ . Indeed,

$$f^* \mathbb{D} \mathbf{Q}_{\mathbf{A}^2} \cong f^* \mathbf{Q}_{\mathbf{A}^2}[4] \cong \mathbf{Q}_X[4]$$

while by using the description of the dualizing sheaf of the nodal singularity given in [Hen22], we can conclude (by looking at the fiber over 0 for example) that it is not isomorphic to  $(\mathbb{D} \mathbf{Q}_Y)[2]$ .

Nevertheless, the morphism  $f^* \mathbb{D} \mathbf{Q}_Y = \mathbf{Q}_X[4] \to (\mathbb{D} \mathbf{Q}_X)[2]$  is given by the composition

$$\mathbf{Q}_X[4] \to \mathbf{Q}_A[4] \oplus \mathbf{Q}_B[4] \to (\mathbb{D} \, \mathbf{Q}_X)[2]$$

of the second morphism in the triangle [Hen22, (0.1)] shifted by 3 with the first morphism of the first triangle in the proof of [Hen22, Proposition 0.2] shifted by 3.

Indeed, by shifting appropriately, we have to describe a morphism of perverse sheaves

$$\mathbf{Q}_X[1] \to \mathbb{D}(\mathbf{Q}_X[1]).$$

In [Hen22], we described  $\mathbf{Q}_X[1]$  as an indecomposable perverse sheaf, with  $\mathbf{Q}_A[1] \oplus \mathbf{Q}_B[1]$  as maximal semisimple quotient, while  $\mathbb{D}(\mathbf{Q}[1])$  is an indecomposable perverse sheaf with  $\mathbf{Q}_A[1] \oplus \mathbf{Q}_B[1]$  as maximal semisimple subobject. The statement can then be verified on the complement of the origin of  $\mathbf{A}^2$  and is then easy.

## REFERENCES

# References

- [Hen22] Lucien Hennecart. "Dualizing sheaf of a nodal singularity". In: (2022).
- [Ols15] Martin Olsson. "Borel–Moore homology, Riemann–Roch transformations, and local terms". In: Advances in Mathematics 273 (2015), pp. 56–123.

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