Exercise sheet 2

Thursday, 19 November 2020

1 Algebraic groups

Exercice 2.1. Show that the conclusion of the theorem of Lie-Kolchin does not hold in general for non-connected solvable algebraic groups.

Exercice 2.2. Borel's theorem.

1. We work over k an algebraically closed field. Let G be a unipotent group and X a quasi-affine (open subvariety of an affine variety) algebraic variety on which G acts. Show that all G-orbits are closed.

2. Show that if X is assumed to be quasi-projective, this is false. (For example, find an action of $\mathbf{G}_a$ on $\mathbf{P}^1$)

Exercice 2.3. Chevalley's theorem. We assume that k is algebraically closed for simplicity. The theorem remains true if it is not the case. We will prove the following result. Let G be a linear algebraic group over k and H an algebraic subgroup. Show that there is a closed immersion $G \to \mathrm{GL}(V)$ for some finite dimensional k vector space V such that $H = \mathrm{Stab}_G(D)$ for some line $D \in \mathbf{P}^1(V)$.

Exercice 2.4. The unipotent radical is connected. Show that the unipotent radical of a soluble linear algebraic group is connected.

Exercice 2.5. Burnside's theorem Assume k is algebraically closed. Let V be a finite dimensional k vector space and $A \subset \mathrm{End}_k(V)$ an associative algebra such that A acts irreducibly on V. Then, show that $A = \mathrm{End}_k(V)$.

2 Lie algebras

Exercice 2.6. The tangent Lie algebra. Let G be an algebraic group over an algebraically closed field k. We let $\Gamma(G, TG)^G$ be the Lie algebra of left-invariant derivations on G and $\mathfrak{g}$ the Lie algebra of derivations of $k[G]$ at the identity. Recall the precise definitions of the objects under consideration. Show that

$$\begin{array}{ccc} \Gamma(G, TG)^G & \to & \mathfrak{g} \\ \delta & \mapsto & e \circ \delta \end{array}$$

(e is the counit of $k[G]$) and

$$\begin{array}{ccc} \mathfrak{g} & \to & \Gamma(G, TG)^G \\ d & \mapsto & (\mathrm{id} \otimes d) \circ \Delta \end{array}$$

are inverse isomorphisms of Lie algebras.

Exercice 2.7. Let G be an algebraic group over an algebraically closed field k.

1. Show that $TG \simeq G \times \mathfrak{g}$.

2. Show that if $\varphi : G \to H$ is a morphism of linear algebraic groups, then $d\varphi(e) : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras.