

Hodge Club - 29 October 2021

Categorification of Hall algebras

I - General formalism

II - 1D CoHA of a quiver

III - The constructible Hall algebra of a finitary category

IV - Perverse sheaves categorification of quantum groups.

I - General formalism

\mathcal{A} exact or abelian category

$\tilde{\mathcal{A}}$ groupoid of objects in \mathcal{A} .

→ category with the same objects as \mathcal{A} but we keep only isomorphisms among morphisms.

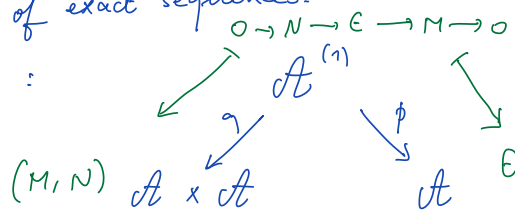
$\mathcal{A}^{(1)}$ category of exact sequences in \mathcal{A}

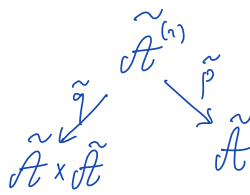
objects : $0 \rightarrow N \xrightarrow{a} E \xrightarrow{b} M \rightarrow 0$ (pairs of morphisms $\begin{matrix} N \xrightarrow{a} E \\ E \xrightarrow{b} M \end{matrix}$)

morphisms $\begin{matrix} 0 \rightarrow N \xrightarrow{a} E \xrightarrow{b} M \rightarrow 0 \\ \quad \quad \quad \downarrow f \quad \downarrow g \quad \downarrow h \\ 0 \rightarrow N' \xrightarrow{a'} E' \xrightarrow{b'} M' \rightarrow 0 \end{matrix}$

$\tilde{\mathcal{A}}^{(1)}$ groupoid of exact sequences.

Natural correspondence :



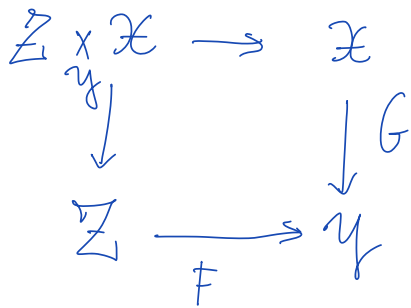


The fiber of \tilde{q} over (M, N) is

$$\text{Ext}^1(M, N) / \text{Hom}(M, N)$$

fiber product of groupoids

(tells you also how to define the fiber product of stacks)



$\mathbb{Z} \times \mathcal{X}$ is the groupoid with objects triple

$$(z, x, f)$$

$z \in \text{object of } \mathbb{Z}$

$x \in \text{ob}(\mathcal{X})$

$f \in \text{Isom}(F(z), G(x))$

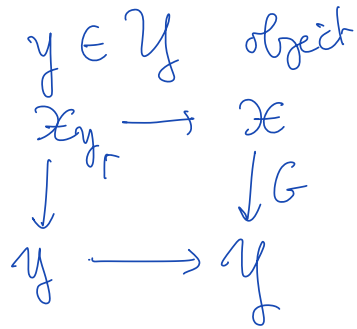
and a morphism of triples $(z, x, f) \xrightarrow{g} (z', x', f')$

is a pair $(g_z, g_x) \in \text{Isom}(z, z') \times \text{Isom}(x, x')$

such that

$$\begin{array}{ccc}
 F(z) & \xrightarrow{F(g_z)} & F(z') \\
 f \downarrow & \circlearrowleft & \downarrow f' \\
 G(x) & \xrightarrow{G(g_x)} & G(x')
 \end{array}$$

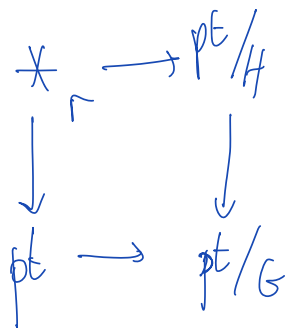
examples



$$\mathcal{X}_y = \left\{ (x, f) \mid y \xrightarrow[f]{} G(x) \right\} + \text{morphisms.}$$

e.g. if $\mathcal{X} = \text{pt}/H$, $\mathcal{Y} = \text{pt}/G$, $H \rightarrow G$ group morphism

then

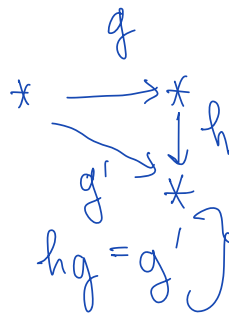


$$\text{ob}(*) = G$$

$g, g' \in G$.

$$\text{Hom}(g, g') = \left\{ h \in H, hg = g' \right\}$$

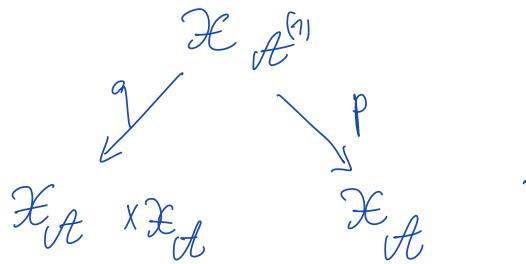
$$\rightarrow * = G/H$$



Very often (if not always), we can upgrade everything to stacks :

$\mathcal{X}_{\mathcal{A}}$ stack of objects in \mathcal{A}

$\mathcal{X}_{\mathcal{A}^{(n)}}$ stack of short exact sequences.



The complexity of the geometry of q is directly related to the homological dimension of \mathcal{A} .

examples to have in mind ① $Q = (I, \Omega)$ quiver

$\mathcal{A} = \text{Rep}_Q(k)$ representations of a quiver
over a field k .

$= \bigsqcup_{d \in \mathbb{N}^I} \text{Ed}/G_d$ stack quotient, or sometimes groupoid.
but never set-theoretic quotient.

$$\text{Ed} = \bigoplus_{i \rightarrow j \in \Omega} \text{Hom}(k^{d_i}, k^{d_j})$$

$$G_d = \prod_{i \in I} \text{GL}_{d_i}$$

or maybe $\mathcal{A} = \text{Rep}_{\Pi_Q}(k)$ $\Pi_Q =$ preprojective
algebra of Q

or maybe $\mathcal{A} = \text{Rep}_{\text{Jac}(Q, w)}(k)$, (Q, w) quiver
with potential and $\text{Jac}(Q, w)$ its Jacobi algebra.

or $\mathcal{A} = \text{Coh}(C)$ for C smooth projective curve

or $\mathcal{A} = \text{Higgs}(C)$

or $\mathcal{A} = \text{Coh}_c(\text{Tot}(w_c \oplus G_c))$.

or $\mathcal{A} = \text{Coh}_c(S)$, S quasiprojective 2-CY surface
(except for $S = T^*C$, C sm. pr. curve, this is very
far from being developed)

We need to associate to the stacks / groupoids \mathcal{Y} under consideration a space $F(\mathcal{Y})$ s.t. we can define a pull-back q^* and a push-forward p_* .

Then, $p_* q^*: F(\mathcal{M}_{\text{ét}}) \times F(\mathcal{M}_{\text{ét}}) \longrightarrow F(\mathcal{M}_{\text{ét}})$

gives an associative multiplication on $F(\mathcal{M}_{\text{ét}})$.

What can be F ?

* If $k = \mathbb{F}_q$ is a finite field, we can consider $\text{Func}(\mathcal{M}_{\text{ét}}(\mathbb{F}_q), \mathbb{C})$ (and it is \mathbb{F}_q -linear)

$\text{Func}(\mathcal{M}_{\text{ét}}(\mathbb{F}_q), \mathbb{C})$

↙
finite support

* We can consider H^* cohomology

* We can consider K -theory

* We the stack under consideration has (a) more structure, for example a "d-critical structure", or can consider the vanishing cycle cohomology.

Actually, the simplest situation is the one with $\text{Rep}_Q(\mathbb{C})$ and cohomology of the moduli spaces.

1D-CoHA of a quiver

[Kontsevich-Sortelmann],
"additive formal group law"

$Q = (I, \Omega)$ quiver

$$\mathcal{M}_d = \bigsqcup_{d \in \mathbb{N}^I} \text{Ed} / G_d$$

$$\text{CoHA}(Q) := H^*(\mathcal{M}_d) = \bigoplus_{d \in \mathbb{N}^I} H^*(\text{Ed} / G_d)$$
$$= H_{G_d}^*(\text{Ed})$$

$$= H_{G_d}^*(\text{pt})$$

$$= \mathbb{C} \left[x_{i,j} \mid \begin{array}{l} i \in I \\ 1 \leq j \leq d_i \end{array} \right]^{G_d}$$

$$G_d = \prod_{i \in I} S_{d_i}$$

product of symmetric groups.

product: This is a shuffle algebra product.

Simpler situation

$$H = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[x_{n1}, \dots, x_{nn}]^{\mathfrak{S}_n}$$

How to build a graded multiplication on H ?

$$g \in \mathcal{F}(x)$$

$$f(x_{n1}, \dots, x_{nm}) \cdot g(x_{m1}, \dots, x_{mm}) \\ = \frac{1}{n!m!} \text{Sym} \left(f(x_{n1}, \dots, x_{nn}) g(x_{m1}, \dots, x_{mm}) \prod_{k=1}^n \prod_{l=1}^m (x_{nk} - x_{ml}) \right)$$

q -weighted shuffle product.

This is the kind of multiplication we obtain on

$$\text{CoHA}(\mathcal{Q}), \quad f \in \text{CoHA}(\mathcal{Q})[d]$$

$$f(x_{ij}) g(y_{ij}) \\ = \frac{1}{d!e!} \text{Sym} \left(f(x_{ij}) g(y_{ij}) \frac{\prod_{u,v \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_v} (y_{vl} - x_{uk})^{d_{uv}}}{\prod_{u \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_u} (y_{ul} - x_{uk})} \right)$$

$$d_{uv} = \# \{ u \rightarrow v \in \mathcal{Q} \}$$

Recent developments: Vertex^(co) algebra structure on $\text{CoHA}(\mathcal{Q})$ + compatibility avec algebra structure

Explanation of the computation of the product for $Q = \bullet$

(computation using basic facts on equivariant cohomology)

$G \curvearrowright X$
reductive group

$H_G^*(X)$ is a $H_G^*(pt)$ -module, using

the map $p^* : H_G^*(pt) \rightarrow H_G^*(X)$; $p : X \rightarrow pt$ G -equivariant.
 $X = pt, T = (\mathbb{C}^*)^n$ $H_T^*(pt) = \mathbb{C}[x_1, \dots, x_n]$; $B = B\mathbb{C}^n$; $H_G(pt) = \mathbb{C}[x_1, \dots, x_n]^{\mathbb{C}^n}$

Theorem: ① $T \subset G$ maximal torus, $W = N_G(T)/T$ Cartan.

$$H_G^*(X) \simeq H_T^*(X)^W$$

② $X \supset T$ torus

X^T fixed points

$X^T \xrightarrow{i} X$ inclusion of fixed points

pull-back in equivariant cohomology

$$i^* : H_T^*(X) \longrightarrow H_T^*(X^T) = H^*(X^T) \otimes H_T^*(pt)$$

push-forward: $i_* : H_T^*(X^T) \longrightarrow H_T^*(X)$

become isomorphism in localized equivariant cohomology:

$$i^* : H_T^*(X) \otimes_{H_T^*(pt)} \text{Frac}(H_T^*(pt)) \xrightarrow{\sim} H^*(X^T) \otimes \text{Frac}(H_T^*(pt)).$$

and $i_* i^* \in \text{End} \left(H_T^*(X) \otimes_{H_T^*(pt)} \text{Frac}(H_T^*(pt)) \right)$ is multiplication by the equivariant

Euler class $e(X^T)$ of X^T the fixed point locus.

We have the tools to compute the multiplication of $\text{CoHA}(\bullet)$.

If G is a group, $BG = pt$ with action of $G = pt/G$.

$$\mathcal{Q} = (\bullet, \emptyset)$$

$$d \in \mathbb{N}, d'+d''=d$$

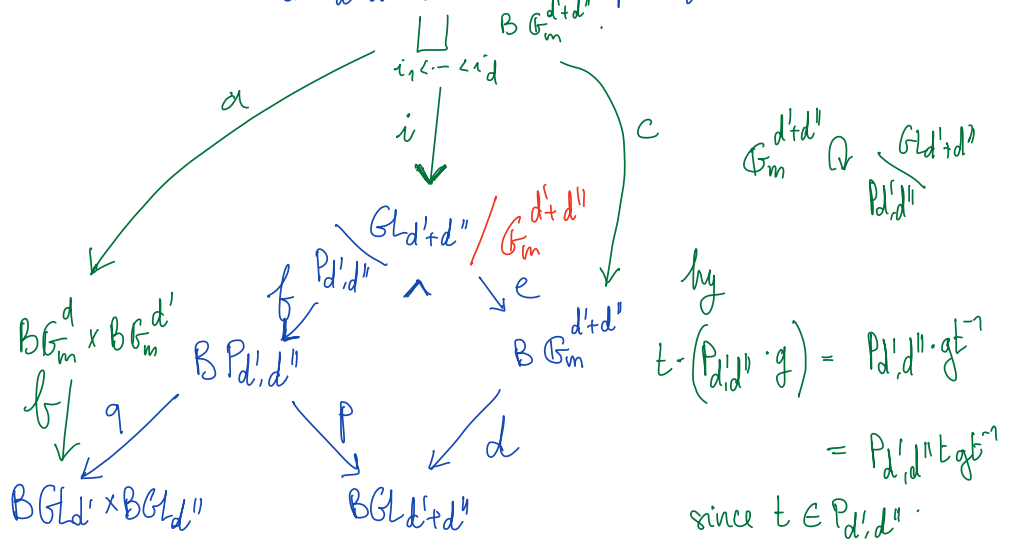
$$\mathcal{U}_d = BG_d$$

$$\mathcal{M}_{d',d''}^{(1)} = BP_{d',d''}$$

(standard)

$P_{d',d''} \subset GL_{d'+d''}$ parabolic, stabilizer of a d' -dimensional subspace of $\mathbb{C}^{d'+d''}$.

Correspondence



To understand $p_* q^*$ acting on cohomology, it suffices to understand

$$\begin{aligned} & d^* p_* q^* \\ &= e_* \circ f^* q^* \\ &= e_* \left(\frac{1}{e(N_i)} \cdot i_* \cdot i^* \cdot f^* q^* \right) \end{aligned}$$

$\underbrace{\quad}_{a^* b^*}$

→ fixed points are given $F \subset \mathbb{C}^{d'+d''}$

$e_i \mapsto e_{d'+d''}$ base de $\mathbb{C}^{d'+d''}$

$$F = \langle e_{i_1}, \dots, e_{i_d} \rangle$$

for $\{i_1, \dots, i_d\} \subset \{1, \dots, d'+d''\}$

split the calculation on each component of $\bigsqcup_{i_1 < \dots < i_d} BGL_m^{d'+d''}$.

On $B \mathbb{G}_m^{d'+d''}$ corresponding to $i_1 < \dots < i_{d'}$, $p \in \mathbb{G}_m^{d'+d''}$ corresponding to $\mathbb{P}^{d'+d''}$

$$Ni|_p \cong T_p \mathbb{P}^{d'+d''} \setminus \mathbb{G}_m^{d'+d''} \quad \text{fixed point.}$$

The weights of $\mathbb{G}_m^{d'+d''}$ on this space are

$$\text{for } \begin{matrix} i \in \{1, \dots, d'+d''\} \setminus \{i_1, \dots, i_{d'}\} = [1, d'+d''] \setminus I \\ j \in \{i_1, \dots, i_{d'}\} = I \end{matrix}$$

$$\text{So } e(Ni)|_p = \prod_{i \in [1, d'+d''] \setminus I} (x_i - \alpha_j) \in H^*_{\mathbb{G}_m^{d'+d''}}(\text{pt}).$$

Altogether, we obtain the desired formula:

$$f(x_1, \dots, x_{d'}) * g(x_{d'+1}, \dots, x_{d'+d''}) = \sum_{\substack{I \cup J = \{1, \dots, d'+d''\} \\ |I| = d'}} f(x_I) g(x_J) \frac{1}{\prod_{i \in J} \prod_{j \in I} (x_i - x_j)}.$$

Next simple situation:

$Q = (\mathbb{I}, \Omega)$ quiver and k -theory

$$KHA(Q) = \bigoplus_{d \in \mathbb{N}^{\mathbb{I}}} K(E_d/G_d)$$

||

$K^{G_d}(\text{pt})$

||

$$\mathbb{Z}[x_{i,k}^{\pm 1}, i \in \mathbb{I}, 1 \leq k \leq d_i] \otimes_{\mathbb{Z}} \mathbb{Q}^d$$

Product: "multiplicative formal group law".

$$f \in KHA(Q)[d], g \in KHA(Q)[e].$$

$$f(x_{i,k})g(y_{i,k}) = \frac{\prod_{u \in \mathbb{I}} \prod_{k=1}^{d_u} \prod_{l=1}^{e_u} (y_{i,k} / x_{i,k})^{a_{u,l}}}{\prod_{u \in \mathbb{I}} \prod_{k=1}^{d_u} \prod_{l=1}^{e_u} y_{i,k} / x_{i,k}}$$

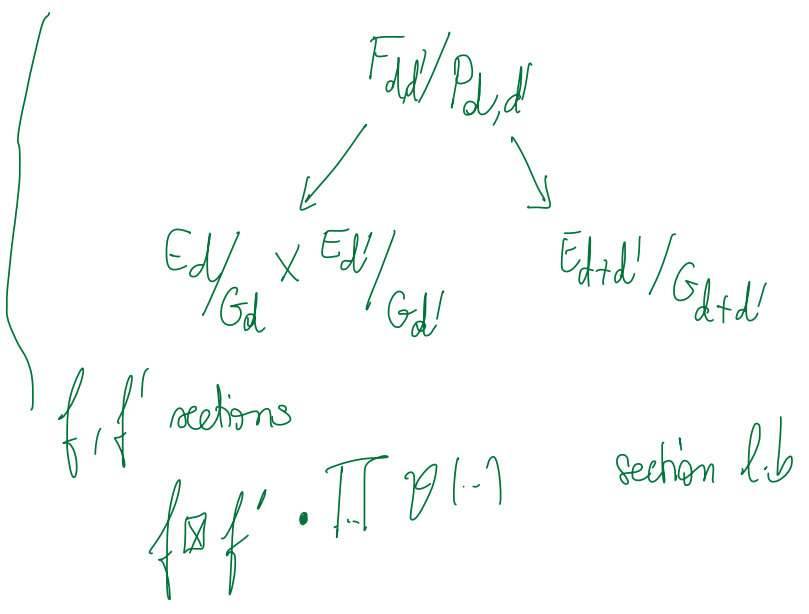
Tricky situation: elliptic cohomology

Guess what the formula is.

$$\text{Ell}(E_d/G_d) = \text{Ell}_{G_d}(\text{pt})$$

$$= (E^d)^{\mathbb{G}_d}$$

L, L' line bundles on $(E^d)^{\mathbb{G}_d}, (E^{d'})^{\mathbb{G}_{d'}}$



Constructible functions on groupoid of \mathbb{F}_q -points

→ Constructible Hall algebra, as originally defined by Ringel for quivers.

\mathcal{A} is a finitary category

i.e. $\forall M, N$ objects of \mathcal{A} , $\text{Ext}_{\mathcal{A}}^0(M, N)$ is a finite set.

Then one can define an associative product on

$$H_{\mathcal{A}} = \bigoplus_{[M] \in \text{ob}(\mathcal{A})/\sim} \mathbb{C} \cdot [M] = \text{Func}(\text{ob}(\mathcal{A})/\sim, \mathbb{C})$$

in the following way $=: \langle M, N \rangle_m$ "multiplicative Euler form".

$$[M] * [N] = \frac{|\text{Hom}(M, N)|}{|\text{Ext}^0(M, N)|} \sum_{[R] \in \text{ob}(\mathcal{A})/\sim} \underbrace{\# \{ N' \subset R \mid N' \simeq N \ \& \ R/N' \simeq M \}}_{F_{MN}^R} [R]$$

to makes the product have more pleasant properties.

Associativity: not hard to check.

One can build a comultiplication in the dual way:

$$\Delta([R]) = \sum_{[M], [N]} \sqrt{\langle M, N \rangle_m} \frac{a_M a_N}{a_R} \cdot [M] \otimes [N]$$

Green's thm: If \mathcal{A} is hereditary ($\text{Ext}^i(-, -) = 0$ for $i \geq 2$) then $H_{\mathcal{A}}$ is a bialgebra:

$$\Delta \circ m = m \otimes m \circ \sigma_{23} \circ \Delta \otimes \Delta$$

↙
swap of 2nd & 3rd factors.

$Q = (I, \Sigma)$ quiver; \mathbb{F}_q finite field

$\mathcal{A} = \text{Rep}_Q(\mathbb{F}_q)$

$H_{\mathcal{A}} =: H_{Q, \mathbb{F}_q}$

Hall algebra of Q over \mathbb{F}_q .

Structure of H_{Q, \mathbb{F}_q} ?

Ringel's theorem

Assume Q loop free (for simplicity)

$U_{\sqrt{q}}(\mathcal{N}^+)$ quantum unipotent enveloping algebra.

$E_i \mapsto [S_i]$ induces an injective algebra morphism

$$U_{\sqrt{q}}(\mathcal{N}^+) \xrightarrow{\phi} H_{Q, \mathbb{F}_q}$$

S_i simple representation of Q
 $\dim S_i = i \in \mathbb{N}^I$

$E_i, i \in I$: Chevalley generators.

$\mathcal{U}_q(\mathfrak{r}^+)$ is the associative algebra generated by $E_i, i \in I$ subject to the Serre relations

$$\sum_{l=0}^{1-a_{ij}} \binom{1-a_{ij}}{l}_q E_i^l E_j E_i^{1-a_{ij}-l} = 0$$

$$\forall i \neq j$$

where $(a_{ij}) = A$ is the Cartan matrix of \mathcal{Q} :

$$a_{ij} = 2\delta_{ij} - \#\{\alpha: i \rightarrow j\} - \#\{\alpha: j \rightarrow i\}$$

* ϕ surjective $\Leftrightarrow \mathcal{Q}$ is of finite type

example: $\mathcal{Q} = \bullet$

$\text{Rep}_{\mathcal{Q}}(\mathbb{F}_q) = \text{Vect}_{\mathbb{F}_q}$
objects = \mathbb{N}
integers

Euler form: $\langle m, n \rangle_m = q^{mn}$

$$\text{Hom}_{\mathbb{F}_q} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[n]$$

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q! [n]_q!}$$

$$[m] \cdot [n] = \begin{bmatrix} m+n \\ m \end{bmatrix}_q \sqrt{q^{mn}} [m+n]$$

$$[m]_q! = \prod_{i=1}^m [i]_q!$$

$$[i]_q = \frac{q^i - 1}{q - 1}$$

$$= \binom{m+n}{m}_{\sqrt{q}} [m+n]$$

symmetrized version of q -binomial coefficients.

Categorification of the quantum group $U_r(\mathfrak{sl}^+)$

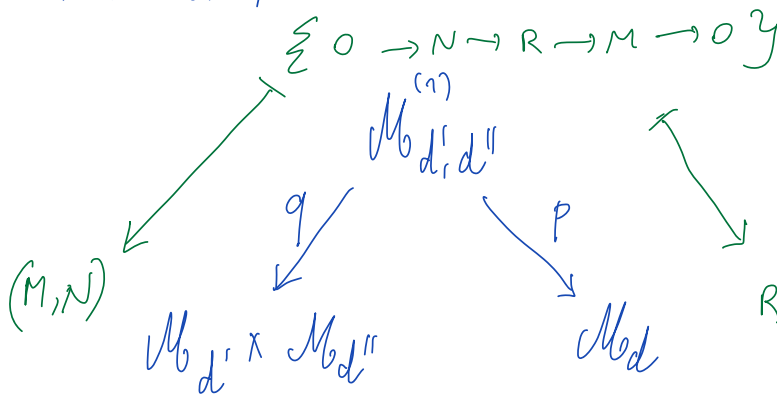
(Lusztig)

Instead of functions on $M_{\mathbb{Q}}(\mathbb{F}_q)$, consider perverse
 (constructible) sheaves on $M_{\mathbb{Q}}$.

constructible

The base field can be \mathbb{C} or $\overline{\mathbb{F}_q}$.

$$d' + d'' = d, \quad d', d', d'' \in \mathbb{N}^I.$$



Remind that q is smooth (of relative dimension $-\langle d', d'' \rangle$) & p is proper.

Define

$$\text{Ind}_{d', d''}^b : \mathcal{D}_c^b(\mathcal{M}_{d'} \times \mathcal{M}_{d''}) \longrightarrow \mathcal{D}_c^b(\mathcal{M}_d)$$

Takes semisimple complexes to semisimple complexes

$$\mathcal{F} \longmapsto p_* q^*[-\langle d', d'' \rangle]$$

Lusztig considers the category \mathcal{Q} of constructible complexes on $M = \bigsqcup_{d \in \mathbb{N}^F} M_d$ containing

* $\bigoplus M_i$ for $i \in I$,

* stable under shifts, direct summands

* stable under $\text{Ind}_{d', d''}^d$.

$\mathcal{P} \subset \mathcal{Q}$

("semisimple") perverse sheaves contained in \mathcal{Q} .

$K_{\oplus}(\mathcal{Q})$ becomes an algebra, the product being induced by $\text{Ind}_{d', d''}^d$.

It is a $\mathbb{Z}[\nu^{\pm 1}]$ -algebra where ν acts as shift [1].

Thm (Lusztig) The map

$$\mathcal{V}_{\nu^{-1}}(\mathbb{Z}^+) \longrightarrow K_{\oplus}(\mathcal{Q})$$

$$E_i \longmapsto \bigoplus M_i [\dim M_i]$$

is well defined and induces an isomorphism of algebras.

$\mathcal{U}_v^{\mathbb{Z}}(\mathbb{Z}^+)$ is Lusztig integral form of $\mathcal{U}_v(\mathbb{Z}^+)$.

In other words, \mathcal{Q} categorifies $\mathcal{U}_v^{\mathbb{Z}}(\mathbb{Z}^+)$.

* If one works over $\overline{\mathbb{F}_q}$, there is an other way to decategorify the category \mathcal{Q} .

* Over \mathbb{C} , one can consider the characteristic cycle of constructible sheaves.

But this is still a long story.